

An encoding of Interval Temporal Logic in Isabelle/HOL

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Abstract

These Isabelle theories introduce the semantics and syntax of Finite Interval Temporal Logic (ITL). The ITL proof system, as introduced in [5, 9], has been encoded and its soundness has been checked. The encoding is shallow using the Intensional Logic technique of [3]. An extensive library of Finite ITL theorems, taken from [8], has been checked.

We also present a theory of first occurrence and use it to derive an algebra of Runtime verification (RV) monitors. Furthermore we provide examples of using quantification over both static (rigid) and state (flexible) variables and several RV examples.

Furthermore the notion of executability of ITL formulae has been formalised.

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1 Finite Intervals

theory *Interval*

imports

Main

begin

An interval is a finite sequence of elements of a particular type. Intervals are similar to list in Isabelle/HOL, the difference is that intervals have instead of nil a single element at the end. So an interval of length zero is a single element whereas for Isabelle's list we have that an empty list is nil (no element present).

The usual operations on intervals are defined: *ilen*, *prefix*, *suffix*, *sub*, *inth*, *ifirst*, *ilast*, *iapp* and *irev*.

We also introduce *index-sequence* which is a sequence of chop (fuse) points. This sequence is used in the old definition of chopstar which is an existential quantification over this sequence. The type *index-sequence* is again of type interval but the elements are natural numbers. Two functions *shift* and *shiftn* are introduced that are used to add (shift) and subtract a natural number of each element in the sequence of chop (fuse) points. The operation *upt* to produce a sequence of consecutive chop points between two natural numbers.

1.1 Definitions

datatype (*iset*: 'a) *interval* =
 INil 'a ([*-*])
 | *ICons* 'a 'a *interval* (**infixr** \odot 65)

for

map: *imap*

rel: *interval-all2*

pred: *interval-all*

type-synonym *index* = *nat interval*

syntax

— interval Enumeration

-interval :: *args* => 'a *interval* (*(((-))*)

translations

$\langle x, xs \rangle == x \odot \langle xs \rangle$

$\langle x \rangle == [x]$

primrec (*nonexhaustive*) *ilen* :: 'a interval \Rightarrow nat **where**

ilen $\langle x \rangle = 0$
| *ilen* ($x \odot xs$) = 1 + (*ilen* *xs*)

primrec (*nonexhaustive*) *inth* :: 'a interval \Rightarrow nat \Rightarrow 'a **where**

inth $\langle x \rangle$ *n* = *x*
| *inth* ($x \odot xs$) *n* = (case *n* of 0 \Rightarrow *x* | Suc *k* \Rightarrow *inth* *xs* *k*)

primrec *prefix*:: nat \Rightarrow 'a interval \Rightarrow 'a interval **where**

prefix *n* $\langle x \rangle = \langle x \rangle$
| *prefix* *n* ($x \odot xs$) = (case *n* of 0 \Rightarrow $\langle x \rangle$ | Suc *m* \Rightarrow $x \odot (\text{prefix } m \text{ } xs)$)

primrec *suffix*:: nat \Rightarrow 'a interval \Rightarrow 'a interval **where**

suffix *n* $\langle x \rangle = \langle x \rangle$
| *suffix* *n* ($x \odot xs$) = (case *n* of 0 \Rightarrow ($x \odot xs$) | Suc *m* \Rightarrow *suffix* *m* *xs*)

definition *sub*:: nat \Rightarrow nat \Rightarrow 'a interval \Rightarrow 'a interval

where

sub *n* *k* *xs* = *prefix* (*k* − *n*) (*suffix* *n* *xs*)

abbreviation *ifirst* :: 'a interval \Rightarrow 'a **where**

ifirst *xs* \equiv (*inth* *xs* 0)

abbreviation *ilast* :: 'a interval \Rightarrow 'a **where**

ilast *xs* \equiv (*inth* *xs* (*ilen* *xs*))

primrec *iapp* :: 'a interval \Rightarrow 'a interval \Rightarrow 'a interval (**infixr** \ominus 65) **where**

iapp-INil: $\langle x \rangle \ominus ys = x \odot ys$ |

iapp-ICons: ($x \odot xs$) $\ominus ys = x \odot (xs \ominus ys)$

primrec *irev* :: 'a interval \Rightarrow 'a interval **where**

irev $\langle x \rangle = \langle x \rangle$

| *irev* ($x \odot xs$) = (*irev* *xs*) $\ominus \langle x \rangle$

definition *index-sequence* :: nat \Rightarrow index \Rightarrow bool **where**

index-sequence *x* *idx* \equiv (*inth* *idx* 0 = *x*) \wedge (\forall *n*. *n* < *ilen* *idx* \longrightarrow *inth* *idx* *n* < *inth* *idx* (Suc *n*))

definition *shift* :: nat \Rightarrow nat \Rightarrow nat **where**

shift *k* = (λ *x*. *x* + *k*)

definition *shiftn* :: nat \Rightarrow nat \Rightarrow nat **where**

shiftn *k* = (λ *x*. *x* − *k*)

primrec *upt* :: nat \Rightarrow nat \Rightarrow nat interval ((1[.. \leq]/- \rfloor))

where

upt-0 : [*i*.. \leq 0] = $\langle 0 \rangle$

| *upt-Suc*: [*i*.. \leq (Suc *j*)] = (if *i* \leq *j* then [*i*.. \leq *j*] $\ominus \langle$ (Suc *j*) \rangle else \langle (Suc *j*) \rangle)

1.2 Lemmas

Basic lemmas are introduced for each of the above operations on intervals.

1.2.1 Shifting indexes to zero

lemma *interval-shift-index-to-zero-a*:

shows $(\forall (i::nat). a \leq i \wedge i < a+b \longrightarrow f (g1(i)) (g2(Suc i))) =$
 $(\forall i. 0 \leq i \wedge i < b \longrightarrow f (g1(i+a)) (g2((Suc i)+a)))$

by *simp (metis add.commute le-Suc-ex le-add2 nat-add-left-cancel-less)*

lemma *interval-shift-index-to-zero-b*:

shows $(\forall (i::nat). a \leq i \wedge i < a+b \longrightarrow f (g(i-a)) (g2((Suc i)-a))) =$
 $(\forall i. 0 \leq i \wedge i < b \longrightarrow f (g(i)) (g2(Suc i)))$ (**is** $?L=?R$)

proof –

have 1: $?L \Longrightarrow ?R$

by (*metis add-Suc add-diff-cancel-left' add-diff-cancel-right' le-add2 less-diff-conv*)

have 2: $?R \Longrightarrow ?L$

by (*metis Nat.add-diff-assoc add-diff-cancel-left' le0 le-add-diff-inverse2 less-diff-conv plus-1-eq-Suc*)

show *?thesis* **using** 1 2 **by** *blast*

qed

1.2.2 Interval Length

lemma *ilen-gr-zero* [*simp*]:

$ilen\ xs \geq 0$

by *auto*

lemma *ilen-ICons* [*simp*]:

$(ilen\ (x \odot xs)) = (ilen\ xs) + 1$

by *simp*

lemma *ilen-ICons-1* :

$ilen\ l > 0 = (\exists\ x\ ls. l = x \odot ls)$

by (*induct l*) *simp-all*

lemma *ilen-imap* [*simp*]:

$ilen\ (imap\ f\ xs) = ilen\ xs$

by (*induct xs*) *simp-all*

lemma *ilen-iapp* [*simp*]:

$ilen\ (xs \ominus ys) = (ilen\ xs) + (ilen\ ys) + 1$

by (*induct xs arbitrary: ys*) *simp-all*

lemma *irev-ilen* [*simp*]:

$ilen\ (irev\ xs) = ilen\ xs$

by (*induct xs*) *simp-all*

1.2.3 inth

lemma *inth-zero* [simp]:

$$\text{inth } (x \odot xs) \ 0 = x$$

by *simp*

lemma *inth-Suc* [simp]:

$$\text{inth } (x \odot xs) \ (\text{Suc } n) = \text{inth } xs \ n$$

by *auto*

lemma *inth-last*:

$$\text{inth } (x \odot xs) \ (\text{ilen } (x \odot xs)) = \text{inth } xs \ (\text{ilen } xs)$$

by *simp*

lemma *inth-last-stutter*:

$$\text{inth } xs \ (\text{ilen } xs + i) = \text{inth } xs \ (\text{ilen } xs)$$

proof (*induction xs arbitrary:i*)

case (*INil x*)

then show ?*case* **by** *simp*

next

case (*ICons x1a xs*)

then show ?*case* **by** *simp*

qed

lemma *inth-ICons-a*:

assumes $0 < i$

shows $\text{inth}(x \odot xs) \ i = \text{inth } xs \ (i-1)$

using *assms* **by** (*metis Suc-diff-1 inth-Suc*)

lemma *inth-ICons-b*:

shows $\text{inth}(x \odot xs) \ (i+1) = \text{inth } xs \ i$

by *simp*

lemma *inth-ICons*:

assumes $0 < i$

shows $\text{inth}(x \odot xs) \ i = \text{inth } xs \ (i-1) \wedge$
 $\text{inth}(x \odot xs) \ (i+1) = \text{inth } xs \ i$

by (*meson assms inth-ICons-a inth-ICons-b*)

lemma *inth-zero-ifirst*:

$$\text{ifirst } xs = \text{inth } xs \ 0$$

by *simp*

lemma *inth-ilen-ilast*:

$$\text{ilast } xs = \text{inth } xs \ (\text{ilen } xs)$$

by *simp*

lemma *INil-ilen* :

$$(xs = \langle x \rangle) \longleftrightarrow \text{ilen } xs = 0 \wedge \text{inth } xs \ 0 = x$$

by (*cases xs*) *auto*

lemma *interval-eq-inth-eq* :
 $(xs = ys) = (ilen\ xs = ilen\ ys \wedge (\forall\ i \leq ilen\ xs.\ inth\ xs\ i = inth\ ys\ i))$
proof
(induct xs arbitrary: ys)
case (*INil* *x*)
then show ?*case* **by** (*metis INil-ilen le-numeral-extra*(3))
next
case (*ICons* *x1a xs*)
then show ?*case*
proof (*cases ys*)
case (*INil* *x1*)
then show ?*thesis* **by** *simp*
next
case (*ICons* *x21 x22*)
then show ?*thesis*
using *ICons.hyps* **by** *fastforce*
qed
qed

lemma *inth-imap* :
 $inth\ (imap\ f\ xs)\ i = f\ (inth\ xs\ i)$
proof
(induct xs arbitrary: i)
case (*INil* *x*)
then show ?*case* **by** *simp*
next
case (*ICons* *x1a xs*)
then show ?*case*
proof (*cases i*)
case 0
then show ?*thesis* **by** *simp*
next
case (*Suc* *nat*)
then show ?*thesis* **using** *ICons.hyps* **by** *simp*
qed
qed

1.2.4 prefix, suffix and sub

lemma *prefix-INil* [*simp*]:
 $prefix\ m\ \langle x \rangle = \langle x \rangle$
by *simp*

lemma *prefix-suc* [*simp*]:
 $prefix\ (Suc\ m)\ (x \odot xs) = x \odot (prefix\ m\ xs)$
by *auto*

lemma *prefix-zero* [*simp*]:
 $prefix\ 0\ (x \odot xs) = \langle x \rangle$

by *auto*

lemma *prefix-zero-ifirst* [*simp*]:

$\text{prefix } 0 \text{ } xs = \langle \text{inth } xs \ 0 \rangle$

by (*induct xs*) *simp-all*

lemma *ifirst-prefix* [*simp*]:

shows $\text{ifirst } (\text{prefix } i \text{ } xs) = \text{ifirst } xs$

proof

(*induct xs arbitrary: i*)

case (*INil x*)

then show *?case* **by** *auto*

next

case (*ICons x1a xs*)

then show *?case*

proof (*cases i*)

case *0*

then show *?thesis* **by** *auto*

next

case (*Suc nat*)

then show *?thesis* **using** *ICons.hyps* **by** *auto*

qed

qed

lemma *ilast-suffix* [*simp*]:

shows $\text{ilast } (\text{suffix } i \text{ } xs) = \text{ilast } xs$

proof

(*induct xs arbitrary: i*)

case (*INil x*)

then show *?case* **by** *auto*

next

case (*ICons x1a xs*)

then show *?case*

proof (*cases i*)

case *0*

then show *?thesis* **by** *auto*

next

case (*Suc nat*)

then show *?thesis* **using** *ICons.hyps* **by** *auto*

qed

qed

lemma *prefix-ilen* [*simp*]:

$(\text{prefix } (\text{ilen } xs) \text{ } xs) = xs$

by (*induct xs*) *simp-all*

lemma *prefix-ilen-gr-1* [*simp*]:

$(\text{prefix } ((\text{ilen } xs) + i) \text{ } xs) = xs$

by (*induct xs simp-all*)

lemma *ilen-prefix-ICons* [*simp*]:

$ilen(\text{prefix } (Suc\ i) (x \odot xs)) = 1 + ilen(\text{prefix } i\ xs)$

using *ilen-ICons* **by** *auto*

lemma *prefix-ilen-code* [*code*]:

$ilen(\text{prefix } i\ xs) = (\text{if } i \leq ilen\ xs \text{ then } i \text{ else } ilen\ xs)$

proof

(*induct xs arbitrary: i*)

case (*INil x*)

then show ?*case* **by** *simp*

next

case (*ICons x1a xs*)

then show ?*case*

proof (*cases i*)

case 0

then show ?*thesis* **by** *auto*

next

case (*Suc nat*)

then show ?*thesis* **using** *ICons.hyps* **by** *auto*

qed

qed

lemma *prefix-ilen-min* [*simp*]:

$ilen(\text{prefix } i\ xs) = \min i (ilen\ xs)$

by (*simp add: prefix-ilen-code min-def*)

lemma *prefix-ilen-good* [*simp*]:

assumes $i \leq ilen\ xs$

shows $(ilen(\text{prefix } i\ xs)) = i$

using *assms* **by** *simp*

lemma *prefix-ilen-bad* :

assumes $i > ilen\ xs$

shows $ilen(\text{prefix } i\ xs) = ilen\ xs$

using *assms* **by** *simp*

lemma *prefix-ilen-bound* :

shows $ilen(\text{prefix } i\ xs) \leq ilen\ xs$

by *simp*

lemma *suffix-ilen-code* [*code*]:

$ilen(\text{suffix } i\ xs) = (\text{if } i \leq ilen\ xs \text{ then } (ilen\ xs) - i \text{ else } 0)$

proof

(*induct xs arbitrary: i*)

case (*INil x*)

then show ?*case* **by** *simp*

next

case (*ICons x1a xs*)

```

then show ?case
proof (cases i)
case 0
then show ?thesis by auto
next
case (Suc nat)
then show ?thesis using ICons.hyps by auto
qed
qed

```

```

lemma suffix-ilen [simp]:
   $ilen\ (suffix\ i\ xs) = (ilen\ xs) - i$ 
by (simp add: suffix-ilen-code)

```

```

lemma suffix-ilen-good [simp]:
  assumes  $i \leq ilen\ xs$ 
  shows  $ilen\ (suffix\ i\ xs) = (ilen\ xs) - i$ 
using assms by simp

```

```

lemma suffix-ilen-bad:
  assumes  $i > ilen\ xs$ 
  shows  $ilen\ (suffix\ i\ xs) = 0$ 
using assms by simp

```

```

lemma suffix-ilen-bound:
   $ilen(suffix\ i\ xs) \leq ilen\ xs$ 
by simp

```

```

lemma inth-prefix [simp]:
  assumes  $k \leq i$ 
  shows  $inth\ (prefix\ i\ xs)\ k = inth\ xs\ k$ 
using assms
proof
  (induct i arbitrary: xs k)
  case 0
  then show ?case
  proof (cases xs)
  case (INil x1)
  then show ?thesis by auto
  next
  case (ICons x21 x22)
  then show ?thesis using 0.prem by auto
  qed
  next
  case (Suc i)
  then show ?case
  proof (cases xs)
  case (INil x1)
  then show ?thesis by auto
  next

```

```

case (ICons x21 x22)
then show ?thesis
proof (cases k)
case 0
then show ?thesis by (simp add: ICons)
next
case (Suc nat)
then show ?thesis
using Suc.hyps Suc.prem1s ICons by auto
qed
qed
qed

```

```

lemma inth-suffix [simp]:
assumes  $k \leq \text{ilen } xs - i$ 
shows  $\text{inth } (\text{suffix } i \text{ } xs) \text{ } k = \text{inth } xs \text{ } (i+k)$ 
using assms
proof (induct xs arbitrary: i k)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
proof (cases i)
case 0
then show ?thesis by auto
next
case (Suc nat)
then show ?thesis
proof auto
assume a0:  $i = \text{Suc } nat$ 
show  $\text{inth } (\text{suffix } nat \text{ } xs) \text{ } k = \text{inth } xs \text{ } (nat + k)$ 
using a0 ICons.hyps ICons.prem1s by auto
qed
qed
qed

```

```

lemma suffix-prefix-help-1:
assumes  $ia+i \leq \text{ilen } xs$ 
 $k \leq ia$ 
shows  $\text{inth } (\text{prefix } ia \text{ } (\text{suffix } i \text{ } xs)) \text{ } k = \text{inth } (\text{suffix } i \text{ } (\text{prefix } (ia+i) \text{ } xs)) \text{ } k$ 
proof -
have 1:  $\text{inth } (\text{prefix } ia \text{ } (\text{suffix } i \text{ } xs)) \text{ } k = \text{inth } (\text{suffix } i \text{ } xs) \text{ } k$ 
using inth-prefix assms by metis
have 2:  $\text{inth } (\text{suffix } i \text{ } xs) \text{ } k = \text{inth } xs \text{ } (i+k)$ 
using inth-suffix assms by (simp add: add-le-imp-le-diff)
have 3:  $\text{inth } xs \text{ } (i+k) = \text{inth } (\text{prefix } (ia+i) \text{ } xs) \text{ } (i+k)$ 
using inth-prefix assms by simp
have 4:  $\text{inth } (\text{prefix } (ia+i) \text{ } xs) \text{ } (i+k) = \text{inth } (\text{suffix } i \text{ } (\text{prefix } (ia+i) \text{ } xs)) \text{ } k$ 
using inth-suffix assms by simp

```

from 1 2 3 4 show ?thesis by auto
qed

lemma suffix-prefix-help-2:

assumes $ia+i \leq \text{ilen } xs$

shows $(\forall k \leq ia . \text{inth } (\text{prefix } ia \ (\text{suffix } i \ xs)) \ k = \text{inth } (\text{suffix } i \ (\text{prefix } (ia+i) \ xs)) \ k)$

using suffix-prefix-help-1 using assms by fastforce

lemma suffix-prefix-help-3:

assumes $ia+i \leq \text{ilen } xs$

shows $\text{ilen } (\text{prefix } ia \ (\text{suffix } i \ xs)) = \text{ilen } (\text{suffix } i \ (\text{prefix } (ia+i) \ xs))$

using assms prefix-ilen-good suffix-ilen-good by auto

lemma suffix-prefix-swap:

assumes $ia+i \leq \text{ilen } xs$

shows $\text{prefix } ia \ (\text{suffix } i \ xs) = \text{suffix } i \ (\text{prefix } (ia+i) \ xs)$

using assms using interval-eq-inth-eq by fastforce

lemma prefix-prefix-zero [simp]:

$\text{prefix } 0 \ (\text{prefix } 0 \ xs) = \text{prefix } 0 \ xs$

by (induct xs) simp-all

lemma pref-pref [simp]:

$(\text{prefix } i \ (\text{prefix } i \ xs)) = \text{prefix } i \ xs$

by (metis prefix-ilen prefix-ilen-gr-1 prefix-ilen-good less-imp-add-positive not-less)

lemma pref-pref-3 [simp]:

$(\text{prefix } i \ (\text{prefix } (i+k) \ xs)) = \text{prefix } i \ xs$

proof

(induct xs arbitrary: i k)

case (INil x)

then show ?case by simp

next

case (ICons x1a xs)

then show ?case

proof (cases i)

case 0

then show ?thesis

by (auto simp add: Nitpick.case-nat-unfold)

next

case (Suc nat)

then show ?thesis

by (simp add: ICons.hyps)

qed

qed

lemma pref-help:

assumes $i \leq \text{ilen } (\text{prefix } (\text{ilen } xs - \text{Suc } 0) \ xs)$

shows $(\text{prefix } i \ (\text{prefix } (\text{ilen } xs - \text{Suc } 0) \ xs)) = (\text{prefix } i \ xs)$

```

using assms
by (metis diff-le-self pref-pref-3 prefix-ilen-good
      ordered-cancel-comm-monoid-diff-class.add-diff-inverse)

lemma pref-pref-help:
  assumes ilen xs > 0
           ia < ilen xs
  shows   (prefix ia (prefix (ilen xs - Suc 0) xs)) = (prefix ia xs)
using assms
by (metis Suc-leI Suc-le-mono Suc-pred diff-le-self pref-help prefix-ilen-good)

lemma interval-pref-pref-help-1:
  assumes i > 0
           i ≤ ilen xs
  shows   (prefix (ilen (prefix i xs) - Suc 0) (prefix i xs)) =
            (prefix (ilen (prefix i xs) - Suc 0) xs)
using assms pref-pref-3 by (metis diff-le-self prefix-ilen-good le-iff-add)

lemma suffix-suc [simp]:
  suffix (Suc m) (x ⊙ xs) = suffix m xs
by auto

lemma suffix-zero [simp]:
  suffix 0 xs = xs
by (induct xs) simp-all

lemma interval-hd-tail:
  assumes ilen xs > 0
  shows   xs = (ifirst xs) ⊙ (suffix 1 xs)
by (metis One-nat-def assms ilen-ICons-1 inth-zero suffix-suc suffix-zero)

lemma suffix-ilen-last [simp]:
  suffix (ilen xs) xs = ⟨(inth xs (ilen xs))⟩
by (induct xs) simp-all

lemma suffix-ilast:
  suffix (ilen xs) xs = ⟨ilast xs⟩
by simp

lemma suffix-suffix [simp]:
  suffix i (suffix j xs) = suffix (i+j) xs
proof
  (induct xs arbitrary: i j)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
  proof (cases i)
  case 0

```



```

then show ?thesis
  by auto
next
case (Suc nat)
then show ?thesis
  by (simp add: Nitpick.case-nat-unfold add.commute ICons)
qed
qed

lemma prefix-suffix-ilen-code [code]:
  ilen (prefix ia (suffix i xs)) =
    (if i ≤ ilen xs then
      (if ia ≤ ilen xs - i then ia else (ilen xs) - i)
      else 0)
using suffix-ilen-code by auto

lemma prefix-suffix-ilen [simp]:
  ilen (prefix ia (suffix i xs)) = min ia (ilen xs - i)
by auto

lemma prefix-suffix-ilen-good [simp]:
  assumes ia ≤ ilen xs - i
    i ≤ ilen xs
  shows ilen (prefix ia (suffix i xs)) = ia
using assms by auto

lemma prefix-suffix-ilen-bad-0:
  assumes i > ilen xs
  shows ilen (prefix ia (suffix i xs)) = 0
using assms by simp

lemma prefix-suffix-ilen-bad-1 :
  assumes i ≤ ilen xs
    ia > ilen xs - i
  shows ilen (prefix ia (suffix i xs)) = (ilen xs) - i
using assms by simp

lemma suffix-suffix-3:
  assumes i > 0
    ia < i
    i ≤ ilen xs
  shows (suffix (i - ia) (suffix ((ilen xs) - i) xs)) = (suffix (((ilen xs) - ia)) xs)
using assms by simp

lemma sub-zero-prefix :
  sub 0 k xs = prefix k xs
by (simp add: sub-def)

lemma sub-suffix :
  assumes i < j

```

$j \leq (\text{ilen } xs) - k$
shows $(\text{sub } (i+k) (j+k) xs) = (\text{sub } i j (\text{suffix } k xs))$
using *assms* **by** (*simp add: sub-def*)

lemma *sub-prefix-suffix-0*:
assumes $0 \leq i$
 $ia + i \leq \text{ilen } xs$
shows $(\text{sub } i (i+ia) xs) = (\text{prefix } (ia) (\text{suffix } i xs))$
using *assms* **by** (*simp add: sub-def*)

lemma *sub-prefix-suffix*:
assumes $0 \leq i$
 $i \leq j$
 $j \leq \text{ilen } xs$
shows $(\text{sub } i j xs) = (\text{prefix } (j-i) (\text{suffix } i xs))$
using *assms* **by** (*simp add: sub-def*)

lemma *ilast-prefix*:
assumes $k \leq \text{ilen } xs$
shows $\text{ilast}(\text{prefix } k xs) = (\text{inth } xs k)$
using *assms prefix-ilen-good* **by** *fastforce*

lemma *ifirst-suffix*:
assumes $k \leq \text{ilen } xs$
shows $\text{ifirst}(\text{suffix } k xs) = (\text{inth } xs k)$
by (*simp add: assms*)

lemma *suffix-gr*:
assumes $i > \text{ilen } xs$
shows $\text{suffix } i xs = \langle \text{ilast}(xs) \rangle$
by (*metis add.commute assms suffix-ilast suffix-suffix less-imp-add-positive suffix.simps(1)*)

lemma *ilast-ifirst*:
 $(\text{ilast } (\text{prefix } i xs)) = (\text{ifirst } (\text{suffix } i xs))$
proof –
have 1: $(\text{ilast } (\text{prefix } i xs)) = (\text{inth } (\text{prefix } i xs) (\text{ilen } (\text{prefix } i xs)))$
by *simp*
have 2: $i \leq \text{ilen } xs \longrightarrow \text{ilen } (\text{prefix } i xs) = i$
using *prefix-ilen-good* **by** *blast*
have 3: $i > \text{ilen } xs \longrightarrow \text{ilen } (\text{prefix } i xs) = \text{ilen } xs$
using *prefix-ilen-bad* **by** *blast*
have 4: $i \leq \text{ilen } xs \longrightarrow$
 $(\text{inth } (\text{prefix } i xs) (\text{ilen } (\text{prefix } i xs))) = (\text{inth } xs i)$
using *ilast-prefix* **by** *auto*
have 5: $i > \text{ilen } xs \longrightarrow$
 $(\text{inth } (\text{prefix } i xs) (\text{ilen } (\text{prefix } i xs))) = (\text{inth } xs (\text{ilen } xs))$
using 3 **by** *auto*
have 6: $(\text{ifirst } (\text{suffix } i xs)) = (\text{inth } (\text{suffix } i xs) 0)$
by *simp*

have 7: $i \leq \text{ilen } xs \longrightarrow$
 $(\text{inth } (\text{suffix } i \text{ } xs) \ 0) = (\text{inth } xs \ i)$
by *simp*
have 8: $i > \text{ilen } xs \longrightarrow$
 $(\text{inth } (\text{suffix } i \text{ } xs) \ 0) = (\text{inth } xs \ (\text{ilen } xs))$
by (*simp add: suffix-gr*)
show ?thesis **using** 4 5 8 **by** (*metis 7 leI*)
qed

lemma *ilen-sub* [*simp*]:
assumes $k \leq n$
 $n \leq \text{ilen } xs$
shows $\text{ilen}(\text{sub } k \ n \ xs) = (n - k)$
using *assms*
by (*metis sub-def prefix-ilen-good suffix-ilen suffix-prefix-swap le-add-diff-inverse2*)

lemma *inth-sub* [*simp*]:
assumes $k \leq n$
 $n \leq \text{ilen } xs$
 $j \leq n - k$
shows $\text{inth}(\text{sub } k \ n \ xs) \ j = (\text{inth } xs \ (k + j))$
proof –
have 1: $\text{inth}(\text{sub } k \ n \ xs) \ j = \text{inth } (\text{prefix } (n - k) \ (\text{suffix } k \ xs)) \ j$
by (*simp add: sub-def*)
have 2: $n - k \leq \text{ilen } (\text{suffix } k \ xs)$
using *sub-def assms* **by** *auto*
have 3: $j \leq (n - k)$
using *assms* **by** *auto*
have 4: $\text{inth } (\text{prefix } (n - k) \ (\text{suffix } k \ xs)) \ j =$
 $\text{inth } (\text{suffix } k \ xs) \ j$
using 2 *assms inth-prefix* **by** *blast*
have 5: $\text{inth } (\text{suffix } k \ xs) \ j = \text{inth } xs \ (k + j)$
using *assms* **by** *auto*
show ?thesis
by (*simp add: 1 4 5*)
qed

lemma *ilast-sub*:
assumes $k \leq n$
 $n \leq \text{ilen } xs$
shows $\text{ilast } (\text{sub } k \ n \ xs) = (\text{inth } xs \ n)$
by (*simp add: assms*)

lemma *ifirst-sub*:
assumes $k \leq n$
 $n \leq \text{ilen } xs$
shows $\text{ifirst } (\text{sub } k \ n \ xs) = (\text{inth } xs \ k)$
by (*simp add: assms*)

lemma *sub-sub*:

assumes $n1 \leq n2$

$n0 \leq n4$

$n2 \leq n4 - n0$

$n4 \leq n3$

$n3 \leq \text{ilen } xs$

shows $(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) = (\text{sub } n1 \ n2 (\text{sub } n0 \ n4 \ xs))$

proof –

have 1: $\text{ilen}(\text{sub } n0 \ n3 \ xs) = n3 - n0$

by (*meson assms ilen-sub le-trans*)

have 2: $\text{ilen}(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) = n2 - n1$

using *ilen-sub assms* **by** *auto*

have 3: $\text{ilen}(\text{sub } n0 \ n4 \ xs) = n4 - n0$

using *assms ilen-sub le-trans* **by** *blast*

have 4: $\text{ilen}(\text{sub } n1 \ n2 (\text{sub } n0 \ n4 \ xs)) = n2 - n1$

by (*simp add: 3 assms*)

have 5: $\bigwedge i. i \leq (n3 - n0) \longrightarrow (\text{inth}(\text{sub } n0 \ n3 \ xs) \ i) = (\text{inth } xs \ (n0 + i))$

using *assms* **by** *auto*

have 6: $\bigwedge i. i \leq (n2 - n1) \longrightarrow (\text{inth}(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) \ i) = (\text{inth}(\text{sub } n0 \ n3 \ xs) \ (n1 + i))$

using *inth-sub assms*

by (*metis 1 Nat.le-diff-conv2 le-trans*)

have 7: $\bigwedge i. i \leq (n2 - n1) \longrightarrow (\text{inth}(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) \ i) = (\text{inth } xs \ (n0 + (n1 + i)))$

using 5 6 *assms* **by** *auto*

have 8: $n0 \leq n4 \wedge n4 \leq \text{ilen } xs$

using *assms le-trans* **by** *blast*

have 9: $\bigwedge i. i \leq (n4 - n0) \longrightarrow (\text{inth}(\text{sub } n0 \ n4 \ xs) \ i) = (\text{inth } xs \ (n0 + i))$

using 8 *inth-sub* **by** *blast*

have 10: $\bigwedge i. i \leq (n2 - n1) \longrightarrow (\text{inth}(\text{sub } n1 \ n2 (\text{sub } n0 \ n4 \ xs)) \ i) = (\text{inth}(\text{sub } n0 \ n4 \ xs) \ (n1 + i))$

by (*simp add: 3 assms*)

have 11: $\bigwedge i. i \leq (n2 - n1) \longrightarrow (\text{inth}(\text{sub } n1 \ n2 (\text{sub } n0 \ n4 \ xs)) \ i) = (\text{inth } xs \ (n0 + (n1 + i)))$

by (*metis 3 Nat.le-diff-conv2 add commute assms inth-sub le-trans*)

have 12: $\bigwedge i. i \leq (n2 - n1) \longrightarrow$

$(\text{inth}(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) \ i) = (\text{inth}(\text{sub } n1 \ n2 (\text{sub } n0 \ n4 \ xs)) \ i)$

by (*simp add: 11 7*)

from 12 2 4 **show** *?thesis* **by** (*simp add: interval-eq-inth-eq*)

qed

lemma *sub-sub-1*:

assumes $n1 \leq n2$

$n0 \leq n3$

$n2 \leq n3 - n0$

$n3 \leq \text{ilen } xs$

shows $(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) = (\text{sub } (n0 + n1) \ (n0 + n2) \ xs)$

proof –

have 1: $\text{ilen}(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) = \text{ilen}(\text{sub } (n1 + n0) \ (n2 + n0) \ xs)$

using *assms* **by** *auto*

have 2: $\bigwedge i. i \leq (n2 - n1) \longrightarrow (\text{inth}(\text{sub } n1 \ n2 (\text{sub } n0 \ n3 \ xs)) \ i) = (\text{inth } xs \ (n0 + (n1 + i)))$

using *assms* **by** *auto*

have 3: $\bigwedge i. i \leq (n2 - n1) \longrightarrow (\text{inth}(\text{sub } (n0 + n1) \ (n0 + n2) \ xs) \ i) = (\text{inth } xs \ (n0 + (n1 + i)))$

using *assms* **by** (*metis* *ab-semigroup-add-class.add-ac(1)* *diff-add-inverse* *inth-sub* *le-Suc-ex* *le-add1*)
have 4: $\bigwedge i. i \leq (n2 - n1) \longrightarrow (inth \ (sub \ n1 \ n2 \ (sub \ n0 \ n3 \ xs)) \ i) = (inth \ (sub \ (n0 + n1) \ (n0 + n2) \ xs) \ i)$
by (*simp* *add: 2 3*)
show *?thesis*
by (*metis* 1 4 *add.commute* *assms* *interval-eq-inth-eq* *ilen-sub*)
qed

lemma *suf-first-upto*:

assumes $(\exists i < k. f \ (suffix \ i \ xs))$
 $k \leq ilen \ xs + 1$

shows $(\exists i < k. f \ (suffix \ i \ xs) \wedge$
 $(\forall j < i. \neg (f \ (suffix \ j \ xs))))$

using *assms*

proof (*induct* *xs* *arbitrary:k*)

case (*INil* *x*)

then show *?case* **by** *auto*

next

case (*ICons* *x1a* *xs*)

then show *?case*

proof –

have 0: $k = 0 \longrightarrow (\exists i < k. f \ (suffix \ i \ (x1a \odot xs)) \wedge (\forall j < i. \neg f \ (suffix \ j \ (x1a \odot xs))))$

using *ICons.prem(1)* **by** *blast*

have 1: $\bigwedge n. k = (Suc \ n) \longrightarrow (\exists i < (Suc \ n). f \ (suffix \ i \ (x1a \odot xs)) \wedge$
 $(\forall j < i. \neg f \ (suffix \ j \ (x1a \odot xs)))) =$
 $(f \ (x1a \odot xs) \vee$
 $(\exists i. 1 \leq i \wedge i < (Suc \ n) \wedge f \ (suffix \ i \ (x1a \odot xs)) \wedge$
 $(\forall j < i. \neg f \ (suffix \ j \ (x1a \odot xs))))$

by *auto*

(*metis* *Nitpick.case-nat-unfold* *less-antisym* *not-le-imp-less* *not-less0*)

have 2: $\bigwedge n. k = (Suc \ n) \longrightarrow (\exists i. 1 \leq i \wedge i < (Suc \ n) \wedge f \ (suffix \ i \ (x1a \odot xs)) \wedge$
 $(\forall j < i. \neg f \ (suffix \ j \ (x1a \odot xs)))) =$
 $(\exists i. i < n \wedge f \ (suffix \ (Suc \ i) \ (x1a \odot xs)) \wedge$
 $(\forall j < (Suc \ i). \neg f \ (suffix \ j \ (x1a \odot xs))))$

by *auto*

(*metis* *Suc-le-D* *nat.case(2)* *not-less* *not-less-eq-eq*)

have 3: $\bigwedge n. k = (Suc \ n) \longrightarrow (\exists i. i < n \wedge f \ (suffix \ (Suc \ i) \ (x1a \odot xs)) \wedge$
 $(\forall j < (Suc \ i). \neg f \ (suffix \ j \ (x1a \odot xs)))) =$
 $(\exists i. i < n \wedge f \ (suffix \ i \ xs) \wedge$
 $\neg(f \ (suffix \ 0 \ (x1a \odot xs))) \wedge$
 $(\forall j. 1 \leq j \wedge j < (Suc \ i) \longrightarrow \neg f \ (suffix \ j \ (x1a \odot xs))))$

by (*metis* *suffix-suc* *le-add1* *less-Suc-eq-0-disj* *plus-1-eq-Suc*)

have 4: $\bigwedge n. k = (Suc \ n) \longrightarrow (\exists i. i < n \wedge f \ (suffix \ i \ xs) \wedge$
 $\neg(f \ (suffix \ 0 \ (x1a \odot xs))) \wedge$
 $(\forall j. 1 \leq j \wedge j < (Suc \ i) \longrightarrow \neg f \ (suffix \ j \ (x1a \odot xs)))) =$
 $(\neg(f \ (x1a \odot xs)) \wedge$
 $(\exists i. i < n \wedge f \ (suffix \ i \ xs) \wedge$
 $(\forall j. j < i \longrightarrow \neg f \ (suffix \ (Suc \ j) \ (x1a \odot xs))))$
using *ICons.hyps* *ICons.prem(1)* **by** (*auto*, *auto* *simp* *add: less-Suc-eq-0-disj*)

have 5: $\bigwedge n. k = (Suc\ n) \longrightarrow (\neg(f\ (x1a \odot xs)) \wedge$
 $(\exists i. i < n \wedge f\ (suffix\ i\ xs) \wedge$
 $(\forall j. j < i \longrightarrow \neg f\ (suffix\ (Suc\ j)\ (x1a \odot xs))))) =$
 $(\neg(f\ (x1a \odot xs)) \wedge$
 $(\exists i < n. f\ (suffix\ i\ xs) \wedge$
 $(\forall j < i. \neg f\ (suffix\ j\ xs)))))$
by auto
have 6: $\bigwedge n. k = (Suc\ n) \longrightarrow (\exists i < (Suc\ n). f\ (suffix\ i\ (x1a \odot xs)) \wedge$
 $(\forall j < i. \neg f\ (suffix\ j\ (x1a \odot xs)))) =$
 $(f\ (x1a \odot xs) \vee (\neg(f\ (x1a \odot xs)) \wedge$
 $(\exists i < n. f\ (suffix\ i\ xs) \wedge$
 $(\forall j < i. \neg f\ (suffix\ j\ xs))))))$
using 1 2 3 4 by auto
have 7: $\bigwedge n. k = (Suc\ n) \longrightarrow (f\ (x1a \odot xs) \vee (\neg(f\ (x1a \odot xs)) \wedge$
 $(\exists i < n. f\ (suffix\ i\ xs) \wedge$
 $(\forall j < i. \neg f\ (suffix\ j\ xs))))) =$
 $(f\ (x1a \odot xs) \vee (\exists i < n. f\ (suffix\ i\ xs) \wedge$
 $(\forall j < i. \neg f\ (suffix\ j\ xs)))))$
by auto
have 8: $\bigwedge n. k = (Suc\ n) \longrightarrow n \leq ilen\ xs + 1$
using ICons.premis(2) by auto
have 9: $\bigwedge n. k = (Suc\ n) \longrightarrow (f\ (x1a \odot xs) \vee (\exists i < n. f\ (suffix\ i\ xs) \wedge$
 $(\forall j < i. \neg f\ (suffix\ j\ xs)))) =$
 $(f\ (x1a \odot xs) \vee (\exists i < n. f\ (suffix\ i\ xs)))$
using 7 8 ICons.hyps by fastforce
have 10: $\bigwedge n. k = (Suc\ n) \longrightarrow (\exists i < k. f\ (suffix\ i\ (x1a \odot xs))) =$
 $(f\ (x1a \odot xs) \vee (\exists i < n. f\ (suffix\ i\ (xs))))$
using Nitpick.case-nat-unfold less-Suc-eq-0-disj by auto
have 11: $\bigwedge n. k = (Suc\ n) \longrightarrow$
 $(\exists i < k. f\ (suffix\ i\ (x1a \odot xs)) \wedge (\forall j < i. \neg f\ (suffix\ j\ (x1a \odot xs))))$
using 9 10 6 ICons.premis(1) by force
show ?thesis
using 0 11 less-imp-Suc-add by blast
qed
qed

1.2.5 iapp

lemma *ilen-snoc-1:*

ilen $l > 0 = (\exists\ x\ ls. l = ls \odot \langle x \rangle)$

proof (*induct* l)

case (*INil* x)

then show *?case* **by fastforce**

next

case (*ICons* $x1a\ l$)

then show *?case*

by (*metis* *Suc-eq-plus1 iapp-ICons iapp-INil interval.exhaust ilen-iapp nat.simps(3) neq0-conv*)

qed

```

lemma prefix-iapp [simp]:
  prefix (ilen xs - k) (xs ⊖ ys) = prefix (ilen xs - k) xs
proof
  (induct xs arbitrary: k)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof (cases k)
    case 0
    then show ?thesis
      by (metis ICons.hyps diff-zero iapp-ICons prefix-suc ilen.simps(2) plus-1-eq-Suc)
    next
    case (Suc nat)
    then show ?thesis
      by (auto simp add: ICons.hyps Nitpick.case-nat-unfold)
    qed
  qed

```

```

lemma prefix-iapp2 [simp]:
  prefix (ilen xs + k + 1) (xs ⊖ ys) = xs ⊖ prefix k ys
proof
  (induct xs arbitrary: k)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof (cases k)
    case 0
    then show ?thesis
      by auto
      (metis ICons.hyps Suc-eq-plus1 add.right-neutral prefix-zero-ifirst)
    next
    case (Suc nat)
    then show ?thesis
      by auto
      (metis ICons.hyps Suc-eq-plus1 add-Suc-right)
    qed
  qed

```

```

lemma suffix-iapp [simp]:
  suffix (ilen xs + m + 1) (xs ⊖ ys) = suffix (m) ys
proof
  (induct xs arbitrary: m)

```

```

case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases m)
    case 0
    then show ?thesis
      by auto
      (metis ICons.hyps One-nat-def Suc-eq-plus1 suffix-zero plus-1-eq-Suc
        semiring-normalization-rules(23))
    next
    case (Suc nat)
    then show ?thesis
      by auto
      (metis ICons.hyps Suc-eq-plus1 suffix-suffix)
  qed
qed

```

```

lemma suffix-iapp2 [simp]:
  (suffix (ilen xs - k) xs)  $\ominus$  ys = suffix (ilen xs - k) (xs  $\ominus$  ys)
proof
  (induct xs)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    by (auto simp add: Nitpick.case-nat-unfold)
  qed

```

```

lemma iapp-assoc [simp]:
  (xs  $\ominus$  ys)  $\ominus$  zs = xs  $\ominus$  (ys  $\ominus$  zs)
by (induct xs simp-all)

```

```

lemma iapp-inth:
  inth (xs  $\ominus$  ys) k = (if k  $\leq$  ilen xs
    then (inth xs k)
    else (inth ys (k - (ilen xs - 1))) )
proof
  (induct xs arbitrary: k)
  case (INil x)
  then show ?case
    proof (cases k)
      case 0
      then show ?thesis by simp
    next
    case (Suc nat)
    then show ?thesis by simp
  qed

```



```

next
case (ICons x1a xs)
then show ?case
proof (cases k)
case 0
then show ?thesis by simp
next
case (Suc nat)
then show ?thesis by (simp add: ICons.hyps)
qed
qed

```

```

lemma irev-iapp [simp]:
  irev (xs  $\ominus$  ys) = (irev ys)  $\ominus$  (irev xs)
by (induct xs) simp-all

```

```

lemma ilast-iapp [simp]:
  ilast(xs  $\ominus$   $\langle x \rangle$ ) = x
by (induct xs) simp-all

```

```

lemma ilast-iapp2 [simp]:
  ilast (xs  $\ominus$  ys) = ilast ys
by (induct xs arbitrary: ys) simp-all

```

```

lemma ifirst-iapp [simp]:
  ifirst ( $\langle x \rangle$   $\ominus$  xs) = x
by (induct xs) simp-all

```

```

lemma ifirst-iapp2 [simp]:
  ifirst(xs  $\ominus$  ys) = ifirst xs
by (induct xs arbitrary: ys) simp-all

```

```

lemma iapp-not-INil [simp]:
  xs  $\ominus$  ys  $\neq$   $\langle x \rangle$ 
by (induct xs arbitrary: ys) simp-all

```

```

lemma iapp-eq-iapp-conv [simp]:
  assumes ilen xs = ilen ys  $\vee$  ilen us = ilen vs
  shows (xs  $\ominus$  us = ys  $\ominus$  vs) = (xs = ys  $\wedge$  us = vs)
using assms
proof
  (induct xs arbitrary: ys)
  case (INil x)
  then show ?case
  proof (cases ys)
  case (INil x1)
  then show ?thesis by simp
  next
  case (ICons x21 x22)

```

```

    then show ?thesis using INil.prem by auto
  qed
next
case (ICons x1a xs)
then show ?case
  proof (cases ys)
    case (INil x1)
    then show ?thesis using ICons.prem by auto
  next
    case (ICons x21 x22)
    then show ?thesis using ICons.hyps ICons.prem by auto
  qed
qed

```

lemma *iapp-eq-iapp-conv2*:

$$\begin{aligned}
 (xs \ominus ys = zs \ominus ts) = \\
 (\exists us. xs = zs \ominus us \wedge us \ominus ys = ts \vee \\
 xs = zs \wedge ys = ts \vee \\
 xs \ominus us = zs \wedge ys = us \ominus ts)
 \end{aligned}$$

proof

(*induct xs arbitrary: ys zs ts*)

case (INil x)

then show ?case

proof (cases zs)

case (INil x1)

then show ?thesis **by** *simp*

next

case (ICons x21 x22)

then show ?thesis **by** *simp*

qed

next

case (ICons x1a xs)

then show ?case

proof (cases zs)

case (INil x1)

then show ?thesis **by** *simp*

next

case (ICons x21 x22)

then show ?thesis **by** (*auto simp add: ICons.hyps*)

qed

qed

lemma *same-iapp-eq[iff, induct-simp]*:

$$(xs \ominus ys = xs \ominus zs) = (ys = zs)$$

using *suffix-iapp* **by** (*metis suffix-zero*)

lemma *iapp-eq-conv[iff]*:

$$(xs \ominus \langle x \rangle = ys \ominus \langle y \rangle) = (xs = ys \wedge x = y)$$

by *auto*

lemma *iapp-same-eq*[*iff*, *induct-simp*]:

$(ys \ominus xs = zs \ominus xs) = (ys = zs)$

by *auto*

lemma *suffix1-iapp*:

$\text{suffix } 1 \ (xs \ominus ys) = (\text{case } xs \text{ of } \langle x \rangle \Rightarrow ys \mid x \odot zs \Rightarrow zs \ominus ys)$

by (*cases xs simp-all*)

lemma *ICons-eq-iapp-conv*:

$(x \odot xs = ys \ominus zs) =$

$((\langle x \rangle = ys \wedge xs = zs) \vee (\exists \ ys'. \ x \odot ys' = ys \wedge xs = ys' \ominus zs))$

by (*cases ys simp-all*)

lemma *iapp-eq-ICons-conv*:

$(ys \ominus zs = x \odot xs) =$

$((\langle x \rangle = ys \wedge zs = xs) \vee (\exists \ ys'. \ ys = x \odot ys' \wedge ys' \ominus zs = xs))$

by (*cases ys auto*)

lemma *ICons-eq-iappI*:

assumes $x \odot xs1 = ys$

$xs = xs1 \ominus zs$

shows $x \odot xs = ys \ominus zs$

using *assms* **by** *auto*

lemma *iapp-eq-iappI*:

assumes $xs \ominus xs1 = zs$

$ys = xs1 \ominus us$

shows $xs \ominus ys = zs \ominus us$

using *assms* **by** *auto*

lemma *iapp-gr-zero*:

$\text{ilen } (xs \ominus ys) > 0$

by *auto*

lemma *iapp-prefix-suffix*:

assumes $i+1 \leq \text{ilen } xs$

$\text{ilen } xs > 0$

shows $xs = (\text{prefix } i \ xs) \ominus (\text{suffix } (i+1) \ xs)$

using *assms*

proof (*induct xs arbitrary:i*)

case (*INil x*)

then show *?case* **by** *simp*

next

case (*ICons x1a xs*)

then show *?case*

proof (*cases i*)

case *0*

```

then show ?thesis by auto
next
case (Suc nat)
then show ?thesis using ICons.hyps ICons.prem1 by auto
qed
qed

```

1.2.6 Reverse

lemma *irev-irev-ident* [simp]:

```

  irev (irev xs) = xs
by (induct xs) auto

```

lemma *irev-swap* :

```

  ((irev xs) = ys) = (xs = irev ys)
by auto

```

lemma *irev-singleton-conv* [simp]:

```

  ( irev xs = ⟨x⟩ ) = (xs = ⟨x⟩)
by (metis irev-irev-ident irev.simps(1))

```

lemma *single-irev-conv* [simp]:

```

  (⟨x⟩ = irev xs) = (⟨x⟩ = xs)
by (metis irev-irev-ident irev.simps(1))

```

lemma *irev-is-irev-conv* [iff]:

```

  (irev xs = irev ys) = (xs = ys)

```

proof

```

  (induct xs arbitrary: ys)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  using irev-swap by force
qed

```

lemma *irev-induct* [case-names INil snoc]:

```

assumes ∧ y. P ⟨y⟩
        ∧ x xs. P xs ⟹ P (xs ⊖ ⟨x⟩)
shows   P xs
using   assms
using   interval.induct[of λ xs. P (irev xs) irev xs]
by simp

```

lemma *irev-exhaust* [case-names INil snoc]:

```

assumes ∧ x. xs = ⟨x⟩ ⟹ P
        ∧ ys y. xs = ys ⊖ ⟨y⟩ ⟹ P
shows   P

```

using *assms*
by (*induct xs rule:irev-induct*) *auto*

lemmas *irev-cases* = *irev-exhaust*

lemma *interval-irev-eq-ICons-iff* [*iff*]:
 $(\text{irev } xs = y \odot ys) = (xs = (\text{irev } ys) \ominus \langle y \rangle)$
by (*metis irev-irev-ident irev.simps(2)*)

lemma *irev-iapp-ICons*:
 $\text{irev } (xs \ominus \langle x \rangle) = x \odot \text{irev } xs$
by (*cases xs*) *simp-all*

lemma *ilast-irev*:
 $\text{ilast } (\text{irev } xs) = \text{ifirst } xs$
proof (*cases xs*)
case (*INil x1*)
then show *?thesis*
by *simp*
next
case (*ICons x21 x22*)
then show *?thesis*
by (*metis ilast-iapp inth-zero irev.simps(2)*)
qed

lemma *ifirst-irev*:
 $\text{ifirst } (\text{irev } xs) = \text{ilast } xs$
proof
 $(\text{induct } xs)$
case (*INil x*)
then show *?case* **by** *simp*
next
case (*ICons x1a xs*)
then show *?case*
by (*metis ifirst-iapp2 inth-last irev.simps(2)*)
qed

lemma *irev-inth*:
assumes $k \leq \text{ilen } (\text{irev } xs)$
shows $(\text{inth } (\text{irev } xs) k) = (\text{inth } xs ((\text{ilen } xs) - k))$
using *assms*
proof
 $(\text{induct } xs)$
case (*INil x*)
then show *?case* **by** *simp*
next
case (*ICons x1a xs*)
then show *?case*

```

proof (cases k)
case 0
then show ?thesis
  by auto
    (metis ICons.hyps diff-zero ilen-gr-zero)
next
case (Suc nat)
then show ?thesis
  using ICons.hyps Suc-diff-le by (auto simp add: iapp-inth) fastforce
qed
qed

```

```

lemma irev-prefix:
assumes  $k \leq \text{ilen } xs$ 
shows  $\text{irev}(\text{prefix } k \text{ } xs) = \text{suffix } ((\text{ilen } xs) - k) (\text{irev } xs)$ 
proof
  (induct xs arbitrary: k)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases k)
    case 0
    then show ?thesis
      by auto
        (metis Suc-eq-plus1 add.right-neutral ilast-iapp ilen-iapp
          irev-ilen suffix-ilast ilen.simps(1))
    next
    case (Suc nat)
    then show ?thesis
      by auto
        (metis ICons.hyps irev-ilen suffix-iapp2)
  qed
qed

```

```

lemma irev-suffix:
assumes  $k \leq \text{ilen } xs$ 
shows  $\text{irev}(\text{suffix } k \text{ } xs) = \text{prefix } ((\text{ilen } xs) - k) (\text{irev } xs)$ 
using assms
proof
  (induct xs arbitrary: k)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case by (simp add: irev-prefix irev-swap)
qed

```

lemma *irev-sub*:
assumes $0 \leq i$
 $i \leq j$
 $j \leq \text{ilen } xs$
shows $\text{irev } (\text{sub } i \ j \ xs) = \text{sub } ((\text{ilen } xs) - j) ((\text{ilen } xs) - i) (\text{irev } xs)$
using *assms*
proof –
have 1: $\text{irev } (\text{sub } i \ j \ xs) = \text{irev } (\text{prefix } (j-i) (\text{suffix } i \ xs))$
using *assms sub-prefix-suffix* **by** (*simp add: sub-prefix-suffix*)
have 2: $\text{irev } (\text{prefix } (j-i) (\text{suffix } i \ xs)) = \text{suffix } ((\text{ilen } xs) - j) (\text{irev } (\text{suffix } i \ xs))$
using *assms irev-prefix[of j-i suffix i xs]* **by** *auto*
have 3: $\text{suffix } ((\text{ilen } xs) - j) (\text{irev } (\text{suffix } i \ xs)) =$
 $\text{suffix } ((\text{ilen } xs) - j) (\text{prefix } ((\text{ilen } xs) - i) (\text{irev } xs))$
using *assms irev-suffix[of i xs]* **by** *auto*
have 4: $\text{suffix } ((\text{ilen } xs) - j) (\text{prefix } ((\text{ilen } xs) - i) (\text{irev } xs)) =$
 $\text{sub } ((\text{ilen } xs) - j) ((\text{ilen } xs) - i) (\text{irev } xs)$
using *assms* **by** (*simp add: diff-le-mono2 sub-prefix-suffix suffix-prefix-swap*)
from 1 2 3 4 **show** *?thesis* **by** *auto*
qed

1.2.7 Induction rule

lemma *interval-ilen-induct*:
assumes $(\bigwedge xs. \forall ys. \text{ilen } ys < \text{ilen } xs \longrightarrow P \ ys \Longrightarrow P \ xs)$
shows $P \ xs$
using *assms* **by** (*fact measure-induct*)

lemma *interval-induct-12*:
assumes $\bigwedge x. P \ \langle x \rangle$
 $\bigwedge x \ y. P \ \langle x, y \rangle$
 $\bigwedge x \ y \ zs. P \ (y \odot zs) \Longrightarrow P \ (x \odot y \odot zs)$
shows $P \ xs$
using *assms*
proof (*induction xs*)
case (*INil x*)
then show *?case* **by** *simp*
next
case (*ICons x1a xs*)
then show *?case*
by *auto*
 $(\text{metis add-cancel-right-right ilen-ICons-1 INil-ilen le-add1 le-eq-less-or-eq less-add-same-cancel1})$
qed

lemma *interval-induct2* [*consumes 1, case-names INil ICons*]:
assumes $\text{ilen } xs = \text{ilen } ys$
 $(\bigwedge x \ y. P \ \langle x \rangle \ \langle y \rangle)$
 $(\bigwedge x \ xs \ y \ ys. \text{ilen } xs = \text{ilen } ys \Longrightarrow P \ xs \ ys \Longrightarrow P \ (x \odot xs) \ (y \odot ys))$
shows $P \ xs \ ys$

```

using assms
proof (induction xs arbitrary: ys)
case (INil x)
then show ?case
by (metis INil-ilen)
next
case (ICons x1a xs)
then show ?case
by (metis ilen-ICons-1 ilen.simps(2) nat.simps(1) plus-1-eq-Suc)
qed

```

1.2.8 Imap

```

lemma imap-ext:
  assumes  $(\forall x. x \in \text{iset } xs \longrightarrow f\ x = g\ x)$ 
  shows  $\text{imap } f\ xs = \text{imap } g\ xs$ 
using assms
by (induct xs simp-all)

```

```

lemma imap-ident [simp]:
   $\text{imap } (\lambda x. x) = (\lambda xs. xs)$ 
proof (rule ext)
  show  $\bigwedge xs. \text{imap } (\lambda x. x)\ xs = xs$ 
  by (simp add: interval.map-ident)
qed

```

```

lemma imap-iapp [simp]:
   $\text{imap } f\ (xs \ominus ys) = \text{imap } f\ xs \ominus \text{imap } f\ ys$ 
by (induct xs auto)

```

```

lemma imap-imap [simp]:
   $\text{imap } f\ (\text{imap } g\ xs) = \text{imap } (f \circ g)\ xs$ 
by (simp add: interval.map-comp)

```

```

lemma imap-comp-imap [simp]:
   $((\text{imap } f) \circ (\text{imap } g)) = \text{imap}(f \circ g)$ 
by (rule ext simp)

```

```

lemma irev-imap:
   $\text{irev } (\text{imap } f\ xs) = \text{imap } f\ (\text{irev } xs)$ 
by (induct xs auto)

```

```

lemma imap-eq-conv [simp]:
   $(\text{imap } f\ xs = \text{imap } g\ xs) = (\forall x \in \text{iset } xs. (f\ x) = (g\ x))$ 
by (induct xs auto)

```

```

lemma imap-cong [fundef-cong]:
  assumes  $xs = ys$ 

```


$(\forall x. x \in \text{iset } ys \longrightarrow f x = g x)$
shows $\text{imap } f xs = \text{imap } g ys$
using *assms* **by** *simp*

lemma *imap-injective*:
assumes $\text{imap } f xs = \text{imap } f ys$
 $\text{inj } f$
shows $xs = ys$
using *assms* **by** (*meson injD interval.inj-map*)

lemma *inj-imap-eq-imap [simp]*:
assumes $\text{inj } f$
shows $(\text{imap } f xs = \text{imap } f ys) = (xs = ys)$
using *assms* **by** (*blast dest: imap-injective*)

lemma *inj-imapI*:
assumes $\text{inj } f$
shows $\text{inj } (\text{imap } f)$
using *assms interval.inj-map* **by** *blast*

lemma *inj-imapD*:
assumes $\text{inj } (\text{imap } f)$
shows $\text{inj } f$
using *assms* **by** (*metis inj-def interval.map(1) interval.simps(1)*)

lemma *inj-imap[iff]*:
 $\text{inj } (\text{imap } f) = \text{inj } f$
by (*blast dest: inj-imapD intro: inj-imapI*)

lemma *imap-idI*:
assumes $(\forall x. x \in \text{iset } xs \longrightarrow f x = x)$
shows $\text{imap } f xs = xs$
using *assms* **by** (*induct xs*) *auto*

lemma *imap-is-INil-conv[iff]*:
 $(\text{imap } f xs = \langle x \rangle) = (\exists y. xs = \langle y \rangle \wedge f y = x)$
proof (*cases xs*)
case (*INil x1*)
then show *?thesis* **by** *simp*
next
case (*ICons x21 x22*)
then show *?thesis* **by** *simp*
qed

lemma *INil-is-imap-conv [iff]*:
 $(\langle x \rangle = \text{imap } f xs) = (\exists y. \langle y \rangle = xs \wedge f y = x)$
proof (*cases xs*)
case (*INil x1*)
then show *?thesis* **by** *auto*
next

case (*ICons* *x21* *x22*)
then show *?thesis* **by** *auto*
qed

lemma *imap-eq-ICons-conv*:
 $(\text{imap } f \text{ } xs = y \odot ys) = (\exists z \text{ } zs. xs = z \odot zs \wedge f \text{ } z = y \wedge \text{imap } f \text{ } zs = ys)$
by (*cases* *xs*) *auto*

lemma *ICons-eq-imap-conv*:
 $(y \odot ys = \text{imap } f \text{ } xs) = (\exists z \text{ } zs. z \odot zs = xs \wedge f \text{ } z = y \wedge ys = \text{imap } f \text{ } zs)$
by (*cases* *xs*) *auto*

lemma *ex-imap-conv*:
 $(\exists xs. ys = \text{imap } f \text{ } xs) = (\forall y \in \text{iset } ys. \exists x. y = f \text{ } x)$
by (*induct* *ys*) (*auto simp add: ICons-eq-imap-conv*)

functor *imap*: *imap*
by (*simp-all add: id-def*)

declare *imap.id* [*simp*]

lemma *ifirst-imap*:
 $\text{ifirst } (\text{imap } f \text{ } xs) = f \text{ } (\text{ifirst } xs)$
by (*cases* *xs*) *simp-all*

lemma *ilast-imap*:
 $\text{ilast } (\text{imap } f \text{ } xs) = f \text{ } (\text{ilast } xs)$
proof (*cases* *xs* *rule: irev-cases*)
case (*INil* *x*)
then show *?thesis* **by** *simp*
next
case (*snoc* *ys* *y*)
then show *?thesis* **by** (*simp add: iapp-inth*)
qed

lemma *imap-tail*:
shows $\text{imap } f \text{ } (\text{suffix } 1 \text{ } xs) = (\text{suffix } 1 \text{ } (\text{imap } f \text{ } xs))$
by (*cases* *xs*) *simp-all*

lemma *imap-eq-imp-ilen-eq*:
assumes $\text{imap } f \text{ } xs = \text{imap } g \text{ } ys$
shows $\text{ilen } xs = \text{ilen } ys$
using *assms*
proof (*induct* *ys* *arbitrary: xs*)
case (*INil* *x*)
then show *?case* **by** *auto*

```

next
case (ICons x1a ys)
then show ?case by (metis ilen-imap)
qed

lemma iset-imap [simp]:
  iset (imap f xs) = f `(iset xs)
by (induct xs) auto

lemma imap-inj-on:
  assumes imap:  $\text{imap } f \text{ xs} = \text{imap } f \text{ ys}$  and
    inj:  $\text{inj-on } f \text{ (iset xs } \cup \text{ iset ys)}$ 
  shows  $\text{xs} = \text{ys}$ 
using imap-eq-imp-ilen-eq[OF imap] assms
proof (induction rule: interval-induct2)
case (INil x y)
then show ?case by auto
next
case (ICons x xs y ys)
then show ?case
  by (simp add: inj-on-Un-image-eq-iff inj-on-eq-iff)
qed

lemma inj-on-imap-eq-imap:
  assumes  $\text{inj-on } f \text{ (iset xs } \cup \text{ iset ys)}$ 
  shows  $(\text{imap } f \text{ xs} = \text{imap } f \text{ ys}) = (\text{xs} = \text{ys})$ 
using assms by (blast dest: imap-inj-on)

lemma inj-on-impI:
  assumes  $\text{inj-on } f \text{ (}\bigcup \text{ (iset 'A))}$ 
  shows  $\text{inj-on (imap } f) \text{ A}$ 
using assms by (blast intro: inj-onI dest: inj-onD imap-inj-on)

lemma imap-prefix:
  assumes  $k \leq \text{ilen } \text{xs}$ 
  shows  $\text{imap } f \text{ (prefix } k \text{ xs)} = \text{prefix } k \text{ (imap } f \text{ xs)}$ 
using assms
proof (induction xs arbitrary: k)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases k)
  case 0
  then show ?thesis by auto
  next
  case (Suc nat)
  then show ?thesis using ICons.IH ICons.prems by auto
  qed
qed

```

qed

```
lemma imap-suffix:
  assumes  $k \leq \text{ilen } xs$ 
  shows  $\text{imap } f (\text{suffix } k \text{ } xs) = \text{suffix } k (\text{imap } f \text{ } xs)$ 
using assms
proof (induction xs arbitrary: k)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
proof (cases k)
case 0
then show ?thesis by auto
next
case (Suc nat)
then show ?thesis using ICons.IH ICons.prem1 by auto
qed
qed
```

1.2.9 index sequence

```
lemma idx-less:
  assumes  $\text{index-sequence } x \text{ } idx$ 
            $n+k < \text{ilen } idx$ 
  shows  $\text{inth } idx \text{ } n < \text{inth } idx (\text{Suc}(n+k))$ 
using index-sequence-def assms by (induct k) auto
```

```
lemma idx-less-eq:
  assumes  $\text{index-sequence } x \text{ } l$ 
            $k \leq j$ 
            $j \leq \text{ilen } l$ 
  shows  $\text{inth } l \text{ } k \leq \text{inth } l \text{ } j$ 
using assms
proof (cases k=j)
show  $\text{index-sequence } x \text{ } l \implies k \leq j \implies j \leq \text{ilen } l \implies k = j \implies \text{inth } l \text{ } k \leq \text{inth } l \text{ } j$ 
  by blast
show  $\text{index-sequence } x \text{ } l \implies k \leq j \implies j \leq \text{ilen } l \implies k \neq j \implies \text{inth } l \text{ } k \leq \text{inth } l \text{ } j$ 
  by (metis Suc-le-lessD idx-less le-SucE le-eq-less-or-eq le-zero-eq
    ordered-cancel-comm-monoid-diff-class.add-diff-inverse zero-induct)
qed
```

```
lemma idx-mono:
  assumes  $\text{index-sequence } x \text{ } l$ 
  shows  $\text{mono } (\lambda x. \text{inth } l \text{ } x)$ 
proof -
  have 1:  $\forall x \ y. x \leq y \longrightarrow \text{inth } l \text{ } x \leq \text{inth } l \text{ } y$ 
```

```

proof
  fix x
  show  $\forall y \geq x. \text{inth } l \ x \leq \text{inth } l \ y$ 
  proof
    fix y
    show  $x \leq y \longrightarrow \text{inth } l \ x \leq \text{inth } l \ y$ 
    proof -
      have 2:  $x \leq y \wedge y \leq \text{ilen } l \implies \text{inth } l \ x \leq \text{inth } l \ y$ 
        using assms idx-less-eq by blast
      have 3:  $x \leq y \wedge x > \text{ilen } l \implies \text{inth } l \ x \leq \text{inth } l \ y$ 
        using assms inth-last-stutter
        by (metis 2 le-cases less-imp-add-positive ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
      have 4:  $x \leq y \wedge x \leq \text{ilen } l \wedge y > \text{ilen } l \implies \text{inth } l \ x \leq \text{inth } l \ y$ 
        by (metis assms idx-less-eq inth-last-stutter less-imp-add-positive order-refl)
      show ?thesis
        using 2 3 4 not-less by blast
    qed
  qed
  qed
  show ?thesis
    by (simp add: 1 monoI)
qed

```

```

lemma idx-less-last :
  assumes index-sequence x idx
     $i < \text{ilen } \text{idx}$ 
     $i + (\text{ilen } \text{idx} - (i + 1)) < \text{ilen } \text{idx}$ 
  shows  $\text{inth } \text{idx } i < \text{inth } \text{idx } (\text{Suc}(i + (\text{ilen } \text{idx} - (i + 1))))$ 
  using assms idx-less by blast

```

```

lemma idx-less-last-1:
  assumes index-sequence x idx
     $i < \text{ilen } \text{idx}$ 
  shows  $\text{inth } \text{idx } i < \text{inth } \text{idx } (\text{ilen } \text{idx})$ 
  using assms idx-less-last by auto

```

```

lemma idx-greater-first:
  assumes index-sequence x idx
     $0 < i$ 
     $i \leq \text{ilen } \text{idx}$ 
  shows  $x < \text{inth } \text{idx } i$ 
  using assms
  proof -
    have 1:  $\text{inth } \text{idx } 0 = x$ 
      using assms by (simp add: index-sequence-def)
    have 2:  $\forall i. 0 < i \wedge i \leq \text{ilen } \text{idx} \longrightarrow (\text{inth } \text{idx } 0) < (\text{inth } \text{idx } i)$ 
      proof
        fix i
        show  $0 < i \wedge i \leq \text{ilen } \text{idx} \longrightarrow \text{inth } \text{idx } 0 < \text{inth } \text{idx } i$ 

```

by (meson Suc-leI assms(1) dual-order.strict-trans1 index-sequence-def idx-less-eq)
 qed
 show ?thesis
 using 1 2 assms(2) assms(3) by blast
 qed

lemma *idx-ICons*:
 $\text{index-sequence } y \ (x \odot ls) =$
 $(x=y \wedge x < \text{inth } ls \ 0 \wedge \text{index-sequence } (\text{inth } ls \ 0) \ ls)$
 using less-Suc-eq-0-disj by (simp add: index-sequence-def) auto

lemma *idx-shift-mono*:
 $\text{mono } (\text{shift } k)$
 by (simp add: shift-def mono-def)

lemma *idx-expand*:
assumes $\text{index-sequence } 0 \ l$
 $(\text{inth } l \ (\text{ilen } l)) = (\text{ilen } xs)$
 $i < (\text{ilen } l)$
shows $(\text{inth } l \ i) \leq (\text{inth } l \ (i+1)) \wedge (\text{inth } l \ (i+1)) \leq (\text{ilen } xs)$
 using assms
 by (metis Suc-eq-plus1 Suc-lessI add.right-neutral idx-less idx-less-last-1
 less-imp-le-nat order-refl)

lemma *idx-shift-idx* [simp]:
 $(\text{index-sequence } (x+k) \ (\text{imap } (\text{shift } k) \ idx)) = (\text{index-sequence } x \ idx)$
 by (simp add: shift-def index-sequence-def inth-imap)

lemma *idx-shiftn* :
assumes $\text{index-sequence } k \ lsk$
shows $\text{index-sequence } 0 \ (\text{imap } (\text{shiftn } k) \ lsk) \wedge k \leq (\text{inth } lsk \ 0)$
 using assms
proof (auto simp add: index-sequence-def shiftn-def inth-imap)
show $\bigwedge n. \forall n < \text{ilen } lsk. \text{inth } lsk \ n < \text{inth } lsk \ (\text{Suc } n) \implies$
 $k = \text{inth } lsk \ 0 \implies$
 $n < \text{ilen } lsk \implies \text{inth } lsk \ n - \text{inth } lsk \ 0 < \text{inth } lsk \ (\text{Suc } n) - \text{inth } lsk \ 0$
 by (metis assms diff-less-mono idx-greater-first le-eq-less-or-eq neq0-conv)
 qed

lemma *idx-shiftn-a* :
assumes $\text{index-sequence } 0 \ (\text{imap } (\text{shiftn } k) \ lsk)$
 $k \leq (\text{inth } lsk \ 0)$
shows $\text{index-sequence } k \ lsk$
 using assms
 by (auto simp add: index-sequence-def shiftn-def inth-imap)

lemma *idx-shiftn-b* :
 $\text{index-sequence } k \ lsk = (\text{index-sequence } 0 \ (\text{imap } (\text{shiftn } k) \ lsk) \wedge k \leq (\text{inth } lsk \ 0))$
 using idx-shiftn idx-shiftn-a by blast

lemma *idx-shiftm-c* :

index-sequence (inth lsk 0) lsk = index-sequence 0 (imap (shiftm (inth lsk 0)) lsk)

using *idx-shiftm-b* **by** *blast*

lemma *lsk-ls* :

(index-sequence k (lsk) \wedge lsk = imap (shift k) ls \wedge index-sequence 0 (ls)) =
(index-sequence k (lsk) \wedge ls = imap (shiftm k) lsk \wedge index-sequence 0 (ls))

proof (*simp add: interval-eq-inth-eq index-sequence-def shift-def shiftm-def inth-imap*)

show (*inth lsk 0 = k \wedge*

($\forall n < \text{ilen lsk}. \text{inth lsk } n < \text{inth lsk (Suc n)}$) \wedge

ilen lsk = ilen ls \wedge

($\forall i \leq \text{ilen lsk}. \text{inth lsk } i = \text{inth ls } i + k$) \wedge

inth ls 0 = 0 \wedge ($\forall n < \text{ilen ls}. \text{inth ls } n < \text{inth ls (Suc n)}$)) =

(inth lsk 0 = k \wedge

($\forall n < \text{ilen lsk}. \text{inth lsk } n < \text{inth lsk (Suc n)}$) \wedge

ilen ls = ilen lsk \wedge

($\forall i \leq \text{ilen ls}. \text{inth ls } i = \text{inth lsk } i - k$) \wedge

inth ls 0 = 0 \wedge ($\forall n < \text{ilen ls}. \text{inth ls } n < \text{inth ls (Suc n)}$)) (is ?L=?R)

proof *rule*

show *?L \implies ?R*

by (*metis (no-types, lifting) add-diff-cancel-right'*)

show *?R \implies ?L*

by (*metis Suc-le-lessD add.commute diff-is-0-eq nat-le-linear not0-implies-Suc*
ordered-cancel-comm-monoid-diff-class.add-diff-inverse zero-order(3))

qed

qed

lemma *idx-link-shiftm*:

(index-sequence k (lsk) \wedge ls = imap (shiftm k) lsk) =
(index-sequence k (lsk) \wedge ls = imap (shiftm k) lsk \wedge
index-sequence 0 (ls) \wedge (ilen ls) =(ilen lsk))

using *idx-shiftm* **using** *ilen-imap* **by** *blast*

lemma *idx-link*:

(lsk = imap (shift k) ls \wedge index-sequence 0 (ls)) =
(lsk = imap (shift k) ls \wedge index-sequence k (lsk) \wedge index-sequence 0 (ls) \wedge
(ilen ls) =(ilen lsk))

by (*metis add.left-neutral idx-shift-idx ilen-imap*)

lemma *idx-bound-0* :

assumes *index-sequence 0 ls*

inth ls (ilen ls) = ilen (suffix k xs)

i \leq ilen ls

shows *inth ls i \leq ilen (suffix k xs)*

using *assms*

by (*metis eq-iff idx-less-last-1 le-neq-implies-less less-imp-le-nat*)

lemma *idx-bound-1*:

(index-sequence 0 (ls) \wedge (inth (ls) (ilen (ls)))) = (ilen (suffix k xs))) =
(index-sequence 0 (ls) \wedge (inth (ls) (ilen (ls)))) = (ilen (suffix k xs)) \wedge

$(\forall i. (i \leq \text{ilen } ls) \longrightarrow ((\text{inth } ls \ i) \leq (\text{ilen } (\text{suffix } k \ xs))))$)
using *idx-bound-0* **by** *blast*

lemma *idx-less-equal*:

assumes *index-sequence 0 l*
 $(\text{inth } l \ (\text{ilen } l)) = \text{ilen } xs$
 $i \leq \text{ilen } l$
 $n \leq \text{ilen } l$

shows $\forall j. j \leq n \longrightarrow \text{inth } l \ j \leq \text{inth } l \ n$

using *assms*

using *idx-less-eq* **by** *blast*

lemma *idx-less-than*:

assumes *index-sequence 0 l*
 $(\text{inth } l \ (\text{ilen } l)) = \text{ilen } xs$
 $i \leq \text{ilen } l$
 $n \leq \text{ilen } l$

shows $\forall j. j > n \wedge j \leq \text{ilen } l \longrightarrow \text{inth } l \ j > \text{inth } l \ n$

by (*meson Suc-leI assms idx-less-equal index-sequence-def less-le-trans*)

lemma *idx-sub*:

assumes $k \leq n$
 $n \leq \text{ilen } l$
index-sequence 0 l

shows *index-sequence* (*inth l k*) (*sub k n l*)

proof –

have 1: *index-sequence* (*inth l k*) (*sub k n l*) =
 $((\text{inth } (\text{sub } k \ n \ l) \ 0) = (\text{inth } l \ k) \wedge$
 $(\forall i. i < \text{ilen}(\text{sub } k \ n \ l) \longrightarrow (\text{inth } (\text{sub } k \ n \ l) \ i) < (\text{inth } (\text{sub } k \ n \ l) \ (\text{Suc } i))))$

using *index-sequence-def* **by** *auto*

have 2: $(\text{inth } (\text{sub } k \ n \ l) \ 0) = (\text{inth } l \ k)$

using *assms ifirst-sub* **by** *auto*

have 3: $\text{ilen}(\text{sub } k \ n \ l) = (n - k)$

by (*simp add: assms*)

have 4: $(\forall i. i < \text{ilen}(\text{sub } k \ n \ l) \longrightarrow (\text{inth } (\text{sub } k \ n \ l) \ i) < (\text{inth } (\text{sub } k \ n \ l) \ (\text{Suc } i)))$

proof

fix *i*

show $i < \text{ilen } (\text{sub } k \ n \ l) \longrightarrow \text{inth } (\text{sub } k \ n \ l) \ i < \text{inth } (\text{sub } k \ n \ l) \ (\text{Suc } i)$

proof –

have 41: $i < (n - k) \longrightarrow \text{inth } (\text{sub } k \ n \ l) \ i = \text{inth } l \ (k + i)$

by (*simp add: assms*)

have 42: $i < (n - k) \longrightarrow \text{inth } (\text{sub } k \ n \ l) \ (\text{Suc } i) = \text{inth } l \ (k + (\text{Suc } i))$

by (*simp add: assms*)

have 43: $i < (n - k) \longrightarrow \text{inth } l \ (k + i) < \text{inth } l \ (k + (\text{Suc } i))$

using *assms*

using *index-sequence-def* **by** *auto*

show *?thesis* **using** 3 41 42 43 **by** *auto*


```

qed
qed
show ?thesis by (simp add: 1 2 4)
qed

```

lemma *idx-split*:

```

assumes  $n \leq \text{ilen } l$ 
shows  $\text{index-sequence } 0 \ l =$ 
       $(\text{index-sequence } 0 \ (\text{prefix } n \ l) \wedge \text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l))$ 
proof -
  have 1:  $\text{index-sequence } 0 \ l \longrightarrow$ 
       $\text{index-sequence } 0 \ (\text{prefix } n \ l) \wedge \text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l)$ 
    using idx-sub using assms index-sequence-def by auto
  have 2:  $\text{index-sequence } 0 \ (\text{prefix } n \ l) =$ 
       $( (\text{inth } (\text{prefix } n \ l) \ 0) = 0 \wedge$ 
         $(\forall i. i < \text{ilen}(\text{prefix } n \ l) \longrightarrow (\text{inth } (\text{prefix } n \ l) \ i) < (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i))))$ 
    using index-sequence-def by blast
  have 3:  $( (\text{inth } (\text{prefix } n \ l) \ 0) = 0 \wedge$ 
       $(\forall i. i < \text{ilen}(\text{prefix } n \ l) \longrightarrow (\text{inth } (\text{prefix } n \ l) \ i) < (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)))) =$ 
       $( (\text{inth } l \ 0) = 0 \wedge$ 
         $(\forall i. i < n \longrightarrow (\text{inth } l \ i) < (\text{inth } l \ (\text{Suc } i))))$ 
    using assms by auto
  have 4:  $\text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l) =$ 
       $( (\text{inth } (\text{suffix } n \ l) \ 0) = (\text{inth } l \ n) \wedge$ 
         $(\forall i. i < \text{ilen}(\text{suffix } n \ l) \longrightarrow (\text{inth } (\text{suffix } n \ l) \ i) < (\text{inth } (\text{suffix } n \ l) \ (\text{Suc } i))))$ 
    using index-sequence-def by blast
  have 5:  $( (\text{inth } (\text{suffix } n \ l) \ 0) = (\text{inth } l \ n) \wedge$ 
       $(\forall i. i < \text{ilen}(\text{suffix } n \ l) \longrightarrow (\text{inth } (\text{suffix } n \ l) \ i) < (\text{inth } (\text{suffix } n \ l) \ (\text{Suc } i)))) =$ 
       $( (\text{inth } l \ n) = (\text{inth } l \ n) \wedge$ 
         $(\forall i. i < \text{ilen } l - n \longrightarrow (\text{inth } l \ (i+n)) < (\text{inth } l \ ((\text{Suc } i)+n))))$ 
    by (simp add: add commute)
  have 6:  $(\forall i. i < \text{ilen } l - n \longrightarrow (\text{inth } l \ (i+n)) < (\text{inth } l \ ((\text{Suc } i)+n))) =$ 
       $(\forall i. n \leq (i+n) \wedge (i+n) < \text{ilen } l \longrightarrow (\text{inth } l \ (i+n)) < (\text{inth } l \ ((\text{Suc } (i+n)))))$ 
    by auto
  have 61:  $(\forall i. n \leq (i+n) \wedge (i+n) < \text{ilen } l \longrightarrow (\text{inth } l \ (i+n)) < (\text{inth } l \ ((\text{Suc } (i+n))))) =$ 
       $(\forall j. n \leq j \wedge j < \text{ilen } l \longrightarrow (\text{inth } l \ j) < (\text{inth } l \ ((\text{Suc } j))))$ 
    by (metis diff-add)
  have 7:  $(\text{index-sequence } 0 \ (\text{prefix } n \ l) \wedge \text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l)) \longrightarrow$ 
       $(\text{inth } l \ 0) = 0 \wedge (\forall i. i < n \longrightarrow (\text{inth } l \ i) < (\text{inth } l \ (\text{Suc } i)))$ 
    using 2 3 by blast
  have 8:  $(\text{index-sequence } 0 \ (\text{prefix } n \ l) \wedge \text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l)) \longrightarrow$ 
       $(\forall i. n \leq i \wedge i < \text{ilen } l \longrightarrow (\text{inth } l \ i) < (\text{inth } l \ ((\text{Suc } i))))$ 
    using 4 5 6 61 by blast
  have 9:  $(\text{index-sequence } 0 \ (\text{prefix } n \ l) \wedge \text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l)) \longrightarrow$ 
       $(\text{inth } l \ 0) = 0 \wedge (\forall i. i < \text{ilen } l \longrightarrow (\text{inth } l \ i) < (\text{inth } l \ (\text{Suc } i)))$ 
    using 7 8 not-le by blast
  have 10:  $(\text{index-sequence } 0 \ (\text{prefix } n \ l) \wedge \text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l)) \longrightarrow$ 
       $\text{index-sequence } 0 \ l$ 
    using 9 index-sequence-def by blast
  from 10 1 show ?thesis by blast

```

qed

lemma *idx-suffixa*:

assumes $n \leq \text{ilen } l$

$\text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l)$

shows $\text{index-sequence } 0 \ ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))$

using *assms idx-shiftn* **by** *blast*

lemma *idx-greater*:

assumes $\text{index-sequence } k \ l$

shows $(\forall i. i \leq \text{ilen } l \longrightarrow k \leq (\text{inth } l \ i))$

by (*metis assms eq-iff index-sequence-def idx-greater-first less-imp-le neq0-conv*)

lemma *idx-suffixb*:

assumes $n \leq \text{ilen } l$

$\text{index-sequence } 0 \ ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))$

shows $\text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l)$

by (*metis assms idx-shiftn-c ilast-ifirst ilast-prefix*)

lemma *idx-suffix*:

assumes $n \leq \text{ilen } l$

shows $\text{index-sequence } (\text{inth } l \ n) \ (\text{suffix } n \ l) =$

$\text{index-sequence } 0 \ ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))$

using *assms idx-shiftn idx-suffixb* **by** *blast*

lemma *idx-ifirst*:

assumes $\text{index-sequence } 0 \ (x1a \odot l)$

shows $x1a < \text{ifirst}(l)$

by (*metis assms idx-ICons*)

lemma *idx-expand1*:

$(\text{index-sequence } x1a \ (x1a \odot l)) = (x1a < \text{ifirst } l \wedge \text{index-sequence } (\text{ifirst } l) \ l)$

using *inth-zero-ifirst less-Suc-eq-0-disj* **by** (*auto simp add: index-sequence-def*)

lemma *idx-ilen-leq-ilast-ifirst*:

assumes $\text{index-sequence } (\text{ifirst } l) \ l$

shows $\text{ilen } (l) \leq (\text{ilast } l - \text{ifirst } l)$

using *assms*

proof

(*induct* l)

case (*INil* x)

then show *?case* **by** *simp*

next

case (*ICons* $x1a \ l$)

then show *?case*

proof –

have 1: $\text{ilen } (x1a \odot l) = \text{ilen } l + 1$

by *simp*

have 2: $\text{index-sequence } (\text{ifirst } l) \ l$

```

  using ICons.premis idx-expand1 by auto
  have 3: ilast (x1a  $\odot$  l) = ilast l
  by simp
  have 4: ifirst (x1a  $\odot$  l) = x1a
  by simp
  have 5: x1a < ifirst l
  using ICons.premis idx-expand1 by auto
  have 6: ilen l  $\leq$  ilast l - ifirst l
  using 2 ICons.hyps by blast
  have 7: ilen l + 1  $\leq$  (ilast l - ifirst l) + 1
  using 6 add-le-cancel-right by blast
  have 8: (ilast l - ifirst l) + 1  $\leq$  ilast l - x1a
  using 5 6 le-less-linear by fastforce
  show ?thesis using 7 8 by auto
qed
qed

```

lemma *idx-ilen-leq*:

assumes *index-sequence* (ifirst l) l

$ilast(l) \leq ilen\ xs$

shows $ilen\ (l) \leq ilen\ (sub\ (ifirst\ l)\ (ilast\ l)\ xs)$

proof –

have 1: ifirst l \leq ilast l

using *assms* by (metis *eq-iff* *gr0I* *index-sequence-def* *idx-less-last-1* *less-imp-le-nat*)

have 2: $ilen\ (sub\ (ifirst\ l)\ (ilast\ l)\ xs) = (ilast\ l - ifirst\ l)$

using 1 *assms* *ilen-sub* by blast

have 3: $ilen\ (l) \leq (ilast\ l - ifirst\ l)$

using *assms* *idx-ilen-leq-ilast-ifirst* by blast

show ?thesis using 2 3 by *linarith*

qed

lemma *idx-shiftm-sub-inth*:

assumes *index-sequence* 0 l

$(inth\ l\ (ilen\ l)) = ilen\ xs$

$k \leq n$

$n \leq ilen\ l$

shows $\forall j. j \leq n-k \longrightarrow$

$inth\ (imap\ (shiftm\ (inth\ l\ k))\ (sub\ k\ n\ l))\ j = inth\ l\ (k+j) - (inth\ l\ k)$

by (*simp* *add*: *Nat.le-diff-conv2* *assms* *inth-imap* *shiftm-def*)

lemma *idx-shiftm-suffix-inth*:

assumes *index-sequence* 0 l

$(inth\ l\ (ilen\ l)) = ilen\ xs$

$n \leq ilen\ l$

shows $\forall j. j \leq ilen\ l - n \longrightarrow$

$inth\ (imap\ (shiftm\ (inth\ l\ n))\ (suffix\ n\ l))\ j = inth\ l\ (n+j) - (inth\ l\ n)$

using *assms* **by** (*metis* *inth-imap* *inth-suffix* *shiftm-def*)

1.2.10 upt

lemma *upt-rec*[*code*]:

$[i.. \leq j] = (\text{if } i < j \text{ then } i \odot [Suc\ i.. \leq j] \text{ else } \langle j \rangle)$

by (*induct j*) *auto*

lemma *upt-conv-INil* [*simp*]:

assumes $j < i$

shows $[i.. \leq j] = \langle j \rangle$

using *assms*

by (*metis upt.simps(1) upt.simps(2) Suc-leD less-Suc-eq-0-disj not-le*)

lemma *upt-same*:

$[i.. \leq i] = \langle i \rangle$

by (*metis upt.simps(1) upt.simps(2) less-Suc-eq less-Suc-eq-0-disj not-le*)

lemma *upt-eq-INil-conv*[*simp*]:

$([i.. \leq j] = \langle j \rangle) = (j \leq i)$

by (*simp add: upt-rec*)

lemma *upt-eq-ICons-conv*:

$([i.. \leq j] = x \odot xs) = (i < j \wedge i = x \wedge [Suc\ i.. \leq j] = xs)$

using *upt-rec* **by** (*induct j arbitrary: x xs*) *auto*

lemma *upt-suc-iapp*:

assumes $i \leq j$

shows $[i.. \leq (Suc\ j)] = [i.. \leq j] \oplus \langle (Suc\ j) \rangle$

using *assms* **by** *simp*

lemma *upt-conv-ICons*:

assumes $i < j$

shows $[i.. \leq j] = i \odot [(Suc\ i).. \leq j]$

using *assms* **by** (*simp add: upt-rec*)

lemma *upt-conv-ICons-ICons*:

$(m \odot n \odot ns = [m.. \leq q]) = (n \odot ns = [(Suc\ m).. \leq q])$

proof (*cases m ≤ q*)

case *True*

then show *?thesis* **by** (*simp add: upt-rec*)

next

case *False*

then show *?thesis* **by** *auto*

qed

lemma *upt-add-eq-iapp*:

assumes $i \leq j$

$k > 0$

shows $[i.. \leq j+k] = [i.. \leq j] \oplus [Suc\ j.. \leq j+k]$

using *assms*

proof

(*induct k*)

```

case 0
then show ?case by blast
next
case (Suc k)
then show ?case using Suc-less-eq le-simps(2) by auto
qed

```

```

lemma upt-ilen:
  ilen [i..≤j] = j - i
by (induct j) (auto simp add: Suc-diff-le)

```

```

lemma upt-inth-help:
  inth [i..≤i + k] k = i + k
proof
  (induct k arbitrary: i)
  case 0
  then show ?case by (simp add: upt-same)
  next
  case (Suc k)
  then show ?case
  by (metis upt-rec add-Suc-shift inth-Suc less-add-same-cancel1 zero-less-Suc)
qed

```

```

lemma upt-inth:
assumes  $i + k \leq j$ 
shows  $(\text{inth } [i..≤j] k) = i + k$ 
using assms
proof
  (induct j arbitrary: k i)
  case 0
  then show ?case by simp
  next
  case (Suc j)
  then show ?case
  by (metis upt.upt-Suc Nat.le-diff-conv2 add commute add-leD1
    iapp-inth le-SucE upt-ilen upt-inth-help)
qed

```

```

lemma upt-ifirst:
assumes  $i \leq j$ 
shows  $\text{ifirst } [i..≤j] = i$ 
using assms by (simp add: upt-rec)

```

```

lemma upt-ilast:
  ilast [i..≤j] = j
by (metis add-diff-inverse-nat INil-ilen order-refl upt-conv-INil upt-ilen upt-inth)

```

```

lemma prefix-upt:
assumes  $i + m \leq n$ 
shows  $\text{prefix } m [i..≤n] = [i..≤i + m]$ 

```

```

using assms
proof
  (induct m arbitrary: i)
  case 0
  then show ?case by (simp add: upt-inth upt-same)
  next
  case (Suc m)
  then show ?case using upt-rec by auto
qed

lemma suffix-upt:
   $\text{suffix } m \ [i..j] = [i+m..j]$ 
proof
  (induct m arbitrary: i j)
  case 0
  then show ?case by simp
  next
  case (Suc j)
  then show ?case using upt-rec
  by (metis add-Suc-shift suffix-suc not-less-eq not-less-iff-gr-or-eq suffix.simps(1))
qed

```

```

lemma imap-suc-upt:
   $\text{imap } \text{Suc} \ [m..n] = [\text{Suc } m.. \text{Suc } n]$ 
proof
  (induct n arbitrary: m)
  case 0
  then show ?case by simp
  next
  case (Suc n)
  then show ?case by simp
qed

```

```

lemma imap-add-upt:
   $\text{imap } (\lambda i. i + n) \ [0..m] = [n..m+n]$ 
proof
  (induct m)
  case 0
  then show ?case by (simp add: upt-same)
  next
  case (Suc m)
  then show ?case by simp
qed

```

1.2.11 Set

```

lemma iset-iapp [simp]:
   $\text{iset } (xs \ominus ys) = (\text{iset } xs \cup \text{iset } ys)$ 
by (induct xs) auto

```

lemma *interval-finite-iset* [iff]:
 finite (iset (xs:: 'a interval))
by (*induct xs*) *auto*

lemma *interval-hd-in-iset* [simp]:
 $x \in \text{iset } (x \odot xs)$
by *simp*

lemma *iset-subset-ICons*:
 $\text{iset } xs \subseteq \text{iset } (x \odot xs)$
by *auto*

lemma *iset-IConsD*:
assumes $y \in \text{iset } (x \odot xs)$
shows $y = x \vee y \in \text{iset } xs$
using *assms* **by** *auto*

lemma *exists-ICons*:
 $(\exists \text{ } ys \in \text{iset } (x \odot xs)). P \text{ } ys) \longleftrightarrow$
 $(P \text{ } x \wedge (\exists \text{ } ys \in \text{iset } xs. P \text{ } ys)) \vee (\neg P \text{ } x \wedge (\exists \text{ } ys \in \text{iset } xs. P \text{ } ys)) \vee$
 $(P \text{ } x \wedge (\forall \text{ } ys \in \text{iset } xs. \neg P \text{ } ys))$
by *auto*

lemma *interval-iset-nonempty*:
 $\text{iset } xs \neq \{\}$
by (*induct xs*) *auto*

lemma *iset-irev* [simp]:
 $\text{iset } (\text{irev } xs) = \text{iset } xs$
by (*induct xs*) *auto*

lemma *split-interval*:
assumes $x \in \text{iset } xs$
shows $\exists \text{ } ys \text{ } zs. xs = ys \ominus (x \odot zs) \vee xs = \langle x \rangle \vee xs = x \odot zs \vee xs = ys \ominus \langle x \rangle$
using *assms*
proof (*induct xs*)
case (*INil* x)
then show ?*case* **by** *simp*
next
case (*ICons* $x1a \text{ } xs$)
then show ?*case*
by (*metis iapp-INil interval.inject(2) interval.set-cases iapp-assoc*)
qed

lemma *interval-in-iset-conv-decomp*:
 $x \in \text{iset } xs =$
 $(\exists \text{ } ys \text{ } zs. xs = ys \ominus (x \odot zs) \vee xs = \langle x \rangle \vee xs = x \odot zs \vee xs = ys \ominus \langle x \rangle)$
by (*auto elim: split-interval*)

lemma *split-interval-first*:

assumes $x \in \text{iset } xs$

shows $\exists ys\ zs. xs = ys \ominus (x \odot zs) \wedge x \notin \text{iset } ys \vee xs = \langle x \rangle \vee xs = x \odot zs \vee$
 $xs = ys \ominus \langle x \rangle \wedge x \notin \text{iset } ys$

using *assms*

proof (*induct xs*)

case (*INil x*)

then show *?case* **by** *simp*

next

case (*ICons x1a xs*)

then show *?case*

proof (*cases x = x1a*)

case *True*

then show *?thesis* **by** *blast*

next

case *False*

then show *?thesis*

using *ICons.premss ICons.hyps*

proof *auto*

show $\bigwedge ys\ zs.$

$x \neq x1a \implies$

$x \notin \text{iset } ys \implies$

$xs = ys \ominus x \odot zs \implies$

$\exists ysa. (\exists zsa. x1a \odot ys \ominus x \odot zs = ysa \ominus x \odot zsa) \wedge x \notin \text{iset } ysa \vee$

$x1a \odot ys \ominus x \odot zs = ysa \ominus \langle x \rangle \wedge x \notin \text{iset } ysa$

by (*meson ICons-eq-iappI iset-IConsD*)

show $x \neq x1a \implies$

$xs = \langle x \rangle \implies$

$\exists ys. (\exists zs. \langle x1a, x \rangle = ys \ominus x \odot zs) \wedge x \notin \text{iset } ys \vee \langle x1a, x \rangle = ys \ominus \langle x \rangle \wedge x \notin \text{iset } ys$

by (*metis False Set.set-insert iapp-INil interval.simps(15) singleton-insert-inj-eq'*)

show $\bigwedge zs. x \neq x1a \implies$

$xs = x \odot zs \implies$

$\exists ys. (\exists zsa. x1a \odot x \odot zs = ys \ominus x \odot zsa) \wedge x \notin \text{iset } ys \vee$

$x1a \odot x \odot zs = ys \ominus \langle x \rangle \wedge x \notin \text{iset } ys$

by (*metis iapp-INil interval.simps(15) singleton-iff*)

show $\bigwedge ys. x \neq x1a \implies$

$x \notin \text{iset } ys \implies$

$xs = ys \ominus \langle x \rangle \implies$

$\exists ysa. (\exists zs. x1a \odot ys \ominus \langle x \rangle = ysa \ominus x \odot zs) \wedge x \notin \text{iset } ysa \vee$

$x1a \odot ys \ominus \langle x \rangle = ysa \ominus \langle x \rangle \wedge x \notin \text{iset } ysa$

by (*meson ICons-eq-iappI iset-IConsD*)

qed

qed

qed

lemma *in-iset-conv-decomp-first*:

$x \in \text{iset } xs =$

$(\exists ys\ zs. xs = ys \ominus (x \odot zs) \wedge x \notin \text{iset } ys \vee xs = \langle x \rangle \vee xs = x \odot zs \vee$
 $xs = ys \ominus \langle x \rangle \wedge x \notin \text{iset } ys)$

by (auto dest!: split-interval-first)

lemma *split-interval-last*:

assumes $x \in \text{iset } xs$

shows $\exists ys\ zs.\ xs = ys \ominus (x \odot zs) \wedge x \notin \text{iset } zs \vee xs = \langle x \rangle \vee xs = x \odot zs \wedge x \notin \text{iset } zs \vee$
 $xs = ys \ominus \langle x \rangle$

using *assms*

proof (*induct xs rule: irev-induct*)

case (*INil y*)

then show ?case **by** *simp*

next

case (*snoc x1a xs*)

then show ?case **proof** (*cases x = x1a*)

case *True*

then show ?thesis **by** *blast*

next

case *False*

then show ?thesis **using** *snoc* **by** *fastforce*

qed

qed

lemma *interval-in-iset-conv-decomp-last*:

$x \in \text{iset } xs =$

$(\exists ys\ zs.\ xs = ys \ominus (x \odot zs) \wedge x \notin \text{iset } zs \vee xs = \langle x \rangle \vee xs = x \odot zs \wedge x \notin \text{iset } zs \vee$
 $xs = ys \ominus \langle x \rangle)$

by (auto dest!: split-interval-last)

lemma *interval-list-prop*:

assumes $\exists x \in \text{iset } xs.\ P\ x$

shows $(\exists ys\ x\ zs.\ (xs = ys \ominus (x \odot zs) \vee xs = \langle x \rangle \vee xs = x \odot zs \vee$
 $xs = ys \ominus \langle x \rangle) \wedge P\ x)$

using *assms*

proof (*induct xs*)

case (*INil x*)

then show ?case **by** *auto*

next

case (*ICons x1a xs*)

then show ?case

by (*meson split-interval*)

qed

lemma *split-interval-propE*:

assumes $\exists x \in \text{iset } xs.\ P\ x$

obtains $ys\ x\ zs$ **where** $xs = ys \ominus (x \odot zs) \vee xs = \langle x \rangle \vee xs = x \odot zs \vee$
 $xs = ys \ominus \langle x \rangle$ **and** $P\ x$

using *interval-list-prop* [*OF assms*] **by** *blast*

lemma *split-interval-first-prop*:

assumes $\exists x \in \text{iset } xs.\ P\ x$

shows $(\exists ys\ x\ zs.\ ((xs = ys \ominus (x \odot zs) \wedge (\forall y \in \text{iset } ys.\ \neg P\ y)) \vee$

```


$$xs = \langle x \rangle \vee xs = x \odot zs \vee$$


$$(xs = ys \ominus \langle x \rangle \wedge (\forall y \in \text{iset } ys. \neg P y))) \wedge P x$$

)
using assms
proof (induct xs)
case (INil x)
then show ?case by auto
next
  case (ICons x1a xs)
  then show ?case proof (cases P x1a)
  case True
  then show ?thesis by blast
  next
  case False
  then show ?thesis
    using ICons
    proof auto
    show  $\bigwedge ys \ x a \ zs.$ 
       $\neg P \ x1a \implies$ 
       $P \ x a \implies$ 
       $\forall y \in \text{iset } ys. \neg P \ y \implies$ 
       $xs = ys \ominus xa \odot zs \implies$ 
       $\exists ysa \ x.$ 
       $(\exists zsa. \ x1a \odot ys \ominus xa \odot zs = ysa \ominus x \odot zsa \wedge (\forall y \in \text{iset } ysa. \neg P \ y) \vee$ 
       $x1a = x \wedge ys \ominus xa \odot zs = zsa \vee x1a \odot ys \ominus xa \odot zs = ysa \ominus \langle x \rangle \wedge (\forall y \in \text{iset } ysa. \neg P \ y)) \wedge$ 
       $P \ x$ 
      by (metis insertE iapp-ICons interval.simps(16))
    show  $\bigwedge x \ ys \ xa \ zs.$ 
       $\neg P \ x1a \implies$ 
       $P \ x \implies$ 
       $P \ xa \implies$ 
       $\forall y \in \text{iset } ys. \neg P \ y \implies$ 
       $xs = ys \ominus xa \odot zs \implies$ 
       $x \in \text{iset } zs \implies$ 
       $\exists ysa \ x.$ 
       $(\exists zsa. \ x1a \odot ys \ominus xa \odot zs = ysa \ominus x \odot zsa \wedge (\forall y \in \text{iset } ysa. \neg P \ y) \vee$ 
       $x1a = x \wedge ys \ominus xa \odot zs = zsa \vee x1a \odot ys \ominus xa \odot zs = ysa \ominus \langle x \rangle \wedge (\forall y \in \text{iset } ysa. \neg P \ y)) \wedge$ 
       $P \ x$ 
      by (metis ICons-eq-iappI iset-IConsD)
    show  $\bigwedge xa. \neg P \ x1a \implies$ 
       $P \ xa \implies$ 
       $xs = \langle xa \rangle \implies$ 
       $\exists ys \ x.$ 
       $(\exists zs. \ \langle x1a, xa \rangle = ys \ominus x \odot zs \wedge (\forall y \in \text{iset } ys. \neg P \ y) \vee$ 
       $x1a = x \wedge \langle xa \rangle = zs \vee \langle x1a, xa \rangle = ys \ominus \langle x \rangle \wedge (\forall y \in \text{iset } ys. \neg P \ y)) \wedge P \ x$ 
      by (metis iapp-INil interval.simps(15) singleton-iff)
    show  $\bigwedge xa \ zs.$ 
       $\neg P \ x1a \implies$ 
       $P \ xa \implies$ 
       $xs = xa \odot zs \implies$ 

```

```

 $\exists ys\ x.$ 
  ( $\exists zsa.\ x1a \odot xa \odot zs = ys \ominus x \odot zsa \wedge (\forall y \in iset\ ys.\ \neg P\ y) \vee$ 
     $x1a = x \wedge xa \odot zs = zsa \vee x1a \odot xa \odot zs = ys \ominus \langle x \rangle \wedge (\forall y \in iset\ ys.\ \neg P\ y)) \wedge$ 
     $P\ x$ 
  by (metis iapp-INil interval.simps(15) singleton-iff)
show  $\bigwedge x\ xa\ zs.$ 
   $\neg P\ x1a \implies$ 
   $P\ x \implies$ 
   $P\ xa \implies$ 
   $xs = xa \odot zs \implies$ 
   $x \in iset\ zs \implies$ 
 $\exists ys\ x.$ 
  ( $\exists zsa.\ x1a \odot xa \odot zs = ys \ominus x \odot zsa \wedge (\forall y \in iset\ ys.\ \neg P\ y) \vee$ 
     $x1a = x \wedge xa \odot zs = zsa \vee x1a \odot xa \odot zs = ys \ominus \langle x \rangle \wedge (\forall y \in iset\ ys.\ \neg P\ y)) \wedge$ 
     $P\ x$ 
  by (metis iapp-INil interval.simps(15) singleton-iff)
show  $\bigwedge ys\ xa.$ 
   $\neg P\ x1a \implies$ 
   $P\ xa \implies$ 
 $\forall y \in iset\ ys.\ \neg P\ y \implies$ 
   $xs = ys \ominus \langle xa \rangle \implies$ 
 $\exists ysa\ x.$ 
  ( $\exists zs.\ x1a \odot ys \ominus \langle xa \rangle = ysa \ominus x \odot zs \wedge (\forall y \in iset\ ysa.\ \neg P\ y) \vee$ 
     $x1a = x \wedge ys \ominus \langle xa \rangle = zs \vee x1a \odot ys \ominus \langle xa \rangle = ysa \ominus \langle x \rangle \wedge (\forall y \in iset\ ysa.\ \neg P\ y)) \wedge$ 
     $P\ x$ 
  by (metis insertE iapp.simps(2) interval.set(2))
qed
qed
qed

```

lemma *split-interval-first-propE*:

assumes $\exists x \in iset\ xs.\ P\ x$

obtains $ys\ x\ zs$ **where** $((xs = ys \ominus (x \odot zs) \wedge (\forall y \in iset\ ys.\ \neg P\ y)) \vee$

$xs = \langle x \rangle \vee xs = x \odot zs \vee$

$(xs = ys \ominus \langle x \rangle \wedge (\forall y \in iset\ ys.\ \neg P\ y)))$ **and** $P\ x$

using *split-interval-first-prop* [*OF assms*] **by** *blast*

lemma *split-first-prop-iff*:

$(\exists x \in iset\ xs.\ P\ x) \longleftrightarrow$

$(\exists ys\ x\ zs.\ ((xs = ys \ominus (x \odot zs) \wedge (\forall y \in iset\ ys.\ \neg P\ y)) \vee$

$xs = \langle x \rangle \vee xs = x \odot zs \vee$

$(xs = ys \ominus \langle x \rangle \wedge (\forall y \in iset\ ys.\ \neg P\ y))) \wedge P\ x$

)

by (*rule*, *erule split-interval-first-prop*) *auto*

lemma *split-interval-last-prop*:

assumes $\exists x \in iset\ xs.\ P\ x$

shows $(\exists ys\ x\ zs.\ ((xs = ys \ominus (x \odot zs) \wedge (\forall y \in iset\ zs.\ \neg P\ y)) \vee$

$xs = \langle x \rangle \vee xs = x \odot zs \wedge (\forall y \in iset\ zs.\ \neg P\ y)) \vee$

$(xs = ys \ominus \langle x \rangle) \wedge P\ x$

```

)
using assms
proof (induct xs rule: irev-induct)
case (INil y)
then show ?case by auto
next
case (snoc x1a xs)
  then show ?case proof (cases P x1a)
  case True
  then show ?thesis by blast
  next
  case False
  then show ?thesis
    using snoc by fastforce
  qed
qed

lemma split-interval-last-propE:
  assumes  $\exists x \in \text{iset } xs. P\ x$ 
  obtains ys x zs where  $((xs = ys \ominus (x \odot zs) \wedge (\forall y \in \text{iset } zs. \neg P\ y)) \vee$ 
     $xs = \langle x \rangle \vee xs = x \odot zs \wedge (\forall y \in \text{iset } zs. \neg P\ y) \vee$ 
     $(xs = ys \ominus \langle x \rangle)) \text{ and } P\ x$ 
  using split-interval-last-prop [OF assms] by blast

lemma split-interval-last-prop-iff:
  ( $\exists x \in \text{iset } xs. P\ x$ )  $\longleftrightarrow$ 
  ( $\exists\ ys\ x\ zs. ((xs = ys \ominus (x \odot zs) \wedge (\forall y \in \text{iset } zs. \neg P\ y)) \vee$ 
     $xs = \langle x \rangle \vee xs = x \odot zs \wedge (\forall y \in \text{iset } zs. \neg P\ y) \vee$ 
     $(xs = ys \ominus \langle x \rangle)) \wedge P\ x$ 
  )
by (rule, erule split-interval-last-prop, auto)

lemma inth-and-iset:
   $x \in \text{iset } xs = (\exists\ i \leq \text{ilen } xs. (\text{inth } xs\ i) = x)$ 
proof (induct xs)
case (INil x)
then show ?case by auto
next
case (ICons x1a xs)
then show ?case
by (metis Suc-le-mono insert-iff interval.simps(16) ilen-ICons ilen-gr-zero
  inth-Suc inth-ICons inth-zero ilen.simps(2) le-diff-conv neq0-conv plus-1-eq-Suc)
qed

lemma card-ilen:
   $\text{card } (\text{iset } xs) \leq \text{ilen } xs + 1$ 
proof (induct xs)
case (INil x)
then show ?case by simp
next

```

```

case (ICons x1a xs)
then show ?case by (simp add: card-insert-le-m1)
qed

```

lemma *iset-inth*:

```

iset xs = { (inth xs k) | k. k ≤ ilen xs }
proof (induct xs)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
proof auto
show iset xs = { inth xs k | k. k ≤ ilen xs } ⇒
  ∃ k. x1a = (case k of 0 ⇒ x1a | Suc x ⇒ inth xs x) ∧ k ≤ Suc (ilen xs)
by force
show ∧k. iset xs = { inth xs k | k. k ≤ ilen xs } ⇒
  k ≤ ilen xs ⇒
  ∃ ka. inth xs k = (case ka of 0 ⇒ x1a | Suc x ⇒ inth xs x) ∧ ka ≤ Suc (ilen xs)
by force
show ∧k. iset xs = { inth xs k | k. k ≤ ilen xs } ⇒
  (case k of 0 ⇒ x1a | Suc x ⇒ inth xs x) ≠ x1a ⇒
  k ≤ Suc (ilen xs) ⇒
  ∃ ka. (case k of 0 ⇒ x1a | Suc x ⇒ inth xs x) = inth xs ka ∧ ka ≤ ilen xs
by (metis (mono-tags, lifting) Nitpick.case-nat-unfold Suc-eq-plus1 le-diff-conv)
qed
qed

```

lemma *prefix-iset*:

```

assumes k ≤ ilen xs
shows iset (prefix k xs) = { (inth xs i) | i. i ≤ k }
proof -
have 1: iset (prefix k xs) = { (inth (prefix k xs) i) | i. i ≤ ilen (prefix k xs) }
by (simp add: iset-inth)
have 2: k ≤ ilen xs
using assms by auto
have 3: { (inth (prefix k xs) i) | i. i ≤ ilen (prefix k xs) } =
  { (inth xs i) | i. i ≤ k }
using 2 by (metis inth-prefix prefix-ilen-good)
show ?thesis using 1 3 by auto
qed

```

lemma *suffix-iset*:

```

assumes k ≤ ilen xs
shows iset (suffix k xs) = { (inth xs (k+i)) | i. i ≤ ilen xs - k }
proof -
have 1: iset (suffix k xs) = { (inth (suffix k xs) i) | i. i ≤ ilen (suffix k xs) }
by (simp add: iset-inth)
have 2: k ≤ ilen xs
using assms by auto

```

have 3: $\{ (inth\ (suffix\ k\ xs)\ i) \mid i.\ i \leq ilen\ (suffix\ k\ xs) \} =$
 $\{ (inth\ xs\ (k+i)) \mid i.\ i \leq ilen\ xs - k \}$
by *force*
show ?thesis **using** 1 3 **by** *auto*
qed

lemma *suffix-iset-a*:
assumes $k \leq ilen\ xs$
shows $iset\ (suffix\ k\ xs) = \{ (inth\ xs\ i) \mid i.\ k \leq i \wedge i \leq ilen\ xs \}$
proof –
have 1: $iset\ (suffix\ k\ xs) = \{ (inth\ xs\ (k+i)) \mid i.\ i \leq ilen\ xs - k \}$
using *assms suffix-iset* **by** *blast*
have 2: $\forall x \in \{ (inth\ xs\ (k+i)) \mid i.\ i \leq ilen\ xs - k \} .$
 $x \in \{ (inth\ xs\ i) \mid i.\ k \leq i \wedge i \leq ilen\ xs \}$
using *assms* **by** *auto*
have 3: $\forall x \in \{ (inth\ xs\ i) \mid i.\ k \leq i \wedge i \leq ilen\ xs \} .$
 $x \in \{ (inth\ xs\ (k+i)) \mid i.\ i \leq ilen\ xs - k \}$
using *nat-le-iff-add* **by** *force*
have 4: $\{ (inth\ xs\ (k+i)) \mid i.\ i \leq ilen\ xs - k \} =$
 $\{ (inth\ xs\ i) \mid i.\ k \leq i \wedge i \leq ilen\ xs \}$
using 2 3 **by** *blast*
show ?thesis **by** (*simp add: 1 4*)
qed

lemma *sub-interval-iset*:
assumes $k \leq n$
 $n \leq ilen\ xs$
shows $iset\ (sub\ k\ n\ xs) = \{ (inth\ xs\ (k+i)) \mid i.\ i \leq n-k \}$
proof –
have 1: $iset\ (sub\ k\ n\ xs) = \{ (inth\ (sub\ k\ n\ xs)\ i) \mid i.\ i \leq ilen\ (sub\ k\ n\ xs) \}$
by (*simp add: iset-inth*)
have 2: $k \leq n$
using *assms* **by** *auto*
have 3: $n \leq ilen\ xs$
using *assms* **by** *auto*
have 4: $\{ (inth\ (sub\ k\ n\ xs)\ i) \mid i.\ i \leq ilen\ (sub\ k\ n\ xs) \} =$
 $\{ (inth\ xs\ (k+i)) \mid i.\ i \leq n-k \}$
using 2 3 **by** *force*
show ?thesis **by** (*simp add: 1 4*)
qed

lemma *sub-iset-a*:
assumes $k \leq n$
 $n \leq ilen\ xs$
shows $iset\ (sub\ k\ n\ xs) = \{ (inth\ xs\ i) \mid i.\ k \leq i \wedge i \leq n \}$
proof –
have 1: $iset\ (sub\ k\ n\ xs) = \{ (inth\ xs\ (k+i)) \mid i.\ i \leq n-k \}$
using *assms sub-interval-iset* **by** *blast*
have 2: $\forall x \in \{ (inth\ xs\ (k+i)) \mid i.\ i \leq n-k \} .$

```

      x ∈ { (inth xs i) | i. k ≤ i ∧ i ≤ n }
    using assms by auto
  have 3: ∀ x ∈ { (inth xs i) | i. k ≤ i ∧ i ≤ n }.
      x ∈ { (inth xs (k+i)) | i. i ≤ n-k }
    using assms using le-Suc-ex by auto fastforce
  have 4: { (inth xs (k+i)) | i. i ≤ n-k } = { (inth xs i) | i. k ≤ i ∧ i ≤ n }
    using 2 3 by blast
  show ?thesis by (simp add: 1 4)
qed

```

```

lemma inth-iset:
  assumes k ≤ ilen xs
  shows (inth xs k) ∈ iset xs
  using assms
  by (meson inth-and-iset)

```

end

2 Finite ITL Semantics

```

theory Semantics
  imports Interval HOL-TLA.Intensional HOL-TLA.Stfun
  begin

```

This theory mechanises a *shallow* embedding of finite ITL using the *Interval* and *Intensional* theories. A shallow embedding represents ITL using Isabelle/HOL predicates, while a *deep* embedding [1] would represent ITL formulas as mutually inductive datatypes. See, e.g., [12] for a discussion about deep vs. shallow embeddings in Isabelle/HOL. The choice of a shallow over a deep embedding is motivated [3, 2] by the following factors: a shallow embedding is usually less involved, and existing Isabelle theories and tools can be applied more directly to enhance automation; due to the lifting in the *Intensional* theory, a shallow embedding can reuse standard logical operators, whilst a deep embedding requires a different set of operators for formulas. Finally, since our target is system verification rather than proving meta-properties of the logic, which requires a deep embedding, a shallow embedding is more fit for purpose.

2.1 Types of Formulas

To mechanise the ITL semantics, the following type abbreviations are used:

```

type-synonym ('a,'b) formfun = 'a interval ⇒ 'b
type-synonym 'a formula      = ('a,bool) formfun
type-synonym ('a,'b) stfun   = 'a ⇒ 'b
type-synonym 'a stpred       = ('a,bool) stfun

```

```

instance
  fun :: (type,type) world ..

```

instance

prod :: (*type, type*) *world* ..

instance

interval :: (*type*) *world* ..

Pair, function, and interval are instantiated to be of type class *world*. This allows use of the lifted Intensional logic for formulas, and standard logical connectives can therefore be used.

2.2 Semantics of ITL

The semantics of ITL is defined. Note chopstar is a derived operator, i.e., it is defined recursively in terms of chop.

definition *skip-d* :: (*'a* :: *world*) *formula*

where *skip-d* $\equiv \lambda s. \text{ilen } s = 1$

definition *chop-d* :: (*'a* :: *world*) *formula* \Rightarrow (*'a* :: *world*) *formula* \Rightarrow (*'a* :: *world*) *formula*

where *chop-d* *F1 F2* $\equiv \lambda s. \exists n. 0 \leq n \wedge n \leq \text{ilen } s \wedge ((\text{prefix } n \ s) \models F1) \wedge ((\text{suffix } n \ s) \models F2)$

definition *current-val-d* :: (*'a* :: *world*, *'b*) *stfun* \Rightarrow (*'a*, *'b*) *formfun*

where *current-val-d* *f* $\equiv \lambda s. (\text{inth } s \ 0) \models f$

definition *next-val-d* :: (*'a* :: *world*, *'b*) *stfun* \Rightarrow (*'a*, *'b*) *formfun*

where *next-val-d* *f* $\equiv \lambda s. \text{if } \text{ilen } s > 0 \text{ then } ((\text{inth } s \ 1) \models f) \text{ else } (\epsilon \ (x :: 'b) . x = x)$

definition *fin-val-d* :: (*'a* :: *world*, *'b*) *stfun* \Rightarrow (*'a*, *'b*) *formfun*

where *fin-val-d* *f* $\equiv \lambda s. (\text{inth } s \ (\text{ilen } s)) \models f$

definition *penult-val-d* :: (*'a* :: *world*, *'b*) *stfun* \Rightarrow (*'a*, *'b*) *formfun*

where *penult-val-d* *f* $\equiv \lambda s. \text{if } \text{ilen } s > 0 \text{ then } (\text{inth } s \ ((\text{ilen } s) - 1)) \models f \text{ else } (\epsilon \ (x :: 'b) . x = x)$

This is the concrete syntax for the (abstract) operators above.

syntax

-skip-d :: *lift* ((*skip*))
-chop-d :: [*lift, lift*] \Rightarrow *lift* ((*;-*) [*84, 84*] *83*)
-current-val-d :: *lift* \Rightarrow *lift* ((*\$-*) [*100*] *99*)
-next-val-d :: *lift* \Rightarrow *lift* ((*-*) [*100*] *99*)
-fin-val-d :: *lift* \Rightarrow *lift* ((*!-*) [*100*] *99*)
-penult-val-d :: *lift* \Rightarrow *lift* ((*!*) [*100*] *99*)
TEMP :: *lift* \Rightarrow *'b* ((*TEMP -*))

syntax (ASCII)

-skip-d :: *lift* ((*skip*))
-chop-d :: [*lift, lift*] \Rightarrow *lift* ((*;-*) [*84, 84*] *83*)
-current-val-d :: *lift* \Rightarrow *lift* ((*\$-*) [*100*] *99*)
-next-val-d :: *lift* \Rightarrow *lift* ((*-*) [*100*] *99*)
-fin-val-d :: *lift* \Rightarrow *lift* ((*!-*) [*100*] *99*)
-penult-val-d :: *lift* \Rightarrow *lift* ((*!*) [*100*] *99*)

translations

$-skip-d \quad \Rightarrow \text{CONST } skip-d$
 $-chop-d \quad \Rightarrow \text{CONST } chop-d$
 $-current-val-d \Rightarrow \text{CONST } current-val-d$
 $-next-val-d \quad \Rightarrow \text{CONST } next-val-d$
 $-fin-val-d \quad \Rightarrow \text{CONST } fin-val-d$
 $-penult-val-d \Rightarrow \text{CONST } penult-val-d$
 $TEMP F \quad \rightarrow (F :: (- \text{ interval}) \Rightarrow -)$

2.3 Abbreviations

Some standard temporal abbreviations, with their concrete syntax.

definition $sometimes-d :: ('a::world) \text{ formula} \Rightarrow 'a \text{ formula}$
where $sometimes-d F \equiv LIFT(\#True; F)$

definition $di-d :: ('a::world) \text{ formula} \Rightarrow 'a \text{ formula}$
where $di-d F \equiv LIFT(F; \#True)$

definition $da-d :: ('a::world) \text{ formula} \Rightarrow 'a \text{ formula}$
where $da-d F \equiv LIFT(\#True; (F; \#True))$

definition $next-d :: ('a::world) \text{ formula} \Rightarrow 'a \text{ formula}$
where $next-d F \equiv LIFT(skip; F)$

definition $prev-d :: ('a::world) \text{ formula} \Rightarrow 'a \text{ formula}$
where $prev-d F \equiv LIFT(F; skip)$

syntax

$-sometimes-d :: lift \Rightarrow lift ((\Diamond-) [88] 87)$
 $-di-d \quad \quad \quad :: lift \Rightarrow lift ((di -) [88] 87)$
 $-da-d \quad \quad \quad :: lift \Rightarrow lift ((da -) [88] 87)$
 $-next-d \quad \quad \quad :: lift \Rightarrow lift ((\bigcirc -) [88] 87)$
 $-prev-d \quad \quad \quad :: lift \Rightarrow lift ((prev -) [88] 87)$

syntax (ASCII)

$-sometimes-d :: lift \Rightarrow lift ((<>-) [88] 87)$
 $-di-d \quad \quad \quad :: lift \Rightarrow lift ((di -) [88] 87)$
 $-da-d \quad \quad \quad :: lift \Rightarrow lift ((da -) [88] 87)$
 $-next-d \quad \quad \quad :: lift \Rightarrow lift ((next -) [88] 87)$
 $-prev-d \quad \quad \quad :: lift \Rightarrow lift ((prev -) [88] 87)$

translations

$-sometimes-d \Rightarrow \text{CONST } sometimes-d$
 $-di-d \quad \quad \quad \Rightarrow \text{CONST } di-d$
 $-da-d \quad \quad \quad \Rightarrow \text{CONST } da-d$
 $-next-d \quad \quad \quad \Rightarrow \text{CONST } next-d$
 $-prev-d \quad \quad \quad \Rightarrow \text{CONST } prev-d$

definition *always-d* :: ('a::world) formula \Rightarrow 'a formula
where *always-d* $F \equiv LIFT(\neg(\Diamond(\neg F)))$

definition *bi-d* :: ('a::world) formula \Rightarrow 'a formula
where *bi-d* $F \equiv LIFT(\neg(di(\neg F)))$

definition *ba-d* :: ('a::world) formula \Rightarrow 'a formula
where *ba-d* $F \equiv LIFT(\neg(da(\neg F)))$

definition *wnext-d* :: ('a::world) formula \Rightarrow 'a formula
where *wnext-d* $F \equiv LIFT(\neg(\bigcirc(\neg F)))$

definition *wprev-d* :: ('a::world) formula \Rightarrow 'a formula
where *wprev-d* $F \equiv LIFT(\neg(prev(\neg F)))$

definition *more-d* :: ('a::world) formula
where *more-d* $\equiv LIFT(\bigcirc(\#True))$

syntax

-*always-d* :: lift \Rightarrow lift $((\Box-) [88] 87)$
-*bi-d* :: lift \Rightarrow lift $((bi-) [88] 87)$
-*ba-d* :: lift \Rightarrow lift $((ba-) [88] 87)$
-*wnext-d* :: lift \Rightarrow lift $((wnext-) [88] 87)$
-*wprev-d* :: lift \Rightarrow lift $((wprev-) [88] 87)$
-*more-d* :: lift $((more))$

syntax (ASCII)

-*always-d* :: lift \Rightarrow lift $(([]-) [88] 87)$
-*bi-d* :: lift \Rightarrow lift $((bi-) [88] 87)$
-*ba-d* :: lift \Rightarrow lift $((ba-) [88] 87)$
-*wnext-d* :: lift \Rightarrow lift $((wnext-) [88] 87)$
-*wprev-d* :: lift \Rightarrow lift $((wprev-) [88] 87)$
-*more-d* :: lift $((more))$

translations

-*always-d* $\equiv CONST$ *always-d*
-*bi-d* $\equiv CONST$ *bi-d*
-*ba-d* $\equiv CONST$ *ba-d*
-*wnext-d* $\equiv CONST$ *wnext-d*
-*wprev-d* $\equiv CONST$ *wprev-d*
-*more-d* $\equiv CONST$ *more-d*

definition *empty-d* :: ('a::world) formula
where *empty-d* $\equiv LIFT(\neg(more))$

definition *dm-d* :: ('a::world) formula \Rightarrow 'a formula
where *dm-d* $F \equiv LIFT(\#True;(more \wedge F))$

syntax

-empty-d :: lift ((empty))
 -dm-d :: lift \Rightarrow lift ((dm -) [88] 87)

syntax (ASCII)

-empty-d :: lift ((empty))
 -dm-d :: lift \Rightarrow lift ((dm -) [88] 87)

translations

-empty-d \Rightarrow CONST empty-d
 -dm-d \Rightarrow CONST dm-d

definition bm-d :: ('a::world) formula \Rightarrow 'a formula

where bm-d F \equiv LIFT(\neg (dm(\neg F)))

definition init-d :: ('a::world) formula \Rightarrow 'a formula

where init-d F \equiv LIFT((empty \wedge F);# True)

definition fin-d :: ('a::world) formula \Rightarrow 'a formula

where fin-d F \equiv LIFT(\Box (empty \longrightarrow F))

definition halt-d :: ('a::world) formula \Rightarrow 'a formula

where halt-d F \equiv LIFT(\Box (empty = F))

definition initonly-d :: ('a::world) formula \Rightarrow 'a formula

where initonly-d F \equiv LIFT(bi(empty = F))

definition keep-d :: ('a::world) formula \Rightarrow 'a formula

where keep-d F \equiv LIFT(ba(skip \longrightarrow F))

definition yields-d :: ('a::world) formula \Rightarrow 'a formula \Rightarrow 'a formula

where yields-d F1 F2 \equiv LIFT(\neg (F1;(\neg F2)))

definition ifthenelse-d :: ('a::world) formula \Rightarrow 'a formula \Rightarrow 'a formula \Rightarrow 'a formula

where ifthenelse-d F G H \equiv LIFT((F \wedge G) \vee (\neg F \wedge H))

primrec power-d :: ('a::world) formula \Rightarrow nat \Rightarrow 'a formula

where pow-0 : (power-d F 0) = LIFT(empty)

| pow-Suc: (power-d F (Suc n)) = LIFT((F);(power-d F n))

syntax

-bm-d :: lift \Rightarrow lift ((bm -) [88] 87)
 -init-d :: lift \Rightarrow lift ((init -) [88] 87)
 -fin-d :: lift \Rightarrow lift ((fin -) [88] 87)
 -halt-d :: lift \Rightarrow lift ((halt -) [88] 87)
 -initonly-d :: lift \Rightarrow lift ((initonly -) [88] 87)
 -keep-d :: lift \Rightarrow lift ((keep -) [88] 87)

$\text{-yields-d} \quad :: [\text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((\text{- yields -}) [88, 88] \ 87)$
 $\text{-ifthenelse-d} \quad :: [\text{lift}, \text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((\text{if}_i \text{ - then - else -}) [88, 88, 88] \ 87)$
 $\text{-power-d} \quad :: [\text{lift}, \text{nat}] \Rightarrow \text{lift} \quad ((\text{power - -}) [88, 88] \ 87)$

syntax (*ASCII*)

$\text{-bm-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{bm -}) [88] \ 87)$
 $\text{-init-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{init -}) [88] \ 87)$
 $\text{-fin-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{fin -}) [88] \ 87)$
 $\text{-halt-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{halt -}) [88] \ 87)$
 $\text{-initonly-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{initonly -}) [88] \ 87)$
 $\text{-keep-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{keep -}) [88] \ 87)$
 $\text{-yields-d} \quad :: [\text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((\text{- yields -}) [88, 88] \ 87)$
 $\text{-ifthenelse-d} \quad :: [\text{lift}, \text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((\text{if}_i \text{ - then - else -}) [88, 88, 88] \ 87)$
 $\text{-power-d} \quad :: [\text{lift}, \text{nat}] \Rightarrow \text{lift} \quad ((\text{power - -}) [88, 88] \ 87)$

translations

$\text{-bm-d} \quad \Rightarrow \text{CONST bm-d}$
 $\text{-init-d} \quad \Rightarrow \text{CONST init-d}$
 $\text{-fin-d} \quad \Rightarrow \text{CONST fin-d}$
 $\text{-halt-d} \quad \Rightarrow \text{CONST halt-d}$
 $\text{-initonly-d} \quad \Rightarrow \text{CONST initonly-d}$
 $\text{-keep-d} \quad \Rightarrow \text{CONST keep-d}$
 $\text{-yields-d} \quad \Rightarrow \text{CONST yields-d}$
 $\text{-ifthenelse-d} \quad \Rightarrow \text{CONST ifthenelse-d}$
 $\text{-power-d} \quad \Rightarrow \text{CONST power-d}$

definition $\text{len-d} :: \text{nat} \Rightarrow ('a::\text{world}) \text{ formula}$

where $\text{len-d } n \equiv \text{LIFT}(\text{power skip } n)$

definition $\text{powerstar-d} :: ('a::\text{world}) \text{ formula} \Rightarrow 'a \text{ formula}$

where $\text{powerstar-d } F \equiv \text{LIFT}(\exists \ k. \text{ power } F \ k)$

syntax

$\text{-len-d} \quad :: \text{nat} \Rightarrow \text{lift} \quad ((\text{len -}) [88] \ 87)$
 $\text{-powerstar-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{powerstar -}) [85] \ 85)$

syntax (*ASCII*)

$\text{-len-d} \quad :: \text{nat} \Rightarrow \text{lift} \quad ((\text{len -}) [88] \ 87)$
 $\text{-powerstar-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{powerstar -}) [85] \ 85)$

translations

$\text{-len-d} \quad \Rightarrow \text{CONST len-d}$
 $\text{-powerstar-d} \quad \Rightarrow \text{CONST powerstar-d}$

definition $\text{chopstar-d} :: ('a::\text{world}) \text{ formula} \Rightarrow 'a \text{ formula}$

where $\text{chopstar-d } F \equiv \text{LIFT}(\text{powerstar } (F \wedge \text{more}))$

syntax

$\text{-chopstar-d} \quad :: \text{lift} \Rightarrow \text{lift} \quad ((\text{-}^*) [85] \ 85)$

syntax (*ASCII*)

-chopstar-d :: *lift* \Rightarrow *lift* ((*chopstar -*) [85] 85)

translations

-chopstar-d \Rightarrow *CONST chopstar-d*

definition *ifthen-d* :: (*'a::world*) *formula* \Rightarrow *'a formula* \Rightarrow *'a formula*

where *ifthen-d* *F G* \equiv *LIFT*(*if_i* *F* then *G* else #*True*)

definition *while-d* :: (*'a::world*) *formula* \Rightarrow *'a formula* \Rightarrow *'a formula*

where *while-d* *F G* \equiv *LIFT*((*F* \wedge *G*)^{*} \wedge (*fin* (\neg *F*))))

syntax

-ifthen-d :: [*lift, lift*] \Rightarrow *lift* ((*if_i* - then -) [88,88] 87)

-while-d :: [*lift, lift*] \Rightarrow *lift* ((*while* - do -) [88,88] 87)

syntax (*ASCII*)

-ifthen-d :: [*lift, lift*] \Rightarrow *lift* ((*if_i* - then -) [88,88] 87)

-while-d :: [*lift, lift*] \Rightarrow *lift* ((*while* - do -) [88,88] 87)

translations

-ifthen-d \Rightarrow *CONST ifthen-d*

-while-d \Rightarrow *CONST while-d*

definition *repeat-d* :: (*'a::world*) *formula* \Rightarrow *'a formula* \Rightarrow *'a formula*

where *repeat-d* *F G* \equiv *LIFT*(*F*; *while* (\neg *G*) do *F*)

syntax

-repeat-d :: [*lift, lift*] \Rightarrow *lift* ((*repeat* - until -) [88,88] 87)

syntax (*ASCII*)

-repeat-d :: [*lift, lift*] \Rightarrow *lift* ((*repeat* - until -) [88,88] 87)

translations

-repeat-d \Rightarrow *CONST repeat-d*

definition *next-assign-d* :: (*'a::world, 'b*) *stfun* \Rightarrow (*'a, 'b*) *formfun* \Rightarrow *'a formula*

where *next-assign-d* *v e* \equiv *LIFT*(*v*\$ = *e*)

definition *prev-assign-d* :: (*'a::world, 'b*) *stfun* \Rightarrow (*'a, 'b*) *formfun* \Rightarrow *'a formula*

where *prev-assign-d* *v e* \equiv *LIFT*(*v*! = *e*)

definition *always-eq-d* :: (*'a::world, 'b*) *stfun* \Rightarrow (*'a, 'b*) *formfun* \Rightarrow *'a formula*

where *always-eq-d* *v e* \equiv $\lambda s. s \models \Box(\$v = e)$

definition *temporal-assign-d* :: (*'a::world, 'b*) *stfun* \Rightarrow (*'a, 'b*) *formfun* \Rightarrow *'a formula*

where *temporal-assign-d* *v e* \equiv $\lambda s. s \models !v = e$

definition *gets-d* :: ('a::world,'b) stfun \Rightarrow ('a,'b) formfun \Rightarrow 'a formula
where *gets-d* v e $\equiv \lambda s. s \models \text{keep}(\text{temporal-assign-d } v \text{ } e)$

definition *stable-d* :: ('a::world,'b) stfun \Rightarrow 'a formula
where *stable-d* v $\equiv \lambda s. s \models \text{gets-d } v \text{ } (\text{current-val-d } v)$

definition *padded-d* :: ('a::world,'b) stfun \Rightarrow 'a formula
where *padded-d* v $\equiv \lambda s. s \models (\text{stable-d } v); \text{skip} \vee \text{empty}$

definition *padded-temp-assign-d* :: ('a::world,'b) stfun \Rightarrow ('a,'b) formfun \Rightarrow 'a formula
where *padded-temp-assign-d* v e $\equiv \lambda s. s \models (\text{temporal-assign-d } v \text{ } e) \wedge (\text{padded-d } v)$

syntax

-next-assign-d :: [lift, lift] \Rightarrow lift ((- := -) [50,51] 50)
-prev-assign-d :: [lift, lift] \Rightarrow lift ((- =: -) [50,51] 50)
-always-eq-d :: [lift, lift] \Rightarrow lift ((- \approx -) [50,51] 50)
-temporal-assign-d :: [lift, lift] \Rightarrow lift ((- \leftarrow -) [50,51] 50)
-gets-d :: [lift, lift] \Rightarrow lift ((- gets -) [50,51] 50)
-stable-d :: lift \Rightarrow lift ((stable -) [51] 50)
-padded-d :: lift \Rightarrow lift ((padded -) [51] 50)
-padded-temp-assign-d :: [lift, lift] \Rightarrow lift ((- $<\sim$ -) [50,51] 50)

syntax (ASCII)

-next-assign-d :: [lift, lift] \Rightarrow lift ((- := -) [50,51] 50)
-prev-assign-d :: [lift, lift] \Rightarrow lift ((- =: -) [50,51] 50)
-always-eq-d :: [lift, lift] \Rightarrow lift ((- alweqv -) [50,51] 50)
-temporal-assign-d :: [lift, lift] \Rightarrow lift ((- <-- -) [50,51] 50)
-gets-d :: [lift, lift] \Rightarrow lift ((- gets -) [50,51] 50)
-stable-d :: lift \Rightarrow lift ((stable -) [51] 50)
-padded-d :: lift \Rightarrow lift ((padded -) [51] 50)
-padded-temp-assign-d :: [lift, lift] \Rightarrow lift ((- $<\sim$ -) [50,51] 50)

translations

-next-assign-d \Rightarrow CONST next-assign-d
-prev-assign-d \Rightarrow CONST prev-assign-d
-always-eq-d \Rightarrow CONST always-eq-d
-temporal-assign-d \Rightarrow CONST temporal-assign-d
-gets-d \Rightarrow CONST gets-d
-stable-d \Rightarrow CONST stable-d
-padded-d \Rightarrow CONST padded-d
-padded-temp-assign-d \Rightarrow CONST padded-temp-assign-d

lemmas *itl-def* = skip-d-def chop-d-def current-val-d-def next-val-d-def fin-val-d-def penult-val-d-def
sometimes-d-def di-d-def da-d-def next-d-def prev-d-def always-d-def bi-d-def ba-d-def wnext-d-def
wprev-d-def more-d-def empty-d-def dm-d-def bm-d-def init-d-def fin-d-def halt-d-def initempty-d-def
keep-d-def yields-d-def ifthenelse-d-def power-d-def len-d-def powerstar-d-def chopstar-d-def
ifthen-d-def while-d-def repeat-d-def next-assign-d-def prev-assign-d-def always-eq-d-def
temporal-assign-d-def gets-d-def stable-d-def padded-d-def padded-temp-assign-d-def

2.4 Properties of Operators

The following lemmas show that above operators have the expected semantics.

lemma *skip-defs* :

$$(w \models \text{skip}) = (\text{ilen } w = 1)$$

by (*simp add: itl-def*)

lemma *chop-defs* :

$$(w \models F1 ; F2) = (\exists n. n \leq \text{ilen } w \wedge ((\text{prefix } n \ w) \models F1) \wedge ((\text{suffix } n \ w) \models F2))$$

by (*simp add: itl-def*)

lemma *sometimes-defs* :

$$(w \models \Diamond F) = (\exists n. n \leq \text{ilen } w \wedge ((\text{suffix } n \ w) \models F))$$

by (*simp add: itl-def chop-defs*)

lemma *always-defs* :

$$(w \models \Box F) = (\forall n. n \leq \text{ilen } w \longrightarrow ((\text{suffix } n \ w) \models F))$$

by (*simp add: itl-def sometimes-defs*)

lemma *di-defs* :

$$(w \models \text{di } F) = (\exists n. n \leq \text{ilen } w \wedge ((\text{prefix } n \ w) \models F))$$

by (*simp add: itl-def chop-defs*)

lemma *bi-defs* :

$$(w \models \text{bi } F) = (\forall n. n \leq \text{ilen } w \longrightarrow ((\text{prefix } n \ w) \models F))$$

by (*simp add: itl-def di-defs*)

lemma *da-defs* :

$$(w \models \text{da } F) = (\exists n \text{ na}. n + \text{na} \leq \text{ilen } w \wedge ((\text{sub } n \ (n + \text{na}) \ w) \models F))$$

proof (*auto simp add: itl-def chop-defs*)

show $\bigwedge n \text{ na}.$

$$n \leq \text{ilen } w \implies$$

$$\text{na} \leq \text{ilen } w - n \implies F (\text{prefix } \text{na} \ (\text{suffix } n \ w)) \implies$$

$$\exists n \text{ na}. n + \text{na} \leq \text{ilen } w \wedge F (\text{sub } n \ (n + \text{na}) \ w)$$

by (*metis sub-def Nat.le-diff-conv2 add.commute add-diff-cancel-left*)

show $\bigwedge n \text{ na}.$

$$n + \text{na} \leq \text{ilen } w \implies F (\text{sub } n \ (n + \text{na}) \ w) \implies$$

$$\exists n \leq \text{ilen } w. \exists \text{na} \leq \text{ilen } w - n. F (\text{prefix } \text{na} \ (\text{suffix } n \ w))$$

by (*metis sub-def add-leD1 ilen-sub prefix-ilen-bound suffix-ilen le-add1*)

qed

lemma *ba-defs* :

$$(w \models \text{ba } F) = (\forall n \text{ na}. n + \text{na} \leq \text{ilen } w \longrightarrow ((\text{sub } n \ (n + \text{na}) \ w) \models F))$$

by (*auto simp add: ba-d-def da-defs*)

lemma *next-defs* :

$$(w \models \bigcirc F) = (\text{ilen } w > 0 \wedge ((\text{suffix } 1 \ w) \models F))$$

using *Suc-le-eq min.absorb1* **by** (*simp add: itl-def chop-defs skip-defs*) *force*

lemma *wnext-defs* :

$(w \models wnext\ F) = (ilen\ w = 0 \vee ((suffix\ 1\ w) \models F))$
by (*simp add: wnext-d-def next-defs*)

lemma *prev-defs* :

$(w \models prev\ F) = (ilen\ w > 0 \wedge ((prefix\ ((ilen\ w)-1)\ w) \models F))$
by (*simp add: itl-def chop-defs skip-defs*)
(metis One-nat-def Suc-leI diff-diff-cancel diff-is-0-eq' diff-le-self neq0-conv zero-neq-one)

lemma *wprev-defs* :

$(w \models wprev\ F) = (ilen\ w = 0 \vee ((prefix\ ((ilen\ w)-1)\ w) \models F))$
by (*metis (mono-tags, lifting) less-le prev-defs unl-lift wprev-d-def zero-le*)

lemma *more-defs* :

$(w \models more) = (ilen\ w > 0)$
by (*simp add: more-d-def next-defs*)

lemma *empty-defs* :

$(w \models empty) = (ilen\ w = 0)$
by (*simp add: empty-d-def more-defs*)

lemma *init-defs* :

$(w \models init\ F) = ((prefix\ 0\ w) \models F)$
using *min.absorb1* **by** (*simp add: init-d-def empty-defs chop-defs*) *force*

lemma *initalt-defs* :

$(w \models bi(empty \longrightarrow F)) = ((prefix\ 0\ w) \models F)$
using *min.absorb1*
by (*simp add: bi-defs empty-defs*)

lemma *fin-defs* :

$(w \models fin\ F) = ((suffix\ (ilen\ w)\ w) \models F)$
by (*simp add: fin-d-def empty-defs always-defs*)

lemma *finalt-defs* :

$(w \models \#True;(F \wedge empty)) = ((suffix\ (ilen\ w)\ w) \models F)$
by (*simp add: chop-defs empty-defs fastforce*)

lemma *halt-defs* :

$(w \models halt(F)) = (\forall n \leq ilen\ w. (ilen\ w = n) = F\ (suffix\ n\ w))$
by (*simp add: halt-d-def empty-defs always-defs*)

lemma *initonly-defs* :

$(w \models initonly(F)) = (\forall n \leq ilen\ w. (n = 0) = F\ (prefix\ n\ w))$
using *min.absorb1* **by** (*simp add: initonly-d-def bi-defs empty-defs*)

lemma *ifthenelse-defs*:

$(w \models if_i\ F\ then\ G\ else\ H) =$
 $((w \models F) \wedge (w \models G)) \vee ((\neg(w \models F) \wedge (w \models H)))$
by (*simp add: itl-def*)

lemma *len-defs* :
 $(w \models \text{len } n) = (\text{ilen } w = n)$
proof
(induct n arbitrary: w)
case 0
then show ?case **by** (*simp add: len-d-def empty-defs*)
next
case (*Suc n*)
then show ?case **by** (*simp add: len-d-def chop-defs skip-defs fastforce*)
qed

lemma *currentval-defs* :
 $(s \models \$v) = (v \text{ (inth } s \ 0))$
by (*simp add: current-val-d-def*)

lemma *nextval-defs* :
 $(s \models v\$) = (\text{if } \text{ilen } s > 0 \text{ then } (v \text{ (inth } s \ 1)) \text{ else } (\epsilon \ x. x=x))$
by (*simp add: itl-def*)

lemma *finval-defs* :
 $(s \models !v) = (v \text{ (inth } s \ (\text{ilen } s)))$
by (*simp add: itl-def*)

lemma *penultval-defs* :
 $(s \models v!) = (\text{if } \text{ilen } s > 0 \text{ then } (v \text{ (inth } s \ ((\text{ilen } s) - 1))) \text{ else } (\epsilon \ x. x=x))$
by (*simp add: itl-def*)

lemma *next-assign-defs* :
assumes *ilen s > 0*
shows $(s \models v := e) = v \text{ (inth } s \ 1) = e \ s$
using *assms* **by** (*auto simp: itl-def*)

lemma *prev-assign-defs* :
assumes *ilen s > 0*
shows $(s \models v =: e) = v \text{ (inth } s \ ((\text{ilen } s) - 1)) = e \ s$
using *assms* **by** (*auto simp: itl-def*)

lemma *always-eqv-defs* :
 $(s \models v \approx e) = (\forall \ i \leq \text{ilen } s. v \text{ (inth } s \ i) = e \ (\text{suffix } i \ s))$
by (*simp add: always-eq-d-def always-defs current-val-d-def*)

lemma *temporal-assign-defs* :
 $(s \models v \leftarrow e) = (v \text{ (inth } s \ (\text{ilen } s)) = e \ s)$
by (*simp add: itl-def*)

lemma *gets-defs* :
 $(s \models v \text{ gets } e) = (\forall \ i < \text{ilen } s. v \text{ (inth } s \ (\text{Suc } i)) = e \ (\text{sub } i \ (i+1) \ s))$
using *Suc-le-eq min.absorb1 add-le-imp-le-diff prefix-suffix-ilen-good*

by (auto simp add: gets-d-def keep-d-def ba-defs skip-defs sub-def temporal-assign-defs)

lemma *stable-defs-helpa*:

assumes $(\forall i < \text{ilen } s. v \text{ (inth } s \text{ (Suc } i))} = v \text{ (inth } s \text{ } i))$

$i \leq \text{ilen } s$

shows $(v \text{ (inth } s \text{ } i) = v \text{ (inth } s \text{ } 0))$

using *assms*

proof (induct s arbitrary:i)

case (INil x)

then show ?case **by** *simp*

next

case (ICons x1a s)

then show ?case

proof (cases i)

case 0

then show ?thesis **by** *blast*

next

case (Suc nat)

then show ?thesis

by (metis ICons.hyps ICons.prem1s(1) ICons.prem1s(2) Suc-le-mono Suc-mono inth-Suc
inth-zero ilen.simp1s(2) plus-1-eq-Suc zero-less-Suc)

qed

qed

lemma *stable-defs-helpb*:

assumes $(\forall i \leq \text{ilen } s. v \text{ (inth } s \text{ } i) = v \text{ (inth } s \text{ } 0))$

$i < \text{ilen } s$

shows $v \text{ (inth } s \text{ (Suc } i))} = v \text{ (inth } s \text{ } i)$

using *assms*

proof (induct s arbitrary:i)

case (INil x)

then show ?case **by** *simp*

next

case (ICons x1a s)

then show ?case

proof (cases i)

case 0

then show ?thesis **using** Suc-leI ICons.prem1s(1) ICons.prem1s(2) **by** *blast*

next

case (Suc nat)

then show ?thesis **using** ICons.prem1s(1) ICons.prem1s(2) Suc-leI less-imp-le-nat **by** *presburger*

qed

qed

lemma *stable-defs-help*:

$(\forall i < \text{ilen } s. v \text{ (inth } s \text{ (Suc } i))} = v \text{ (inth } s \text{ } i)) \longleftrightarrow$

$(\forall i \leq \text{ilen } s. v \text{ (inth } s \text{ } i) = v \text{ (inth } s \text{ } 0))$

proof –

have 1: $(\forall i < \text{ilen } s. v \text{ (inth } s \text{ (Suc } i))} = v \text{ (inth } s \text{ } i)) \longrightarrow$

```

      (∀ i ≤ ilen s. v ( inth s i) = v ( inth s 0))
using stable-defs-helpa by auto
have 2: (∀ i ≤ ilen s. v ( inth s i) = v ( inth s 0)) →
      (∀ i < ilen s. v ( inth s (Suc i)) = v ( inth s i))
using stable-defs-helpb by blast
show ?thesis using 1 2 by blast
qed

```

lemma *stable-defs*:

```

(s ⊨ stable v) = (∀ i ≤ ilen s. (v (inth s i)) = (v (inth s 0)))
by (simp add: stable-d-def gets-defs current-val-d-def sub-def stable-defs-help)

```

lemma *padded-defs* :

```

(s ⊨ padded v) = ((∀ i < ilen s. (v (inth s i)) = (v (inth s 0))) ∨ ilen s = 0)
proof (simp add: padded-d-def stable-defs chop-defs skip-defs empty-defs)
show ((∃ n ≤ ilen s.
      (∀ i. i ≤ n ∧ i ≤ ilen s → v (inth s i) = v (inth s 0)) ∧ ilen s - n = Suc 0) ∨
      ilen s = 0) =
      ((∀ i < ilen s. v ( inth s i) = v ( inth s 0)) ∨ ilen s = 0)
proof rule+
show ∧ i. (∃ n ≤ ilen s.
      (∀ i. i ≤ n ∧ i ≤ ilen s → v (inth s i) = v (inth s 0)) ∧ ilen s - n = Suc 0) ∨
      ilen s = 0 ⇒
      i < ilen s ⇒ v (inth s i) = v (inth s 0)
by (metis One-nat-def Suc-leI Suc-le-mono le-add-diff-inverse2 less-imp-le-nat not-less-zero
      plus-1-eq-Suc)
show (∀ i < ilen s. v (inth s i) = v (inth s 0)) ∨ ilen s = 0 ⇒
      (∃ n ≤ ilen s.
      (∀ i. i ≤ n ∧ i ≤ ilen s → v (inth s i) = v (inth s 0)) ∧ ilen s - n = Suc 0) ∨
      ilen s = 0
by (metis Suc-leI Suc-pred diff-diff-cancel diff-le-self gr-zeroI le-imp-less-Suc)
qed
qed

```

lemma *padded-temporal-assign-defs* :

```

(s ⊨ v <~ e) =
((s ⊨ padded v) ∧ (v ( inth s (ilen s)) = e s ))
by (auto simp add: padded-temp-assign-d-def padded-defs temporal-assign-defs)

```

lemmas *itl-defs* = *skip-defs chop-defs sometimes-defs always-defs di-defs bi-defs da-defs ba-defs next-defs*
wnext-defs prev-defs wprev-defs more-defs empty-defs init-defs initalt-defs fin-defs finalt-defs
halt-defs initonly-defs ifthenelse-defs len-defs currentval-defs nextval-defs finval-defs
penultval-defs next-assign-defs prev-assign-defs always-eqv-defs temporal-assign-defs
gets-defs stable-defs padded-defs padded-temporal-assign-defs

2.5 Soundness of Finite ITL Axioms

2.5.1 ChopAssoc

lemma *ChopAssocSemHelpa*:

assumes $(\exists i \text{ ia} . i \leq \text{ilen } \sigma \wedge \text{ia} \leq \text{ilen } \sigma - i \wedge (\text{prefix } i \text{ } \sigma \models f) \wedge$
 $(\text{prefix } \text{ia} (\text{suffix } i \text{ } \sigma) \models g) \wedge (\text{suffix } (\text{ia} + i) \text{ } \sigma \models h))$
shows $(\exists j \text{ ja} . j \leq \text{ilen } \sigma \wedge \text{ja} \leq j \wedge (\text{prefix } \text{ja} (\text{prefix } j \text{ } \sigma) \models f) \wedge$
 $(\text{suffix } \text{ja} (\text{prefix } j \text{ } \sigma) \models g) \wedge (\text{suffix } j \text{ } \sigma \models h))$
proof –
have 1: $(\exists i \text{ ia} . i \leq \text{ilen } \sigma \wedge \text{ia} \leq \text{ilen } \sigma - i \wedge (\text{prefix } i \text{ } \sigma \models f) \wedge$
 $(\text{prefix } \text{ia} (\text{suffix } i \text{ } \sigma) \models g) \wedge (\text{suffix } (\text{ia} + i) \text{ } \sigma \models h))$
using *assms* **by** *auto*
obtain *i ia* **where** 2: $i \leq \text{ilen } \sigma \wedge \text{ia} \leq \text{ilen } \sigma - i \wedge (\text{prefix } i \text{ } \sigma \models f) \wedge$
 $(\text{prefix } \text{ia} (\text{suffix } i \text{ } \sigma) \models g) \wedge (\text{suffix } (\text{ia} + i) \text{ } \sigma \models h)$
using 1 **by** *auto*
have 3: $(\text{suffix } (\text{ia} + i) \text{ } \sigma \models h)$
using 2 **by** *auto*
have 4: $\text{ia} + i \leq \text{ilen } \sigma$
using 2 *Nat.le-diff-conv2* **by** *blast*
have 5: $i \leq \text{ia} + i$
by *simp*
have 6: $(\text{suffix } i (\text{prefix } (\text{ia} + i) \text{ } \sigma) \models g)$
using 2 4 *suffix-prefix-swap* **by** *force*
have 7: $(\text{prefix } i (\text{prefix } (\text{ia} + i) \text{ } \sigma) \models f)$
by (*simp add: 2 add commute*)
show *?thesis* **using** 2 4 5 6 7 **by** *blast*
qed

lemma *ChopAssocSemHelpb*:

assumes $(\exists j \text{ ja} . j \leq \text{ilen } \sigma \wedge \text{ja} \leq j \wedge (\text{prefix } \text{ja} (\text{prefix } j \text{ } \sigma) \models f) \wedge$
 $(\text{suffix } \text{ja} (\text{prefix } j \text{ } \sigma) \models g) \wedge (\text{suffix } j \text{ } \sigma \models h))$
shows $(\exists i \text{ ia} . i \leq \text{ilen } \sigma \wedge \text{ia} \leq \text{ilen } \sigma - i \wedge (\text{prefix } i \text{ } \sigma \models f) \wedge$
 $(\text{prefix } \text{ia} (\text{suffix } i \text{ } \sigma) \models g) \wedge (\text{suffix } (\text{ia} + i) \text{ } \sigma \models h))$
proof –
have 1: $(\exists j \text{ ja} . j \leq \text{ilen } \sigma \wedge \text{ja} \leq j \wedge (\text{prefix } \text{ja} (\text{prefix } j \text{ } \sigma) \models f) \wedge$
 $(\text{suffix } \text{ja} (\text{prefix } j \text{ } \sigma) \models g) \wedge (\text{suffix } j \text{ } \sigma \models h))$
using *assms* **by** *auto*
obtain *j ja* **where** 2: $j \leq \text{ilen } \sigma \wedge \text{ja} \leq j \wedge (\text{prefix } \text{ja} (\text{prefix } j \text{ } \sigma) \models f) \wedge$
 $(\text{suffix } \text{ja} (\text{prefix } j \text{ } \sigma) \models g) \wedge (\text{suffix } j \text{ } \sigma \models h)$
using 1 **by** *auto*
have 3: $\text{ja} \leq \text{ilen } \sigma$
using 2 *le-trans* **by** *blast*
have 4: $j - \text{ja} \leq \text{ilen } \sigma - \text{ja}$
by (*simp add: 2 diff-le-mono*)
have 5: $(\text{prefix } \text{ja} \text{ } \sigma \models f)$
by (*metis 2 pref-pref-3 le-add-diff-inverse*)
have 6: $(\text{prefix } (j - \text{ja}) (\text{suffix } \text{ja} \text{ } \sigma) \models g)$
by (*simp add: 2 suffix-prefix-swap*)
have 7: $(\text{suffix } ((j - \text{ja}) + \text{ja}) \text{ } \sigma \models h)$
by (*simp add: 2*)
show *?thesis* **using** 3 4 5 6 7 **by** *blast*
qed

lemma *ChopAssocSemHelp*:

$(\exists i \text{ ia} . i \leq \text{ilen } \sigma \wedge \text{ia} \leq \text{ilen } \sigma - i \wedge (\text{prefix } i \sigma \models f) \wedge$
 $(\text{prefix } \text{ia} (\text{suffix } i \sigma) \models g) \wedge (\text{suffix } (\text{ia} + i) \sigma \models h)) =$
 $(\exists j \text{ ja} . j \leq \text{ilen } \sigma \wedge \text{ja} \leq j \wedge (\text{prefix } \text{ja} (\text{prefix } j \sigma) \models f) \wedge$
 $(\text{suffix } \text{ja} (\text{prefix } j \sigma) \models g) \wedge (\text{suffix } j \sigma \models h))$

using *ChopAssocSemHelpa*[of $\sigma f g h$]
ChopAssocSemHelpb[of $\sigma f g h$] **by** *auto*

lemma *ChopAssocSemHelp2*:

$(\sigma \models f ; (g ; h)) = (\sigma \models (f;g);h)$

proof –

have $(\sigma \models f ; (g ; h)) =$
 $((\exists i \leq \text{ilen } \sigma . (\text{prefix } i \sigma \models f) \wedge (\exists \text{ia} \leq \text{ilen } (\text{suffix } i \sigma).$
 $(\text{prefix } \text{ia} (\text{suffix } i \sigma) \models g) \wedge (\text{suffix } (\text{ia} + i) \sigma \models h))))$

by (*simp add: chop-defs*)

also have ... =

$(\exists i \text{ ia} . i \leq \text{ilen } \sigma \wedge \text{ia} \leq \text{ilen } \sigma - i \wedge (\text{prefix } i \sigma \models f) \wedge$
 $(\text{prefix } \text{ia} (\text{suffix } i \sigma) \models g) \wedge (\text{suffix } (\text{ia} + i) \sigma \models h))$

by *fastforce*

also have ... =

$(\exists j \text{ ja} . j \leq \text{ilen } \sigma \wedge \text{ja} \leq j \wedge (\text{prefix } \text{ja} (\text{prefix } j \sigma) \models f) \wedge$
 $(\text{suffix } \text{ja} (\text{prefix } j \sigma) \models g) \wedge (\text{suffix } j \sigma \models h))$

using *ChopAssocSemHelp*[of $\sigma f g h$] **by** *blast*

also have ... =

$(\exists i \leq \text{ilen } \sigma . (\exists \text{ia} \leq \text{ilen } (\text{prefix } i \sigma) . (\text{prefix } \text{ia} (\text{prefix } i \sigma) \models f) \wedge$
 $(\text{suffix } \text{ia} (\text{prefix } i \sigma) \models g)) \wedge (\text{suffix } i \sigma \models h))$

by *fastforce*

also have ... =

$(\sigma \models (f;g);h)$ **by** (*simp add: chop-defs*)

finally show $(\sigma \models f ; (g ; h)) = (\sigma \models (f;g);h)$.

qed

lemma *ChopAssocSem*:

$(\sigma \models f ; (g ; h)) = (f;g);h$

using *ChopAssocSemHelp2* **using** *unl-lift2* **by** *blast*

2.5.2 OrChopImp

lemma *OrChopImpSem*:

$(\sigma \models (f \vee g);h \longrightarrow f;h \vee g;h)$

by (*simp add: itl-defs*) *blast*

2.5.3 ChopOrImp

lemma *ChopOrImpSem*:

$(\sigma \models f;(g \vee h) \longrightarrow f;g \vee f;h)$

by (*simp add: itl-defs*) *blast*

2.5.4 EmptyChop

lemma *EmptyChopSem*:

$(\sigma \models \text{empty} ; f = f)$

using *min.absorb1* by (simp add: *itl-defs*) force

2.5.5 ChopEmpty

lemma *ChopEmptySem*:
 $(\sigma \models f; \text{empty} = f)$
 by (simp add: *itl-defs*) auto

2.5.6 StateImpBi

lemma *StateImpBiSem*:
 $(\sigma \models \text{init } f \longrightarrow \text{bi } (\text{init } f))$
 by (simp add: *itl-defs*)

2.5.7 NextImpNotNextNot

lemma *NextImpNotNextNotSem*:
 $(\sigma \models \bigcirc f \longrightarrow \neg (\bigcirc (\neg f)))$
 by (simp add: *itl-defs*)

2.5.8 BiBoxChopImpChop

lemma *BiBoxChopImpChopSem*:
 $(\sigma \models \text{bi } (f \longrightarrow f1) \wedge \Box (g \longrightarrow g1) \longrightarrow f; g \longrightarrow f1; g1)$
 by (simp add: *itl-defs*) fastforce

2.5.9 BoxInduct

lemma *box-induct-help-1* :
 assumes $(f \ \sigma)$
 $(\forall i. \text{Suc } 0 \leq \text{ilen } \sigma - i \longrightarrow$
 $i \leq \text{ilen } \sigma \longrightarrow (\text{suffix } i \ \sigma \models f) \longrightarrow f (\text{suffix } (\text{Suc } i) \ \sigma))$
 shows $(\forall j. j \leq \text{ilen } \sigma \longrightarrow f (\text{suffix } j \ \sigma))$
 proof
 fix j
 show $j \leq \text{ilen } \sigma \longrightarrow f (\text{suffix } j \ \sigma)$
 using *assms*
 proof
 (induct j arbitrary: σ)
 case 0
 then show ?case by simp
 next
 case (Suc j)
 then show ?case
 by (metis *Nat.le-diff-conv2 One-nat-def Suc-eq-plus1-left Suc-leD*)
 qed
 qed

lemma *BoxInductSem*:
 $(\sigma \models \Box (f \longrightarrow \text{wnext } f) \wedge f \longrightarrow \Box f)$
 proof –
 have 1: $(\sigma \models \Box (f \longrightarrow \text{wnext } f) \wedge f \longrightarrow \Box f) =$

```

      (( $\forall n \leq \text{ilen } \sigma. f (\text{suffix } n \ \sigma) \longrightarrow \text{ilen } \sigma = n \vee f (\text{suffix } (\text{Suc } n) \ \sigma)$ )  $\wedge f \ \sigma \longrightarrow$ 
       ( $\forall n \leq \text{ilen } \sigma. f (\text{suffix } n \ \sigma)$ ))
    by (simp add: always-defs wnext-defs)
  from 1 show ?thesis using box-induct-help-1[of f  $\sigma$ ] by fastforce
qed

```

2.5.10 ChopStarEqv

lemma *ChopExist*:

```

 $\vdash (\exists k. f;g \ k) = f;(\exists k. g \ k)$ 
by (auto simp add: itl-defs Valid-def)

```

lemma *ExistChop*:

```

 $\vdash (\exists k. (g \ k);f) = (\exists k. g \ k);f$ 
by (auto simp add: itl-defs Valid-def)

```

lemma *powersem1*:

```

( $\sigma \models (\exists k. \text{power } f \ k) = (\text{empty} \vee (\exists k. \text{power } f (\text{Suc } k)))$ )
proof auto
  show  $\bigwedge x. \sigma \models (\text{power } f \ x) \implies \forall k. \neg(\sigma \models (f;\text{power } f \ k)) \implies \sigma \models \text{empty}$ 
  by (metis not0-implies-Suc pow-0 pow-Suc)
  show  $\sigma \models \text{empty} \implies \exists x. \sigma \models (\text{power } f \ x)$ 
  by (metis pow-0)
  show  $\bigwedge k. \sigma \models (f;\text{power } f \ k) \implies \exists x. \sigma \models (\text{power } f \ x)$ 
  by (metis pow-Suc)
qed

```

lemma *powersem*:

```

 $\vdash (\exists k. \text{power } f \ k) = (\text{empty} \vee (f);(\exists k. (\text{power } f \ k)))$ 
proof -
  have 1:  $\vdash (\exists k. \text{power } f \ k) = (\text{empty} \vee (\exists k. \text{power } f (\text{Suc } k)))$ 
  using powersem1 by blast
  have 2:  $\vdash (\exists k. \text{power } f (\text{Suc } k)) = (\exists k. (f);\text{power } f \ k)$ 
  by simp
  have 3:  $\vdash (\exists k. (f);(\text{power } f \ k)) = (f);(\exists k. (\text{power } f \ k))$ 
  using ChopExist by blast
  from 1 2 3 show ?thesis by fastforce
qed

```

lemma *PowerstarEqvSem*:

```

( $\sigma \models (\text{powerstar } f) = (\text{empty} \vee f;(\text{powerstar } f))$ )
proof -
  have 1:  $(\sigma \models (\text{powerstar } f)) =$ 
    ( $\sigma \models (\exists k. \text{power } f \ k)$ )
  by (simp add: powerstar-d-def)
  have 2:  $(\sigma \models (\exists k. \text{power } f \ k)) =$ 
    ( $\sigma \models (\text{empty} \vee f;(\exists k. (\text{power } f \ k)))$ )
  using powersem by (metis inteq-reflection)
  from 1 2 show ?thesis by (simp add: powerstar-d-def)
qed

```

lemma *ChopstarEqvSem*:

$(\sigma \models f^* = (\text{empty} \vee (f \wedge \text{more}); f^*))$

by (*metis PowerstarEqvSem chopstar-d-def*)

2.6 Quantification over State (Flexible) Variables

The hidden state approach, as used in the embedding of TLA in Isabelle/HOL TLA embedding [3, 2], is used. Here [3, 2], a state space is defined by its projections, and everything else is unknown. Thus, a variable is a projection of the state space, and has the same type as a state function. Moreover, strong typing is achieved, since the projection function may have any result type. To achieve this, the state space is represented by an undefined type, which is an instance of the *world* class to enable use with the *Intensional* theory.

type-synonym *'a statefun* = (*state*, *'a*) *stfun*

type-synonym *statepred* = *bool statefun*

type-synonym *'a tempfun* = (*state*, *'a*) *formfun*

type-synonym *temporal* = *state formula*

Similar to [3, 2] we define a state to be an anonymous type whose only purpose is to provide Skolem IConstants. Similarly, we do not define a type of state variables separate from that of arbitrary state functions, again in order to simplify the definition of flexible quantification later on. Nevertheless, we need to distinguish state variables. Note we deviate from [3, 2] in that we do not use axioms but use definitions and lemmas.

2.7 Temporal Quantifiers

definition *exist-state-d* :: (*'a statefun* \Rightarrow *temporal*) \Rightarrow *temporal* (**binder** *Eex* 10)

where *exist-state-d* *F* \equiv ($\lambda s. (\exists x. s \models F x)$)

syntax

-Eex :: [*idts*, *lift*] \Rightarrow *lift* $((\exists \exists \exists \text{ -./ -}) [0,10] 10)$

translations

-Eex v A == *Eex v. A*

definition *forall-state-d* :: (*'a statefun* \Rightarrow *temporal*) \Rightarrow *temporal* (**binder** *Aall* 10)

where *forall-state-d* *F* \equiv *LIFT*($\neg(\exists \exists \exists x. \neg(F x))$)

syntax

-Aall :: [*idts*, *lift*] \Rightarrow *lift* $((\exists \forall \forall \text{ -./ -}) [0,10] 10)$

translations

-Aall v A == *Aall v. A*

end

3 Fuse operator

```
theory Fuse
imports Semantics
begin
```

This theory introduces the fuse operator.

3.1 Definitions

```
primrec fuse :: 'a interval  $\Rightarrow$  'a interval  $\Rightarrow$  'a interval
where fuse-INil : fuse  $\langle x \rangle$  ys = ys
| fuse-ICons : fuse (x  $\odot$  xs) ys = x  $\odot$  (fuse xs ys)
```

```
primrec lfuse :: 'a interval interval  $\Rightarrow$  'a interval
where lfuse-INil : lfuse  $\langle xs \rangle$  = xs
| lfuse-ICons : lfuse (x  $\odot$  xs) = fuse x (lfuse xs)
```

```
primrec lastfirst :: 'a interval interval  $\Rightarrow$  bool
where lastfirst  $\langle xs \rangle$  = True
| lastfirst (xs  $\odot$  xxs) =
  ( ((ilast xs) = (ifirst (ifirst xxs)))  $\wedge$  (lastfirst xxs))
```

3.2 Lemmas

```
lemma fuse-ilen :
  assumes ilast xs = ifirst ys
  shows ilen (fuse xs ys) = (ilen xs) + (ilen ys)
using assms by (induct xs) simp-all
```

```
lemma fuse-ilen-a:
  ilen(fuse xs ys) = ilen xs + ilen ys
proof
  (induct xs arbitrary: ys)
  case (INil x)
  then show ?case by simp
next
  case (ICons x1a xs)
  then show ?case by simp
qed
```

```
lemma fuse-inth:
  assumes  $i \leq \text{ilen} (fuse\ xs\ ys)$ 
  ilast xs = ifirst ys
  shows ( $i \leq \text{ilen}\ xs \longrightarrow \text{inth} (fuse\ xs\ ys)\ i = \text{inth}\ xs\ i$ )
   $\wedge$ 
  ( $\text{ilen}\ xs \leq i \wedge i \leq \text{ilen} (fuse\ xs\ ys) \longrightarrow \text{inth} (fuse\ xs\ ys)\ i = \text{inth}\ ys\ (i - \text{ilen}\ xs)$ )
```

```
using assms
proof
```

```

(induct xs arbitrary: i)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
using less-Suc-eq-0-disj less-Suc-eq-le by fastforce
qed

```

```

lemma fuse-inth-a:
  assumes  $j \leq \text{ilen } ys$ 
          $\text{ilast } xs = \text{ifirst } ys$ 
  shows  $\text{inth } (\text{fuse } xs \ ys) \ (\text{ilen } xs + j) = (\text{inth } ys \ j)$ 
using assms
by (simp add: fuse-ilen-a fuse-inth)

```

```

lemma fuse-leftneutral :
   $\text{fuse } (\text{INil } (\text{ifirst } xs)) \ xs = xs$ 
by simp

```

```

lemma fuse-rightneutral :
   $\text{fuse } xs \ (\text{INil } (\text{ilast } xs)) = xs$ 
by (induct xs) simp-all

```

```

lemma ifirst-fuse :
  assumes  $\text{ilast } xs = \text{ifirst } ys$ 
  shows  $\text{ifirst } (\text{fuse } xs \ ys) = \text{ifirst } xs$ 
using assms by (induct xs) simp-all

```

```

lemma ilast-fuse :
  assumes  $\text{ilast } xs = \text{ifirst } ys$ 
  shows  $\text{ilast } (\text{fuse } xs \ ys) = \text{ilast } ys$ 
using assms by (induct xs) simp-all

```

```

lemma FusionAssoc :
  assumes  $(\text{ilast } xs) = (\text{ifirst } ys)$ 
          $(\text{ilast } ys) = (\text{ifirst } zs)$ 
  shows  $(\text{fuse } xs \ (\text{fuse } ys \ zs)) = (\text{fuse } (\text{fuse } xs \ ys) \ zs)$ 
using assms by (induct xs) simp-all

```

```

lemma ilast-ifirst:
   $(\text{ilast } (\text{prefix } i \ xs)) = (\text{ifirst } (\text{suffix } i \ xs))$ 
using ilast-ifirst by blast

```

```

lemma prefix-fuse :
  assumes  $\text{ilast } xs = \text{ifirst } ys$ 
  shows  $(\text{prefix } (\text{ilen } xs) \ (\text{fuse } xs \ ys)) = xs$ 
using assms by (induct xs arbitrary: ys) simp-all

```

lemma *suffix-fuse* :
assumes $ilast\ xs = ifirst\ ys$
shows $(suffix\ (ilen\ xs)\ (fuse\ xs\ ys)) = ys$
using *assms* **by** (*induct xs arbitrary: ys simp-all*)

lemma *fuse-prefix-suffix-ilen* :
assumes $n \leq ilen\ xs$
shows $ilen\ (fuse\ (prefix\ n\ xs)\ (suffix\ n\ xs)) = ilen\ xs$
using *assms*
by (*metis fuse-ilen-a prefix-ilen-good suffix-ilen-good le-add-diff-inverse*)

lemma *fuse-prefix-suffix-inth* :
assumes $n \leq ilen\ xs$
 $i \leq ilen\ xs$
shows $inth\ (fuse\ (prefix\ n\ xs)\ (suffix\ n\ xs))\ i = inth\ xs\ i$
using *assms fuse-inth[of i (prefix n xs) (suffix n xs)]*
 $ilast-ifirst[of\ n\ xs]$
 $fuse-prefix-suffix-ilen[of\ n\ xs]$
by (*metis diff-add inth-prefix inth-suffix prefix-ilen-good le-cases*
 $le-diff-conv\ ordered-cancel-comm-monoid-diff-class.add-diff-inverse$)

lemma *fuse-prefix-suffix*:
assumes $n \leq ilen\ xs$
shows $fuse\ (prefix\ n\ xs)\ (suffix\ n\ xs) = xs$
using *assms*
by (*simp add: fuse-prefix-suffix-ilen fuse-prefix-suffix-inth interval-eq-inth-eq*)

lemma *chop-fuse-1* :
 $(\exists\ \sigma 1\ \sigma 2. \sigma = fuse\ \sigma 1\ \sigma 2 \wedge$
 $(\sigma 1 \models f) \wedge (\sigma 2 \models g) \wedge$
 $(ilast\ \sigma 1 = ifirst\ \sigma 2)) =$
 $(\exists\ i. 0 \leq i \wedge i \leq ilen\ \sigma \wedge (prefix\ i\ \sigma \models f) \wedge (suffix\ i\ \sigma \models g))$
by (*metis fuse-ilen fuse-prefix-suffix ilast-ifirst*
 $ilen-gr-zero\ prefix-fuse\ suffix-fuse\ le-add-same-cancel1$)

lemma *chop-fuse-2* :
 $(\exists\ \sigma 1\ \sigma 2. \sigma = fuse\ \sigma 1\ \sigma 2 \wedge$
 $(\sigma 1 \in X) \wedge (\sigma 2 \in Y) \wedge$
 $(ilast\ \sigma 1 = ifirst\ \sigma 2)) =$
 $(\exists\ i \leq ilen\ \sigma. (prefix\ i\ \sigma) \in X \wedge (suffix\ i\ \sigma) \in Y)$
by (*metis fuse-ilen fuse-prefix-suffix ilast-ifirst prefix-fuse suffix-fuse le-add1*)

lemma *chop-fuse*:
 $(\exists\ \sigma 1\ \sigma 2. \sigma = fuse\ \sigma 1\ \sigma 2 \wedge$
 $(\sigma 1 \models f) \wedge (\sigma 2 \models g) \wedge$
 $(ilast\ \sigma 1 = ifirst\ \sigma 2)) =$
 $(\sigma \models f;g)$
by (*metis chop-defs fuse-ilen-a fuse-prefix-suffix ilast-ifirst*
 $prefix-fuse\ suffix-fuse\ le-add1$)

lemma *sub-fuse*:

assumes $k \leq n$

$n \leq m$

$m \leq \text{ilen } xs$

shows $\text{fuse } (\text{sub } k \ n \ xs) \ (\text{sub } n \ m \ xs) = (\text{sub } k \ m \ xs)$

proof –

have 1: $\text{ilast}(\text{sub } k \ n \ xs) = (\text{inth } xs \ n)$

using *assms ilast-sub le-trans less-imp-le-nat* **by** *blast*

have 2: $\text{ifirst}(\text{sub } n \ m \ xs) = (\text{inth } xs \ n)$

using *assms ifirst-sub less-imp-le-nat* **by** *blast*

have 3: $\text{ilen}(\text{fuse } (\text{sub } k \ n \ xs) \ (\text{sub } n \ m \ xs)) = \text{ilen}(\text{sub } k \ m \ xs)$

by (*metis Nat.add-diff-assoc2 assms(1) assms(2) assms(3) fuse-ilen-a ilen-sub le-trans ordered-cancel-comm-monoid-diff-class.add-diff-inverse*)

have 4: $(\forall \ i. \ i \leq \text{ilen}(\text{sub } k \ m \ xs) \longrightarrow (\text{inth } (\text{fuse } (\text{sub } k \ n \ xs) \ (\text{sub } n \ m \ xs)) \ i) = (\text{inth } xs \ (k+i)))$

proof

fix i

show $i \leq \text{ilen}(\text{sub } k \ m \ xs) \longrightarrow \text{inth } (\text{fuse } (\text{sub } k \ n \ xs) \ (\text{sub } n \ m \ xs)) \ i = \text{inth } xs \ (k + i)$

proof –

have 41: $\text{ilen}(\text{sub } k \ m \ xs) = (m-k)$

using *assms ilen-sub le-trans less-imp-le-nat* **by** *metis*

have 42: $i \leq \text{ilen}(\text{sub } k \ m \ xs) \longrightarrow \text{inth } (\text{fuse } (\text{sub } k \ n \ xs) \ (\text{sub } n \ m \ xs)) \ i =$
 $(\text{if } i \leq \text{ilen}(\text{sub } k \ n \ xs) \text{ then } (\text{inth } (\text{sub } k \ n \ xs) \ i)$
 $\text{else } (\text{inth } (\text{sub } n \ m \ xs) \ (i - \text{ilen}(\text{sub } k \ n \ xs))))$

by (*metis 1 2 3 fuse-inth le-cases*)

have 43: $i \leq (m-k) \longrightarrow \text{inth } (\text{fuse } (\text{sub } k \ n \ xs) \ (\text{sub } n \ m \ xs)) \ i =$
 $(\text{if } i \leq (n-k) \text{ then } (\text{inth } xs \ (k+i)) \text{ else } (\text{inth } xs \ (n+(i-(n-k)))))$

using 42 *Nat.le-diff-conv2 assms(1) assms(2) assms(3)* **by** *auto*

have 44: $i \leq (m-k) \longrightarrow \text{inth } (\text{fuse } (\text{sub } k \ n \ xs) \ (\text{sub } n \ m \ xs)) \ i =$
 $(\text{inth } xs \ (k+i))$

by (*simp add: 43 add.commute assms less-imp-le-nat*)

show *?thesis*

by (*simp add: 41 44*)

qed

qed

have 5: $(\forall \ i. \ i \leq \text{ilen}(\text{sub } k \ m \ xs) \longrightarrow (\text{inth } (\text{sub } k \ m \ xs) \ i) = (\text{inth } xs \ (k+i)))$

using *assms(1) assms(2) assms(3)* **by** *auto*

show *?thesis*

by (*simp add: 3 4 5 interval-eq-inth-eq*)

qed

lemma *sub-fuse-idx*:

assumes *index-sequence 0 l*

$\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma$

$(\text{Suc } i) < \text{ilen } l$

shows $\text{fuse } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \ (\text{sub } (\text{inth } l \ (\text{Suc } i)) \ (\text{inth } l \ (\text{ilen } l)) \ \sigma) =$
 $(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{ilen } l)) \ \sigma)$

proof –

have 1: $\text{ilast}(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) = (\text{inth } \sigma \ (\text{inth } l \ (\text{Suc } i)))$

by (*metis assms idx-less-equal idx-less-last-1 ilast-sub le-add2 less-imp-le-nat plus-1-eq-Suc*)

```

have 2:  $\text{ifirst}(\text{sub } (\text{inth } l \text{ (Suc } i)) (\text{inth } l \text{ (ilen } l)) \sigma) = (\text{inth } \sigma (\text{inth } l \text{ (Suc } i)))$ 
  by (metis assms dual-order.strict-iff-order eq-imp-le idx-less-last-1 ifirst-sub)
have 3:  $(\text{inth } l \text{ } i) < (\text{inth } l \text{ (Suc } i))$ 
  using Suc-lessD assms idx-less-than less-imp-le-nat by blast
have 4:  $(\text{inth } l \text{ (Suc } i)) < (\text{inth } l \text{ (ilen } l))$ 
  using assms idx-less-last-1 by blast
show ?thesis using 3 4 assms by (simp add: sub-fuse)
qed

```

lemma *idx-fuse-ifirst-ilast*:

```

assumes index-sequence 0 l1
  index-sequence 0 l2
   $\text{inth } l1 \text{ (ilen } l1) = cp$ 
   $\text{inth } l2 \text{ (ilen } l2) = \text{ilen } \sigma - cp$ 
   $cp \leq \text{ilen } \sigma$ 
   $l = \text{fuse } l1 \text{ (imap (shift } cp) \text{ } l2)$ 
shows  $\text{ilast } l1 = \text{ifirst } (\text{imap (shift } cp) \text{ } l2)$ 
using assms
by (metis shift-def add.left-neutral index-sequence-def inth-imap )

```

lemma *idx-fuse-inth-cp*:

```

assumes index-sequence 0 l1
  index-sequence 0 l2
   $\text{inth } l1 \text{ (ilen } l1) = cp$ 
   $\text{inth } l2 \text{ (ilen } l2) = \text{ilen } \sigma - cp$ 
   $cp \leq \text{ilen } \sigma$ 
   $l = \text{fuse } l1 \text{ (imap (shift } cp) \text{ } l2)$ 
   $i \leq \text{ilen } l2$ 
shows  $\text{inth } l \text{ (ilen } l1 + i) = cp + \text{inth } l2 \text{ } i$ 
proof -
have 1:  $\text{ilast } l1 = \text{ifirst } (\text{imap (shift } cp) \text{ } l2)$ 
  using assms idx-fuse-ifirst-ilast by blast
have 2:  $\text{inth } l \text{ (ilen } l1 + i) = \text{inth } (\text{imap (shift } cp) \text{ } l2) \text{ } i$ 
  using assms by (metis 1 fuse-inth-a ilen-imap)
have 3:  $\text{inth } (\text{imap (shift } cp) \text{ } l2) \text{ } i = \text{inth } l2 \text{ } i + cp$ 
  by (simp add: shift-def inth-imap)
show ?thesis using 2 3 by auto
qed

```

lemma *idx-fuse-idx*:

```

assumes index-sequence 0 l1
  index-sequence 0 l2
   $\text{inth } l1 \text{ (ilen } l1) = cp$ 
   $\text{inth } l2 \text{ (ilen } l2) = \text{ilen } \sigma - cp$ 
   $cp \leq \text{ilen } \sigma$ 
   $l = \text{fuse } l1 \text{ (imap (shift } cp) \text{ } l2)$ 
   $i \leq \text{ilen } l2$ 
shows index-sequence 0 l

```

proof –

have 1: $ilast\ l1 = ifirst\ (imap\ (shift\ cp)\ l2)$
using *assms idx-fuse-ifirst-ilast* **by** *blast*
have 2: $inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ 0 = inth\ l1\ 0$
using 1 *fuse-inth* **by** *blast*
have 3: $ilen\ (fuse\ l1\ (imap\ (shift\ cp)\ l2)) = ilen\ l1 + ilen\ l2$
by (*simp add: fuse-ilen-a*)
have 4: $\forall i. 0 \leq i \wedge i \leq ilen\ l1 \longrightarrow$
 $inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ i =$
 $inth\ l1\ i$
by (*metis 1 inth-prefix prefix-fuse*)
have 5: $\forall i. ilen\ l1 \leq i \wedge i \leq ilen\ l1 + ilen\ l2 \longrightarrow$
 $inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ i =$
 $cp + inth\ l2\ (i - ilen\ l1)$
by (*metis (no-types, lifting) 1 3 shift-def add commute fuse-inth inth-imap*)
have 6: $\forall i. 0 \leq i \wedge i < ilen\ l1 + ilen\ l2 \longrightarrow$
 $inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ i < inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ (Suc\ i)$
using *assms* **by** (*metis 1 3 4 add.right-neutral index-sequence-def idx-link*
idx-split ilen-gr-zero prefix-fuse suffix-fuse le-add1)
have 7: $index-sequence\ 0\ l =$
 $((inth\ l1\ 0) = 0 \wedge$
 $(\forall i. 0 \leq i \wedge i < ilen\ l1 + ilen\ l2 \longrightarrow$
 $inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ i < inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ (Suc\ i)))$
by (*simp add: 2 3 assms index-sequence-def*)
from 7 6 2 **show** *?thesis*
using *assms index-sequence-def* **by** *auto*
qed

lemma *idx-fuse-ilen*:

assumes *index-sequence 0 l1*
index-sequence 0 l2
 $inth\ l1\ (ilen\ l1) = cp$
 $inth\ l2\ (ilen\ l2) = ilen\ \sigma - cp$
 $cp \leq ilen\ \sigma$
 $l = fuse\ l1\ (imap\ (shift\ cp)\ l2)$
 $i \leq ilen\ l2$
shows $inth\ l\ (ilen\ l) = ilen\ \sigma$
using *assms idx-fuse-inth-cp[of l1 l2 cp σ]* **by** (*simp add: fuse-ilen-a*)

lemma *ifirst-lfuse-ifirst*:

assumes *lastfirst (xs \odot xxs)*
shows $ifirst(lfuse\ xxs) = ifirst(ifirst\ xxs)$
using *assms*
proof
(induct xxs)
case (*INil x*)
then show *?case* **by** *simp*
next
case (*ICons x1a xxs*)

```

then show ?case
  proof (cases x1a)
    case (INil x1)
    then show ?thesis using ICons.hyps ICons.prem by auto
  next
    case (ICons x21 x22)
    then show ?thesis by simp
  qed
qed

```

```

lemma ifirst-lfuse:
assumes lastfirst (xs  $\odot$  xxs)
shows (ifirst (lfuse (xs  $\odot$  xxs)) ) = (ifirst xs)
proof -
  have 1: lastfirst (xs  $\odot$  xxs)
  using assms by auto
  have 2: ( (ilast xs) = (ifirst (ifirst xxs)))  $\wedge$  (lastfirst xxs)
  using 1 by simp
  have 3: (ilast xs) = (ifirst (ifirst xxs))
  using 2 by auto
  have 4: ifirst (lfuse (xs  $\odot$  xxs)) = ifirst(fuse xs (lfuse xxs))
  by simp
  have 5: ifirst(lfuse xxs) = ifirst(ifirst xxs)
  using assms ifirst-lfuse-ifirst by blast
  have 6: ifirst(fuse xs (lfuse xxs)) = ifirst xs
  by (metis 3 5 ifirst-fuse)
  show ?thesis using 6 by auto
qed

```

```

lemma lastfirst-lfuse-ilast:
assumes lastfirst xxs
shows ilast(lfuse xxs) = ilast(ilast xxs)
using assms
proof
  (induct xxs)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xxs)
  then show ?case
  by (metis fuse-ilen-a fuse-inth-a ifirst-lfuse-ifirst
    inth-last lastfirst.simps(2) lfuse-ICons order.order-iff-strict)
qed

```

```

lemma lastfirst-lfuse:
assumes lastfirst xxs
shows ifirst (lfuse xxs) = ifirst(ifirst(xxs))
using assms
proof
  (cases xxs)

```

```

case (INil x1)
then show ?thesis by simp
next
case (ICons x21 x22)
then show ?thesis
using assms ifirst-lfuse by auto
qed

```

lemma *lfuse-ilen* :

assumes *lastfirst xs*

shows $ilen (lfuse\ xs) = (\sum k::nat = 0..(ilen\ xs). ilen(inth\ xs\ k))$

using *assms*

proof

(*induct xs*)

case (INil *x*)

then show ?*case* **by** *simp*

next

case (ICons *x1a xs*)

then show ?*case* **proof** –

have 1 : $ilen (lfuse (x1a \odot xs)) = ilen (fuse\ x1a\ (lfuse\ xs))$

by *simp*

have 2 : *lastfirst* (*x1a* \odot *xs*) **using** *ICons.prem*s **by** *auto*

have 3 : *ilast* *x1a* = *ifirst*(*lfuse xs*)

using *ICons.prem*s *ifirst-lfuse-ifirst* **by** *fastforce*

have 4 : $ilen (fuse\ x1a\ (lfuse\ xs)) = (ilen\ x1a) + ilen(lfuse\ xs)$

using 3 *fuse-ilen* **by** *blast*

have 5 : $(ilen\ x1a) + ilen(lfuse\ xs) =$

$(ilen\ x1a) + (\sum k::nat = 0..(ilen\ xs). ilen(inth\ xs\ k))$

using *ICons.hyps* *ICons.prem*s **by** *auto*

have 6 : $(\sum k = 0..ilen\ (x1a \odot xs). ilen (inth (x1a \odot xs) k)) =$

$(ilen(inth (x1a \odot xs) 0)) +$

$(\sum k = 1..1+ilen\ (xs). ilen (inth (x1a \odot xs) k))$

by (*simp add: sum.atLeast-Suc-atMost*)

have 7 : $(ilen(inth (x1a \odot xs) 0)) = ilen(x1a)$

by *simp*

have 8 : $(\sum k = 1..1+ilen\ (xs). ilen (inth (x1a \odot xs) k)) =$

$(\sum k = 0..ilen\ (xs). ilen (inth (x1a \odot xs) (k+1)))$

using *sum.shift-bounds-cl-nat-ivl*[of $\lambda k. ilen (inth (x1a \odot xs) k)$ 0 1 *ilen(xs)*]

by *simp*

have 9 : $(\sum k = 0..ilen\ (xs). ilen (inth (x1a \odot xs) (k+1))) =$

$(\sum k = 0..ilen\ (xs). ilen (inth (xs) (k)))$

by *auto*

have 10 : $(ilen\ x1a) + (\sum k::nat = 0..(ilen\ xs). ilen(inth\ xs\ k)) =$

$(\sum k = 0..ilen\ (x1a \odot xs). ilen (inth (x1a \odot xs) k))$

using 6 7 8 9 **by** *linarith*

show ?*thesis*

by (*simp add: 10 4 5*)

qed

qed


```

lemma idx-fuse:
  assumes ilast l1 = ifirst l2
  shows
    (index-sequence (ifirst l1) (fuse l1 l2)) =
      ( index-sequence (ifirst l1) l1  $\wedge$  index-sequence (ifirst l2) l2 )
using assms
proof
  (induct l1 arbitrary: l2)
  case (INil x)
  then show ?case by (simp add: index-sequence-def)
  next
  case (ICons x1a l1)
  then show ?case
    using idx-expand1 ifirst-fuse by force
qed

lemma idx-lfuse-help1:
  assumes ( $\forall k. k < \text{ilen } (\text{lfuse } l) \longrightarrow$ 
     $\text{inth } (\text{fuse } x1a (\text{lfuse } l)) (\text{ilen } x1a + k) <$ 
     $\text{inth } (\text{fuse } x1a (\text{lfuse } l)) (\text{ilen } x1a + \text{Suc } k)$ )

     $\text{ilen } x1a \leq n$ 
     $n < \text{ilen } x1a + \text{ilen } (\text{lfuse } l)$ 
  shows  $\text{inth } (\text{fuse } x1a (\text{lfuse } l)) n < \text{inth } (\text{fuse } x1a (\text{lfuse } l)) (\text{Suc } n)$ 
  using assms
  by (metis add-Suc-right add-less-imp-less-left le-Suc-ex)

lemma idx-lfuse:
  assumes lastfirst l
  shows (index-sequence (ifirst (lfuse l)) (lfuse l)) =
    ( $\forall i \leq \text{ilen } l. \text{index-sequence } (\text{ifirst } (\text{inth } l i)) (\text{inth } l i)$ )
using assms
proof
  (induct l)
  case (INil x)
  then show ?case by (simp add: index-sequence-def)
  next
  case (ICons x1a l)
  then show ?case
    proof –
      have 0: lastfirst l
      using ICons.prem lastfirst.simps(2) by blast
      have 1: index-sequence (ifirst (lfuse (x1a  $\odot$  l))) (lfuse (x1a  $\odot$  l)) =
        (inth (fuse x1a (lfuse l)) 0 = ifirst (fuse x1a (lfuse l))  $\wedge$ 
          ( $\forall n < \text{ilen } (\text{fuse } x1a (\text{lfuse } l)).$ 
             $\text{inth } (\text{fuse } x1a (\text{lfuse } l)) n < \text{inth } (\text{fuse } x1a (\text{lfuse } l)) (\text{Suc } n)$ ))
        by (simp add: index-sequence-def)
      have 2: (inth (fuse x1a (lfuse l)) 0) = ifirst (fuse x1a (lfuse l))
        by simp
      have 3:  $\text{ilen } (\text{fuse } x1a (\text{lfuse } l)) = \text{ilen } x1a + \text{ilen } (\text{lfuse } l)$ 

```

```

    by (simp add: fuse-ilen-a)
  have 4: (∀ n < ilen (fuse x1a (lfuse l)).
    inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n)) =
    (∀ n < ilen x1a + ilen (lfuse l).
    inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n))
    by (simp add: 3)
  have 5: (∀ n < ilen x1a + ilen (lfuse l).
    inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n)) =
    ((∀ n < ilen x1a. inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n)) ∧
    (∀ n. 0 ≤ n - ilen x1a ∧ n - ilen x1a < ilen (lfuse l) →
    inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n)))
    by auto
    (metis add.commute less-diff-conv2 not-less)
  have 6: (∀ n < ilen x1a. inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n)) =
    index-sequence (ifirst x1a) x1a
    proof (simp add: index-sequence-def)
      have ilast x1a = ifirst (lfuse l)
      using ICons.premis ifirst-lfuse-ifirst by force
      then show (∀ n < ilen x1a. inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n)) = (∀ n < ilen
        x1a. inth x1a n < inth x1a (Suc n))
        by (metis (no-types) Suc-leI inth-prefix less-or-eq-imp-le prefix-fuse)
      qed
  have 7: (∀ n. ilen x1a ≤ n ∧ n < ilen x1a + ilen (lfuse l) →
    inth (fuse x1a (lfuse l)) n < inth (fuse x1a (lfuse l)) (Suc n)) =
    (∀ k. k < ilen (lfuse l) →
    inth (fuse x1a (lfuse l)) (ilen x1a + k) <
    inth (fuse x1a (lfuse l)) (ilen x1a + Suc k))
    using idx-lfuse-help1 by auto
  have 8: (∀ k. k < ilen (lfuse l) →
    inth (fuse x1a (lfuse l)) (ilen x1a + k) <
    inth (fuse x1a (lfuse l)) (ilen x1a + Suc k)) =
    (∀ k. k < ilen (lfuse l) →
    inth ( (lfuse l)) (k) <
    inth ( (lfuse l)) (Suc k)) (is ?L = ?R)
    proof
      show ?L ⇒ ?R
      by (metis ICons.premis Suc-leI fuse-inth-a ifirst-lfuse-ifirst lastfirst.simps(2)
        le-simps(1))
      show ?R ⇒ ?L
      by (metis ICons.premis Suc-leI add-Suc-right fuse-inth-a
        ifirst-lfuse-ifirst lastfirst.simps(2) less-imp-le-nat)
    qed
  have 9: (∀ k. k < ilen (lfuse l) →
    inth ( (lfuse l)) (k) <
    inth ( (lfuse l)) (Suc k)) = index-sequence (ifirst (lfuse l)) (lfuse l)
    by (simp add: index-sequence-def)
  have 91: index-sequence (ifirst (lfuse (x1a ⊙ l))) (lfuse (x1a ⊙ l)) =
    (index-sequence (ifirst x1a) x1a ∧ index-sequence (ifirst (lfuse l)) (lfuse l))
    using 1 2 4 5 6 7 8 9
    by (metis ICons.premis idx-fuse ifirst-lfuse

```

```

      ifirst-lfuse-ifirst lastfirst.simps(2) lfuse-ICons)
have 10: index-sequence (ifirst (lfuse l)) (lfuse l) =
  (∀ i ≤ ilen l. index-sequence (ifirst (inth l i)) (inth l i))
  using 0 ICons.hyps by blast
have 11: (∀ i ≤ ilen(x1a⊙l). index-sequence (ifirst (inth (x1a⊙l) i)) (inth (x1a⊙l) i)) =
  (∀ i ≤ 1 + ilen l. index-sequence (ifirst (inth (x1a⊙l) i)) (inth (x1a⊙l) i))
  by auto
have 12: (∀ i ≤ 1 + ilen l. index-sequence (ifirst (inth (x1a⊙l) i)) (inth (x1a⊙l) i)) =
  ( index-sequence (ifirst (x1a)) (x1a) ∧
    (∀ i. 1 ≤ i ∧ i ≤ 1 + ilen l →
      index-sequence (ifirst (inth (x1a ⊙ l) i)) (inth (x1a ⊙ l) i)))
  by (metis One-nat-def Suc-leI ilen-gr-zero inth-zero
    prefix-ilen-good ilen.simps(2) order.strict-iff-order)
have 13: (∀ i. 1 ≤ i ∧ i ≤ 1 + ilen l →
  index-sequence (ifirst (inth (x1a ⊙ l) i)) (inth (x1a ⊙ l) i)) =
  (∀ j. j ≤ ilen l →
    index-sequence (ifirst (inth (x1a ⊙ l) (1+j))) (inth (x1a ⊙ l) (1+j)))
  by auto
  (simp add: Nitpick.case-nat-unfold)
have 14: (∀ j. j ≤ ilen l →
  index-sequence (ifirst (inth (x1a ⊙ l) (1+j))) (inth (x1a ⊙ l) (1+j))) =
  (∀ j. j ≤ ilen l →
    index-sequence (ifirst (inth (l) (j))) (inth (l) (j)))
  by simp
show ?thesis
using 10 12 13 91 by auto
qed
qed

```

lemma *lfuse-ilen-a*:

assumes *lastfirst xs*

shows (∀ i. i ≤ ilen (xs) →
 (∀ j ≤ ilen (inth (xs) (i)) . j ≤ ilen(lfuse xs)))

using *assms*

proof (*induct xs*)

case (*INil x*)

then show ?*case* **by** *simp*

next

case (*ICons x1a xs*)

then show ?*case*

proof –

have 0: *ilast x1a* = *ifirst(ifirst xs)*

using *ICons.prem*s *lastfirst.simps(2)* **by** *blast*

have 1: (∀ i. i ≤ ilen (x1a ⊙ xs) →

(∀ j ≤ ilen (inth (x1a ⊙ xs) i). j ≤ ilen (lfuse (x1a ⊙ xs))))

=

((∀ j ≤ ilen (inth (x1a ⊙ xs) 0). j ≤ ilen (lfuse (x1a ⊙ xs))) ∧

(∀ i. 1 ≤ i ∧ i-1 ≤ ilen (xs) →

$(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xs) i). j \leq \text{ilen} (\text{lfuse} (x1a \odot xs)))$)
by *auto*
 $(\text{metis One-nat-def add.commute le-diff-conv le-zero-eq not-less-eq-eq old.nat.simps}(4)$
 $\text{plus-1-eq-Suc})$
have 2: $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xs) 0). j \leq \text{ilen} (\text{lfuse} (x1a \odot xs))) =$
 $(\forall j \leq \text{ilen} (x1a). j \leq \text{ilen} (\text{fuse } x1a (\text{lfuse } xs)))$
by *simp*
have 3: $(\forall j \leq \text{ilen} (x1a). j \leq \text{ilen} (\text{fuse } x1a (\text{lfuse } xs))) =$
 $(\forall j \leq \text{ilen} (x1a). j \leq \text{ilen} (x1a) + \text{ilen} (\text{lfuse } xs))$
by $(\text{simp add: fuse-ilen-a})$
have 4: $(\forall j \leq \text{ilen} (x1a). j \leq \text{ilen} (x1a) + \text{ilen} (\text{lfuse } xs))$
by *linarith*
have 5: $(\forall i. 1 \leq i \wedge i-1 \leq \text{ilen} (xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xs) i). j \leq \text{ilen} (\text{lfuse} (x1a \odot xs))))$
 $=$
 $(\forall i. 0 \leq i \wedge i \leq \text{ilen} (xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xs) (\text{Suc } i)). j \leq \text{ilen} (\text{lfuse} (x1a \odot xs))))$
by $(\text{metis add-diff-cancel-left' ilen-gr-zero prefix-ilen-good le-add1}$
 $\text{ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc})$
have 6: $(\forall i. 0 \leq i \wedge i \leq \text{ilen} (xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xs) (\text{Suc } i)). j \leq \text{ilen} (\text{lfuse} (x1a \odot xs)))) =$
 $(\forall i. 0 \leq i \wedge i \leq \text{ilen} (xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)). j \leq \text{ilen } x1a + \text{ilen} (\text{lfuse} (xs))))$
by $(\text{simp add: fuse-ilen-a})$
have 7: $\forall i. 0 \leq i \wedge i \leq \text{ilen } xs \longrightarrow (\forall j \leq \text{ilen} (\text{inth } xs i). j \leq \text{ilen} (\text{lfuse } xs))$
using *ICons.hyps ICons.premis lastfirst.simps*(2) **by** *blast*
have 8: $(\forall i. 0 \leq i \wedge i \leq \text{ilen} (xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)). j \leq \text{ilen } x1a + \text{ilen} (\text{lfuse} (xs))))$
by $(\text{simp add: 7 trans-le-add2})$
show *?thesis* **using** 1 3 5 6 8 **by** *auto*
qed
qed

lemma *lfuse-split*:

assumes *lastfirst xs* $\wedge (\forall j \leq \text{ilen} (xs). \text{ilen}(\text{inth} (xs) j) > 0)$

shows $(\forall i \leq \text{ilen}(xs).$

$$\begin{aligned}
& (\forall ia < \text{ilen} ((\text{inth} (xs) i)). \\
& \quad f (\text{sub} (\text{inth} (\text{inth} (xs) i) ia) \\
& \quad \quad (\text{inth} (\text{inth} (xs) i) (\text{Suc } ia)) \\
& \quad \quad \sigma))) = \\
& (\forall j < \text{ilen} (\text{lfuse } xs)). \\
& \quad f (\text{sub} (\text{inth} (\text{lfuse } xs)) j) \\
& \quad \quad (\text{inth} (\text{lfuse } xs) (\text{Suc } j)) \\
& \quad \quad \sigma))
\end{aligned}$$

using *assms*

proof $(\text{induct } xs)$

case $(\text{INil } x)$

then show *?case* **by** *auto*

next

```

case (ICons x1a xxs)
then show ?case
proof -
  have 1: (∀ i ≤ ilen (x1a ⊙ xxs).
    ∀ ia < ilen (inth (x1a ⊙ xxs) i).
    f (sub (inth (inth (x1a ⊙ xxs) i) ia) (inth (inth (x1a ⊙ xxs) i) (Suc ia)) σ)) =
    (∀ i ≤ ilen (xxs)+1.
    ∀ ia < ilen (inth (x1a ⊙ xxs) i).
    f (sub (inth (inth (x1a ⊙ xxs) i) ia) (inth (inth (x1a ⊙ xxs) i) (Suc ia)) σ))
  by simp
  have 2: ... =
    ( (∀ ia < ilen (inth (x1a ⊙ xxs) 0).
      f (sub (inth (inth (x1a ⊙ xxs) 0) ia) (inth (inth (x1a ⊙ xxs) 0) (Suc ia)) σ))
      ∧
      (∀ i. 1 ≤ i ∧ i ≤ ilen (xxs)+1 →
        (∀ ia < ilen (inth (x1a ⊙ xxs) i).
          f (sub (inth (inth (x1a ⊙ xxs) i) ia) (inth (inth (x1a ⊙ xxs) i) (Suc ia)) σ))))
  by (metis One-nat-def Suc-leI add-nonneg-nonneg gr-zeroI ilen-gr-zero zero-le-one)
  have 3: (∀ ia < ilen (inth (x1a ⊙ xxs) 0).
    f (sub (inth (inth (x1a ⊙ xxs) 0) ia) (inth (inth (x1a ⊙ xxs) 0) (Suc ia)) σ)) =
    (∀ ia < ilen x1a.
      f (sub (inth (x1a) ia) (inth (x1a) (Suc ia)) σ))
  by simp
  have 4: (∀ i. 1 ≤ i ∧ i ≤ ilen (xxs)+1 →
    (∀ ia < ilen (inth (x1a ⊙ xxs) i).
      f (sub (inth (inth (x1a ⊙ xxs) i) ia) (inth (inth (x1a ⊙ xxs) i) (Suc ia)) σ))) =
    (∀ i. 0 ≤ i-1 ∧ i-1 ≤ ilen (xxs) →
      (∀ ia < ilen (inth (x1a ⊙ xxs) ((i-1)+1)).
        f (sub (inth (inth (x1a ⊙ xxs) ((i-1)+1)) ia)
          (inth (inth (x1a ⊙ xxs) ((i-1)+1)) (Suc ia)) σ)))
  by (auto simp add: Nitpick.case-nat-unfold)
  have 5: ... =
    (∀ i. 0 ≤ i ∧ i ≤ ilen (xxs) →
      (∀ ia < ilen (inth (x1a ⊙ xxs) ((i)+1)).
        f (sub (inth (inth (x1a ⊙ xxs) ((i)+1)) ia)
          (inth (inth (x1a ⊙ xxs) ((i)+1)) (Suc ia)) σ)))
  using 4 by auto
  have 6: ... =
    (∀ i. 0 ≤ i ∧ i ≤ ilen (xxs) →
      (∀ ia < ilen (inth (xxs) ((i))).
        f (sub (inth (inth (xxs) ((i))) ia) (inth (inth (xxs) ((i))) (Suc ia)) σ)))
  by simp
  have 7: lastfirst xxs
    using ICons.prem lastfirst.simps(2) by blast
  have 8: ilast x1a = ifirst(ifirst xxs)
    using ICons.prem lastfirst.simps(2) by blast
  have 9: (∀ j ≤ ilen (x1a ⊙ xxs). ilen (inth (x1a ⊙ xxs) j) > 0)
    using ICons.prem by blast
  have 10: ilen (inth (x1a ⊙ xxs) 0) > 0

```

using *ICons.prem*s by *blast*
 have 11: $(\forall j. 1 \leq j \wedge j \leq \text{ilen } (xs)+1 \longrightarrow \text{ilen}(\text{inth } (x1a \odot xs) j) > 0)$
 using *ICons.prem*s by *auto*
 have 12: $(\forall j. 0 \leq j-1 \wedge j-1 \leq \text{ilen } (xs) \longrightarrow \text{ilen}(\text{inth } (x1a \odot xs) ((j-1)+1)) > 0)$
 using *ICons.prem*s by *auto*
 have 13: $(\forall j. j \leq \text{ilen } (xs) \longrightarrow \text{ilen}(\text{inth } (x1a \odot xs) ((j)+1)) > 0)$
 using *ICons.prem*s by *auto*
 have 14: $(\forall j. j \leq \text{ilen } (xs) \longrightarrow \text{ilen}(\text{inth } (xs) ((j))) > 0)$
 using 13 by *simp*
 have 15: $(\forall i. i \leq \text{ilen } (xs) \longrightarrow$
 $(\forall ia < \text{ilen } (\text{inth } (xs) ((i))).$
 $f(\text{sub } (\text{inth } (\text{inth } (xs) ((i))) ia) (\text{inth } (\text{inth } (xs) ((i))) (\text{Suc } ia)) \sigma)) =$
 $(\forall j < \text{ilen } (\text{lfuse } (xs))).$
 $f(\text{sub } (\text{inth } (\text{lfuse } (xs)) j)$
 $(\text{inth } (\text{lfuse } (xs)) (\text{Suc } j))$
 $\sigma))$
 by (*simp add: 14 7 ICons.hyps*)
 have 16: $(\forall j < \text{ilen } (\text{lfuse } (x1a \odot xs))).$
 $f(\text{sub } (\text{inth } (\text{lfuse } (x1a \odot xs)) j) (\text{inth } (\text{lfuse } (x1a \odot xs)) (\text{Suc } j)) \sigma) =$
 $(\forall j < \text{ilen } (\text{fuse } x1a (\text{lfuse } (xs))).$
 $f(\text{sub } (\text{inth } (\text{lfuse } (x1a \odot xs)) j) (\text{inth } (\text{lfuse } (x1a \odot xs)) (\text{Suc } j)) \sigma))$
 by *simp*
 have 17: ... =
 $(\forall j < \text{ilen } x1a + \text{ilen } ((\text{lfuse } (xs))).$
 $f(\text{sub } (\text{inth } (\text{lfuse } (x1a \odot xs)) j) (\text{inth } (\text{lfuse } (x1a \odot xs)) (\text{Suc } j)) \sigma))$
 by (*simp add: fuse-ilen-a*)
 have 18: ... =
 $((\forall j < \text{ilen } x1a.$
 $f(\text{sub } (\text{inth } (\text{lfuse } (x1a \odot xs)) j) (\text{inth } (\text{lfuse } (x1a \odot xs)) (\text{Suc } j)) \sigma)) \wedge$
 $(\forall j. \text{ilen } x1a \leq j \wedge j < \text{ilen } x1a + \text{ilen } ((\text{lfuse } (xs))) \longrightarrow$
 $f(\text{sub } (\text{inth } (\text{lfuse } (x1a \odot xs)) j) (\text{inth } (\text{lfuse } (x1a \odot xs)) (\text{Suc } j)) \sigma)))$
 using *le-add1 less-le-trans not-less* by *blast*
 have 19: $(\forall j < \text{ilen } x1a.$
 $f(\text{sub } (\text{inth } (\text{lfuse } (x1a \odot xs)) j) (\text{inth } (\text{lfuse } (x1a \odot xs)) (\text{Suc } j)) \sigma) =$
 $(\forall j < \text{ilen } x1a.$
 $f(\text{sub } (\text{inth } (\text{fuse } x1a (\text{lfuse } (xs))) j) (\text{inth } (\text{fuse } x1a (\text{lfuse } (xs))) (\text{Suc } j)) \sigma))$
 by *simp*
 have 20: ... =
 $(\forall j < \text{ilen } x1a.$
 $f(\text{sub } (\text{inth } (x1a) j) (\text{inth } (x1a) (\text{Suc } j)) \sigma))$
 proof -
 have *prefix* $(\text{ilen } x1a) (\text{fuse } x1a (\text{lfuse } xs)) = x1a$
 by (*metis* (*no-types*) *ICons.prem*s *ifirst-lfuse-ifirst lastfirst.simps(2) prefix-fuse*)
 then show ?thesis
 by (*metis* (*no-types*) *Suc-leI inth-prefix le-simps(1)*)
 qed
 have 21: $(\forall j. \text{ilen } x1a \leq j \wedge j < \text{ilen } x1a + \text{ilen } ((\text{lfuse } (xs))) \longrightarrow$
 $f(\text{sub } (\text{inth } (\text{lfuse } (x1a \odot xs)) j) (\text{inth } (\text{lfuse } (x1a \odot xs)) (\text{Suc } j)) \sigma) =$
 $(\forall j. \text{ilen } x1a \leq j \wedge j < \text{ilen } x1a + \text{ilen } ((\text{lfuse } (xs))) \longrightarrow$
 $f(\text{sub } (\text{inth } (\text{fuse } x1a (\text{lfuse } (xs))) j) (\text{inth } (\text{fuse } x1a (\text{lfuse } (xs))) (\text{Suc } j)) \sigma))$

```

    by simp
  have 22: ... =
    (∀ j. ilen x1a ≤ j ∧ j < ilen x1a + ilen ((lfuse (xs))) →
      f (sub (inth ((lfuse (xs))) (j - ilen x1a))
        (inth (fuse x1a (lfuse (xs))) ((j)+1)) σ)) (is ?L=?R)
  proof rule
    show ?L⇒?R
      by auto
      (metis 8 ICons.prem fuse-ilen-a fuse-inth
        ifirst-lfuse-ifirst less-imp-le-nat)
    show ?R ⇒ ?L
      by auto
      (metis 8 ICons.prem fuse-ilen-a fuse-inth ifirst-lfuse-ifirst less-imp-le-nat)
  qed
  have 23: ... =
    (∀ j. ilen x1a ≤ j ∧ j < ilen x1a + ilen ((lfuse (xs))) →
      f (sub (inth ((lfuse (xs))) (j - ilen x1a))
        (inth ( (lfuse (xs))) (((Suc j) - ilen x1a))) σ)) (is ?L=?R)
  proof
    show ?L⇒?R
      by auto
      (metis 8 ICons.prem Suc-leI fuse-ilen-a fuse-inth ifirst-lfuse-ifirst le-SucI)
    show ?R ⇒ ?L
      by auto
      (metis 8 ICons.prem Suc-leI fuse-ilen-a fuse-inth ifirst-lfuse-ifirst le-SucI)
  qed
  have 24: ... =
    (∀ j. 0 ≤ j ∧ j < ilen ((lfuse (xs))) →
      f (sub (inth ((lfuse (xs))) (j) ) (inth ( (lfuse (xs))) (((Suc j) ))) σ))
    by (rule interval-shift-index-to-zero-b)
  show ?thesis
  using 15 17 18 2 20 22 23 24 4 5 by auto
  qed
  qed
end

```

4 Finite ITL: Axioms and Rules

```

theory ITL
imports
  Semantics
begin

```

The Finite ITL axiom and proof rules are introduced (taken from [5]). The soundness of the rules and axioms are checked using the lemmas of Semantics.thy.

4.1 Rules

lemma *MP* :
 assumes $\vdash f \longrightarrow g$
 $\vdash f$
 shows $\vdash g$
using *assms(1) assms(2) by fastforce*

lemma *BoxGen* :
 assumes $\vdash f$
 shows $\vdash \Box f$
using *assms by (auto simp: itl-defs)*

lemma *BiGen*:
 assumes $\vdash f$
 shows $\vdash bi\ f$
using *assms by (auto simp: itl-defs)*

4.2 Axioms

lemma *ChopAssoc* :
 $\vdash f ; (g ; h) = (f;g);h$
using *ChopAssocSem Valid-def by blast*

lemma *OrChopImp* :
 $\vdash (f \vee g);h \longrightarrow f;h \vee g;h$
using *OrChopImpSem Valid-def by blast*

lemma *ChopOrImp* :
 $\vdash f;(g \vee h) \longrightarrow f;g \vee f;h$
using *ChopOrImpSem Valid-def by blast*

lemma *EmptyChop* :
 $\vdash empty ; f = f$
using *EmptyChopSem Valid-def by blast*

lemma *ChopEmpty* :
 $\vdash f;empty = f$
using *ChopEmptySem Valid-def by blast*

lemma *StateImpBi* :
 $\vdash init\ f \longrightarrow bi\ (init\ f)$
using *StateImpBiSem Valid-def by blast*

lemma *NextImpNotNextNot* :
 $\vdash \bigcirc f \longrightarrow \neg (\bigcirc (\neg f))$
using *NextImpNotNextNotSem Valid-def by blast*

lemma *BiBoxChopImpChop* :
 $\vdash bi\ (f \longrightarrow f1) \wedge \Box(g \longrightarrow g1) \longrightarrow f;g \longrightarrow f1;g1$
using *BiBoxChopImpChopSem Valid-def by blast*


```

lemma BoxInduct :
   $\vdash \Box (f \longrightarrow \text{wnext } f) \wedge f \longrightarrow \Box f$ 
using BoxInductSem Valid-def by blast

```

```

lemma ChopstarEqv :
   $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$ 
using ChopstarEqvSem Valid-def by blast

```

4.3 Additional Lemmas

The following is again from [3, 2] but adapted for our need.

```

lemma int-eq-true:
assumes  $\vdash P$ 
shows  $\vdash P = \# \text{True}$ 
using assms by auto

```

```

lemma int-eq:
assumes  $\vdash X = Y$ 
shows  $X = Y$ 
using assms by (auto simp: inteq-reflection)

```

```

lemma int-iffI:
assumes  $\vdash F \longrightarrow G$ 
            $\vdash G \longrightarrow F$ 
shows  $\vdash F = G$ 
using assms by force

```

```

lemma int-iffD1:
assumes h:  $\vdash F = G$ 
shows  $\vdash F \longrightarrow G$ 
using h by auto

```

```

lemma int-iffD2:
assumes h:  $\vdash F = G$ 
shows  $\vdash G \longrightarrow F$ 
using h by auto

```

```

lemma lift-imp-trans:
assumes  $\vdash A \longrightarrow B$ 
            $\vdash B \longrightarrow C$ 
shows  $\vdash A \longrightarrow C$ 
using assms by force

```

```

lemma lift-imp-neg:
assumes  $\vdash A \longrightarrow B$ 
shows  $\vdash \neg B \longrightarrow \neg A$ 
using assms by auto

```

```

lemma lift-and-com:  $\vdash (A \wedge B) = (B \wedge A)$ 

```

by *auto*

4.4 Quantification

lemma *EExI* :
 $\vdash F\ y \longrightarrow (\exists\exists\ x.\ F\ x)$
 by (*auto simp add: exist-state-d-def Valid-def*)

lemma *EExE*:
 assumes $\bigwedge x.\ \vdash F\ x \longrightarrow G$
 shows $\vdash (\exists\exists\ x.\ F\ x) \longrightarrow G$
 using *assms* by (*metis (mono-tags, lifting) Valid-def exist-state-d-def unl-lift2*)

lemma *EExVal*:
 $(w \models (\exists\exists\ x.\ F\ x)) =$
 $(\exists\ x\ (val :: 'a\ interval).\ (\ (val = (imap\ x\ w) \wedge (w \models F\ x))))$
 by (*simp add: exist-state-d-def*)

lemma *AAxDef*:
 $\vdash (\forall\forall\ x.\ F\ x) = (\neg(\exists\exists\ x.\ \neg(F\ x)))$
 by (*simp add: Valid-def forall-state-d-def exist-state-d-def*)

lemma *ExEqvRule*:
 assumes $\bigwedge x.\ \vdash (f\ x) = (g\ x)$
 shows $\vdash (\exists\ x.\ f\ x) = (\exists\ x.\ g\ x)$
 using *assms* by *fastforce*

4.5 Lemmas about *current-val*

lemma *current-const*: $\vdash \$(\#c) = \#c$
 by (*auto simp: itl-def*)

lemma *current-fun1*: $\vdash \$ (f\<x>) = f\<\$x>$
 by (*auto simp: itl-def*)

lemma *current-fun2*: $\vdash \$ (f\<x,y>) = f\<\$x,\$y>$
 by (*auto simp: itl-def*)

lemma *current-fun3*: $\vdash \$ (f\<x,y,z>) = f\<\$x,\$y,\$z>$
 by (*auto simp: itl-def*)

lemma *current-forall*: $\vdash \$(\forall\ x.\ P\ x) = (\forall\ x.\ \$ (P\ x))$
 by (*auto simp: itl-def*)

lemma *current-exists*: $\vdash \$(\exists\ x.\ P\ x) = (\exists\ x.\ \$ (P\ x))$
 by (*auto simp: itl-def*)

lemma *current-exists1*: $\vdash \$(\exists!\ x.\ P\ x) = (\exists!\ x.\ \$ (P\ x))$
 by (*auto simp: itl-def*)

lemmas *all-current* = *current-const* *current-fun1* *current-fun2* *current-fun3*
current-forall *current-exists* *current-exists1*

lemmas *all-current-unl* = *all-current*[*THEN intD*]
lemmas *all-current-eq* = *all-current*[*THEN inteq-reflection*]

4.6 Lemmas about *next-val*

lemma *next-const*: $\vdash \text{more} \longrightarrow (\#c)\$ = \#c$
by (*auto simp: itl-defs*)

lemma *next-fun1*: $\vdash \text{more} \longrightarrow f\langle x \rangle\$ = f\langle x\$ \rangle$
by (*auto simp: itl-defs*)

lemma *next-fun2*: $\vdash \text{more} \longrightarrow f\langle x, y \rangle\$ = f\langle x\$, y\$ \rangle$
by (*auto simp: itl-defs*)

lemma *next-fun3*: $\vdash \text{more} \longrightarrow f\langle x, y, z \rangle\$ = f\langle x\$, y\$, z\$ \rangle$
by (*auto simp: itl-defs*)

lemma *next-forall*: $\vdash \text{more} \longrightarrow (\forall x. P\ x)\$ = (\forall x. (P\ x)\$)$
by (*auto simp: itl-defs*)

lemma *next-exists*: $\vdash \text{more} \longrightarrow (\exists x. P\ x)\$ = (\exists x. (P\ x)\$)$
by (*auto simp: itl-defs*)

lemma *next-exists1*: $\vdash \text{more} \longrightarrow (\exists! x. P\ x)\$ = (\exists! x. (P\ x)\$)$
by (*auto simp: itl-defs*)

lemmas *all-next* = *next-const* *next-fun1* *next-fun2* *next-fun3*
next-forall *next-exists* *next-exists1*

lemmas *all-next-unl* = *all-next*[*THEN intD*]

4.7 Lemmas about *fin-val*

lemma *fin-const*: $\vdash !(\#c) = \#c$
by (*auto simp: itl-def*)

lemma *fin-fun1*: $\vdash !(f\langle x \rangle) = f\langle !x \rangle$
by (*auto simp: itl-def*)

lemma *fin-fun2*: $\vdash !(f\langle x, y \rangle) = f\langle !x, !y \rangle$
by (*auto simp: itl-def*)

lemma *fin-fun3*: $\vdash !(f\langle x, y, z \rangle) = f\langle !x, !y, !z \rangle$
by (*auto simp: itl-def*)

lemma *fin-forall*: $\vdash !(\forall x. P\ x) = (\forall x. !(P\ x))$

by (auto simp: itl-def)

lemma *fin-exists*: $\vdash !(\exists x. P x) = (\exists x. !(P x))$
 by (auto simp: itl-def)

lemma *fin-exists1*: $\vdash !(\exists! x. P x) = (\exists! x. !(P x))$
 by (auto simp: itl-def)

lemmas *all-fin* = *fin-const* *fin-fun1* *fin-fun2* *fin-fun3*
fin-forall *fin-exists* *fin-exists1*

lemmas *all-fin-unl* = *all-fin*[*THEN* *intD*]
lemmas *all-fin-eq* = *all-fin*[*THEN* *inteq-reflection*]

4.8 Lemmas about *penult-val*

lemma *penult-const*: $\vdash \text{more} \longrightarrow (\#c)! = \#c$
 by (auto simp: itl-defs)

lemma *penult-fun1*: $\vdash \text{more} \longrightarrow f\langle x \rangle! = f\langle x! \rangle$
 by (auto simp: itl-defs)

lemma *penult-fun2*: $\vdash \text{more} \longrightarrow f\langle x, y \rangle! = f\langle x!, y! \rangle$
 by (auto simp: itl-defs)

lemma *penult-fun3*: $\vdash \text{more} \longrightarrow f\langle x, y, z \rangle! = f\langle x!, y!, z! \rangle$
 by (auto simp: itl-defs)

lemma *penult-forall*: $\vdash \text{more} \longrightarrow (\forall x. P x)! = (\forall x. (P x)!)$
 by (auto simp: itl-def)

lemma *penult-exists*: $\vdash \text{more} \longrightarrow (\exists x. P x)! = (\exists x. (P x)!)$
 by (auto simp: itl-def)

lemma *penult-exists1*: $\vdash \text{more} \longrightarrow (\exists! x. P x)! = (\exists! x. (P x)!)$
 by (auto simp: itl-defs)

lemmas *all-penult* = *penult-const* *penult-fun1* *penult-fun2* *penult-fun3*
penult-forall *penult-exists* *penult-exists1*

lemmas *all-penult-unl* = *all-penult*[*THEN* *intD*]

4.9 Basic temporal variables properties

lemma *empty-imp-fin-equiv-curr*:
 $\vdash \text{empty} \longrightarrow !v = \v
 by (simp add: Valid-def itl-defs)

lemma *skip-imp-fin-equiv-next*:
 $\vdash \text{skip} \longrightarrow !v = v\$$

by (*simp add: Valid-def itl-defs*)

lemma *skip-imp-penult-equiv-curr*:

$\vdash \text{skip} \longrightarrow v! = \v

by (*simp add: Valid-def itl-defs*)

end

5 Finite ITL theorems

theory *Theorems*

imports

ITL

begin

We give the proofs of a list of Finite ITL theorems. These proofs and theorems were from [8].

5.1 Propositional reasoning

This is a list of propositional logic theorems used in the proofs of the ITL theorems.

lemma *IfThenElseImp*:

$\vdash (\text{if}_i \ g \ \text{then} \ f \ \text{else} \ f1) \longrightarrow ((g \longrightarrow f) \wedge (\neg g \longrightarrow f1))$

by (*simp add: itl-defs Valid-def*)

lemma *Prop01*:

assumes $\vdash f \longrightarrow \neg g \vee h$

shows $\vdash g \wedge f \longrightarrow h$

using *assms* **by** *auto*

lemma *Prop02*:

assumes $\vdash f \longrightarrow g$

$\vdash f1 \longrightarrow g$

shows $\vdash f \vee f1 \longrightarrow g$

using *assms(1) assms(2)* **by** *fastforce*

lemma *Prop03*:

assumes $\vdash f = (g \vee h)$

shows $\vdash h \longrightarrow f$

using *assms* **by** *auto*

lemma *Prop04*:

assumes $\vdash f = h$

$\vdash f = h1$

shows $\vdash h1 = h$

using *assms(1) assms(2)* **using** *int-eq* **by** *auto*

lemma *Prop05*:

assumes $\vdash f \longrightarrow g$
shows $\vdash f \longrightarrow h \vee g$
using *assms* **by** *auto*

lemma *Prop06*:

assumes $\vdash f = (g \vee h)$
 $\vdash h = h1$
shows $\vdash f = (g \vee h1)$
using *assms*(1) *assms*(2) **by** *fastforce*

lemma *Prop07*:

assumes $\vdash f \longrightarrow g \vee h$
shows $\vdash f \wedge \neg g \longrightarrow h$
using *assms* **by** *auto*

lemma *Prop08*:

assumes $\vdash f \longrightarrow g \vee h$
 $\vdash h \longrightarrow h1$
shows $\vdash f \longrightarrow g \vee h1$
using *assms*(1) *assms*(2) **by** *fastforce*

lemma *Prop09*:

assumes $\vdash f \wedge g \longrightarrow h$
shows $\vdash f \longrightarrow (g \longrightarrow h)$
using *assms* **by** *auto*

lemma *Prop10*:

assumes $\vdash f \longrightarrow g$
shows $\vdash f = (f \wedge g)$
using *assms* **by** *auto*

lemma *Prop11*:

$(\vdash f = f1) = ((\vdash f \longrightarrow f1) \wedge (\vdash f1 \longrightarrow f))$
by (*auto simp: Valid-def*)

lemma *Prop12*:

$(\vdash f \longrightarrow (f1 \wedge f2)) = ((\vdash f \longrightarrow f1) \wedge (\vdash f \longrightarrow f2))$
by (*auto simp: Valid-def*)

5.2 State formulas

The *init* operator denotes state formulas, i.e., ITL formula that only constrain the first state of an interval.

lemma *Initprop* :

$\vdash ((init\ f) \wedge (init\ g)) = init(f \wedge g)$
 $\vdash (\neg (init\ f)) = init(\neg f)$
 $\vdash ((init\ f) \vee (init\ g)) = init(f \vee g)$
 $\vdash init \# True$

by (*auto simp: itl-defs*)

lemma *Finprop* :

$\vdash ((\#True;(f \wedge empty)) \wedge (\#True;(g \wedge empty))) = (\#True;((f \wedge g) \wedge empty))$

$\vdash ((\#True;(f \wedge empty)) \vee (\#True;(g \wedge empty))) = (\#True;((f \vee g) \wedge empty))$

$\vdash (\#True;((\#True) \wedge empty))$

$\vdash (\neg (\#True;(f \wedge empty))) = (\#True;(\neg f \wedge empty))$

by (*auto simp: finalt-defs*) (*simp add: itl-defs, fastforce*)

5.3 Basic Theorems

lemma *BiChopImpChop* :

$\vdash bi(f \longrightarrow f1) \longrightarrow f;g \longrightarrow f1;g$

proof –

have 1: $\vdash g \longrightarrow g$ **by** *auto*

hence 2: $\vdash \Box (g \longrightarrow g)$ **by** (*rule BoxGen*)

have 3: $\vdash bi(f \longrightarrow f1) \wedge \Box(g \longrightarrow g) \longrightarrow f;g \longrightarrow f1;g$ **by** (*rule BiBoxChopImpChop*)

from 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *AndChopA*:

$\vdash (f \wedge f1);g \longrightarrow f;g$

proof –

have 1: $\vdash f \wedge f1 \longrightarrow f$ **by** *auto*

hence 2: $\vdash bi(f \wedge f1 \longrightarrow f)$ **by** (*rule BiGen*)

have 3: $\vdash bi(f \wedge f1 \longrightarrow f) \longrightarrow (f \wedge f1);g \longrightarrow f;g$ **by** (*rule BiChopImpChop*)

from 2 3 **show** *?thesis* **using** *MP* **by** *blast*

qed

lemma *AndChopB*:

$\vdash (f \wedge f1);g \longrightarrow f1;g$

proof –

have 1: $\vdash f \wedge f1 \longrightarrow f1$ **by** *auto*

hence 2: $\vdash bi(f \wedge f1 \longrightarrow f1)$ **by** (*rule BiGen*)

have 3: $\vdash bi(f \wedge f1 \longrightarrow f1) \longrightarrow (f \wedge f1);g \longrightarrow f1;g$ **by** (*rule BiChopImpChop*)

from 2 3 **show** *?thesis* **using** *MP* **by** *blast*

qed

lemma *NextChop*:

$\vdash (\bigcirc f);g = \bigcirc(f;g)$

proof –

have 1: $\vdash skip;(f;g) = (skip;f);g$ **by** (*rule ChopAssoc*)

show *?thesis* **by** (*metis 1 int-eq next-d-def*)

qed

lemma *BoxChopImpChop* :

$\vdash \Box (g \longrightarrow g1) \longrightarrow f;g \longrightarrow f;g1$

proof –

have 1: $\vdash g \longrightarrow g$ **by** *auto*

hence 2: $\vdash bi(g \longrightarrow g)$ **by** (*rule BiGen*)

have 3: $\vdash bi (f \longrightarrow f) \wedge \Box(g \longrightarrow g1) \longrightarrow f;g \longrightarrow f;g1$ **by** (rule *BiBoxChopImpChop*)
from 2 3 **show** ?thesis **by** fastforce
qed

lemma *LeftChopImpChop*:
assumes $\vdash f \longrightarrow f1$
shows $\vdash f;g \longrightarrow f1;g$
proof –
have 1: $\vdash f \longrightarrow f1$ **using** *assms* **by** *auto*
hence 2: $\vdash bi (f \longrightarrow f1)$ **by** (rule *BiGen*)
have 3: $\vdash bi (f \longrightarrow f1) \longrightarrow f;g \longrightarrow f1;g$ **by** (rule *BiChopImpChop*)
from 2 3 **show** ?thesis **using** *MP* **by** *blast*
qed

lemma *RightChopImpChop*:
assumes $\vdash g \longrightarrow g1$
shows $\vdash f;g \longrightarrow f;g1$
proof –
have 1: $\vdash g \longrightarrow g1$ **using** *assms* **by** *auto*
hence 2: $\vdash \Box (g \longrightarrow g1)$ **by** (rule *BoxGen*)
have 3: $\vdash \Box (g \longrightarrow g1) \longrightarrow f;g \longrightarrow f;g1$ **by** (rule *BoxChopImpChop*)
from 2 3 **show** ?thesis **using** *MP* **by** *blast*
qed

lemma *RightChopEqvChop*:
assumes $\vdash g = g1$
shows $\vdash (f;g) = (f;g1)$
using *assms* *RightChopImpChop*[of *g g1 f*] *RightChopImpChop*[of *g1 g f*]
by *fastforce*

lemma *ChopOrEqv*:
 $\vdash f;(g \vee g1) = (f;g \vee f;g1)$
proof –
have 1: $\vdash g \longrightarrow g \vee g1$ **by** *auto*
hence 2: $\vdash f;g \longrightarrow f;(g \vee g1)$ **by** (rule *RightChopImpChop*)
have 3: $\vdash g1 \longrightarrow g \vee g1$ **by** *auto*
hence 4: $\vdash f;g1 \longrightarrow f;(g \vee g1)$ **by** (rule *RightChopImpChop*)
from 2 4 **show** ?thesis **by** (meson *ChopOrImp Prop02 Prop11*)
qed

lemma *OrChopEqv*:
 $\vdash (f \vee f1);g = (f;g \vee f1;g)$
proof –
have 1: $\vdash f \longrightarrow f \vee f1$ **by** *auto*
hence 2: $\vdash f;g \longrightarrow (f \vee f1);g$ **by** (rule *LeftChopImpChop*)
have 3: $\vdash f1 \longrightarrow f \vee f1$ **by** *auto*
hence 4: $\vdash f1;g \longrightarrow (f \vee f1);g$ **by** (rule *LeftChopImpChop*)
from 2 4 **show** ?thesis
by (meson *OrChopImp int-iffI Prop02*)

qed

lemma *OrChopImpRule*:

assumes $\vdash f \longrightarrow f1 \vee f2$

shows $\vdash f;g \longrightarrow (f1;g) \vee (f2;g)$

proof –

have 1: $\vdash f \longrightarrow f1 \vee f2$ **using** *assms* **by** *auto*

hence 2: $\vdash f;g \longrightarrow (f1 \vee f2);g$ **by** (*rule LeftChopImpChop*)

have 3: $\vdash (f1 \vee f2);g = (f1;g \vee f2;g)$ **by** (*rule OrChopEqv*)

from 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *LeftChopEqvChop*:

assumes $\vdash f = f1$

shows $\vdash f;g = (f1;g)$

proof –

have 1: $\vdash f = f1$ **using** *assms* **by** *auto*

hence 2: $\vdash f \longrightarrow f1$ **by** *auto*

hence 3: $\vdash f;g \longrightarrow f1;g$ **by** (*rule LeftChopImpChop*)

have $\vdash f1 \longrightarrow f$ **using** 1 **by** *auto*

hence 4: $\vdash f1;g \longrightarrow f;g$ **by** (*rule LeftChopImpChop*)

from 3 4 **show** *?thesis* **by** (*simp add: int-iffI*)

qed

lemma *OrChopEqvRule*:

assumes $\vdash f = (f1 \vee f2)$

shows $\vdash f;g = ((f1;g) \vee (f2;g))$

proof –

have 1: $\vdash f = (f1 \vee f2)$ **using** *assms* **by** *auto*

hence 2: $\vdash f;g = ((f1 \vee f2);g)$ **by** (*rule LeftChopEqvChop*)

have 3: $\vdash (f1 \vee f2);g = (f1;g \vee f2;g)$ **by** (*rule OrChopEqv*)

from 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *NextImpNext*:

assumes $\vdash f \longrightarrow g$

shows $\vdash \bigcirc f \longrightarrow \bigcirc g$

proof –

have 1: $\vdash f \longrightarrow g$ **using** *assms* **by** *auto*

hence 2: $\vdash \Box (f \longrightarrow g)$ **by** (*rule BoxGen*)

have 3: $\vdash \Box (f \longrightarrow g) \longrightarrow (skip;f) \longrightarrow (skip;g)$ **by** (*rule BoxChopImpChop*)

have 4: $\vdash (skip;f) \longrightarrow (skip;g)$ **by** (*metis* 2 3 *MP*)

from 4 **show** *?thesis* **by** (*metis next-d-def*)

qed

lemma *ChopOrImpRule*:

assumes $\vdash g \longrightarrow g1 \vee g2$

shows $\vdash f;g \longrightarrow (f;g1) \vee (f;g2)$

proof –

have 1: $\vdash g \longrightarrow g1 \vee g2$ **using** *assms* **by** *auto*

hence 2: $\vdash f;g \longrightarrow f;(g1 \vee g2)$ **by** (rule *RightChopImpChop*)
 have 3: $\vdash f;(g1 \vee g2) = (f;g1 \vee f;g2)$ **by** (rule *ChopOrEqv*)
 from 2 3 **show** ?thesis **by** fastforce
 qed

lemma *NextImpDist*:

$\vdash \bigcirc (f \longrightarrow g) \longrightarrow \bigcirc f \longrightarrow \bigcirc g$

proof –

have 1: $\vdash (\neg (f \longrightarrow g)) = (f \wedge \neg g)$ **by** auto
 hence 2: $\vdash \text{skip};(\neg (f \longrightarrow g)) = \text{skip};(f \wedge \neg g)$ **by** (rule *RightChopEqvChop*)
 have 3: $\vdash f \longrightarrow g \vee (f \wedge \neg g)$ **by** auto
 hence 4: $\vdash \text{skip};f \longrightarrow (\text{skip};g) \vee (\text{skip};(f \wedge \neg g))$ **by** (rule *ChopOrImpRule*)
 hence 5: $\vdash \neg (\text{skip};(f \wedge \neg g)) \longrightarrow (\text{skip};f) \longrightarrow (\text{skip};g)$ **by** auto
 have 6: $\vdash \neg (\text{skip};(\neg (f \longrightarrow g))) \longrightarrow (\text{skip};f) \longrightarrow (\text{skip};g)$ **using** 2 5 **by** fastforce
 hence 7: $\vdash \neg (\bigcirc(\neg (f \longrightarrow g))) \longrightarrow (\bigcirc f) \longrightarrow (\bigcirc g)$ **by** (simp add: next-d-def)
 have 8: $\vdash \bigcirc(f \longrightarrow g) \longrightarrow \neg (\bigcirc(\neg (f \longrightarrow g)))$ **by** (rule *NextImpNotNextNot*)
 from 7 8 **show** ?thesis **using** lift-imp-trans **by** blast
 qed

lemma *ChopImpDiamond*:

$\vdash f;g \longrightarrow \Diamond g$

proof –

have 1: $\vdash f \longrightarrow \# \text{True}$ **by** auto
 hence 2: $\vdash f;g \longrightarrow \# \text{True};g$ **by** (rule *LeftChopImpChop*)
 from 2 **show** ?thesis **by** (simp add: sometimes-d-def)
 qed

lemma *NowImpDiamond*:

$\vdash f \longrightarrow \Diamond f$

proof –

have 1: $\vdash \text{empty};f = f$ **by** (rule *EmptyChop*)
 have 2: $\vdash \text{empty} \longrightarrow \# \text{True}$ **by** auto
 hence 3: $\vdash \text{empty};f \longrightarrow \# \text{True};f$ **by** (rule *LeftChopImpChop*)
 have 4: $\vdash f \longrightarrow \# \text{True};f$ **using** 1 3 **by** fastforce
 from 4 **show** ?thesis **by** (simp add: sometimes-d-def)
 qed

lemma *BoxElim*:

$\vdash \Box f \longrightarrow f$

proof –

have 1: $\vdash \neg f \longrightarrow \Diamond (\neg f)$ **by** (rule *NowImpDiamond*)
 hence 2: $\vdash \neg (\Diamond (\neg f)) \longrightarrow f$ **by** auto
 from 2 **show** ?thesis **by** (metis always-d-def)
 qed

lemma *NextDiamondImpDiamond*:

$\vdash \bigcirc (\Diamond f) \longrightarrow \Diamond f$

proof –

have 1: $\vdash \text{skip};(\# \text{True};f) = ((\text{skip};\# \text{True});f)$ **by** (rule *ChopAssoc*)

hence 2: $\vdash (\text{skip}; \# \text{True}); f = \text{skip}; (\# \text{True}; f)$ **by** *auto*
hence 3: $\vdash (\text{skip}; \# \text{True}); f = \bigcirc(\Diamond f)$ **by** (*simp add: next-d-def sometimes-d-def*)
have 4: $\vdash (\text{skip}; \# \text{True}); f \longrightarrow \Diamond f$ **by** (*rule ChopImpDiamond*)
from 3 4 **show** ?thesis **by** *fastforce*
qed

lemma *BoxImpNowAndWeakNext*:

$\vdash \Box f \longrightarrow (f \wedge \text{wnext} (\Box f))$

proof –

have 1: $\vdash \neg f \longrightarrow \Diamond (\neg f)$ **by** (*rule NowImpDiamond*)
hence 2: $\vdash \neg (\Diamond (\neg f)) \longrightarrow f$ **by** *auto*
hence 3: $\vdash \Box f \longrightarrow f$ **by** (*metis always-d-def*)
have 4: $\vdash \bigcirc (\Diamond (\neg f)) \longrightarrow \Diamond (\neg f)$ **by** (*rule NextDiamondImpDiamond*)
have 5: $\vdash \neg \neg (\Diamond (\neg f)) \longrightarrow \Diamond (\neg f)$ **by** *auto*
hence 6: $\vdash \bigcirc (\neg \neg (\Diamond (\neg f))) \longrightarrow \bigcirc (\Diamond (\neg f))$ **by** (*rule NextImpNext*)
have 7: $\vdash \bigcirc (\neg \neg (\Diamond (\neg f))) \longrightarrow \Diamond (\neg f)$ **using** 4 6 **by** *auto*
hence 8: $\vdash \bigcirc (\neg (\Box f)) \longrightarrow \Diamond (\neg f)$ **by** (*simp add: always-d-def*)
hence 9: $\vdash \neg (\Diamond (\neg f)) \longrightarrow \neg (\bigcirc (\neg (\Box f)))$ **by** *auto*
hence 10: $\vdash \Box f \longrightarrow \text{wnext} (\Box f)$ **by** (*simp add: always-d-def wnext-d-def*)
from 3 10 **show** ?thesis **by** *fastforce*

qed

lemma *BoxImpBoxRule*:

assumes $\vdash f \longrightarrow g$

shows $\vdash \Box f \longrightarrow \Box g$

proof –

have 1: $\vdash f \longrightarrow g$ **using** *assms* **by** *auto*
hence 2: $\vdash \neg g \longrightarrow \neg f$ **by** *auto*
hence 3: $\vdash \Box (\neg g \longrightarrow \neg f)$ **by** (*rule BoxGen*)
have 4: $\vdash \Box (\neg g \longrightarrow \neg f) \longrightarrow (\# \text{True}; (\neg g)) \longrightarrow (\# \text{True}; (\neg f))$ **by** (*rule BoxChopImpChop*)
have 5: $\vdash (\# \text{True}; (\neg g)) \longrightarrow (\# \text{True}; (\neg f))$ **using** 3 4 *MP* **by** *blast*
hence 6: $\vdash \Diamond (\neg g) \longrightarrow \Diamond (\neg f)$ **by** (*simp add: sometimes-d-def*)
hence 7: $\vdash \neg (\Diamond (\neg f)) \longrightarrow \neg (\Diamond (\neg g))$ **by** *auto*
from 7 **show** ?thesis **by** (*simp add: always-d-def*)

qed

lemma *BoxImpDist*:

$\vdash \Box (f \longrightarrow g) \longrightarrow \Box f \longrightarrow \Box g$

proof –

have 1: $\vdash (f \longrightarrow g) \longrightarrow (\neg g \longrightarrow \neg f)$ **by** *auto*
hence 2: $\vdash \Box (f \longrightarrow g) \longrightarrow \Box (\neg g \longrightarrow \neg f)$ **by** (*rule BoxImpBoxRule*)
have 3: $\vdash \Box (\neg g \longrightarrow \neg f) \longrightarrow (\# \text{True}; (\neg g)) \longrightarrow (\# \text{True}; (\neg f))$ **by** (*rule BoxChopImpChop*)
have 4: $\vdash \Box (f \longrightarrow g) \longrightarrow (\# \text{True}; (\neg g)) \longrightarrow (\# \text{True}; (\neg f))$
using 2 3 *lift-imp-trans* **by** *blast*
hence 5: $\vdash \Box (f \longrightarrow g) \longrightarrow \Diamond (\neg g) \longrightarrow \Diamond (\neg f)$ **by** (*simp add: sometimes-d-def*)
hence 6: $\vdash \Box (f \longrightarrow g) \longrightarrow \neg (\Diamond (\neg f)) \longrightarrow \neg (\Diamond (\neg g))$ **by** *auto*
from 6 **show** ?thesis **by** (*simp add: always-d-def*)

qed

lemma *DiamondEmpty*:

```

  ⊢ ◇ empty
proof -
  have 1: ⊢ #True by auto
  have 2: ⊢ #True; empty = #True by (rule ChopEmpty)
  have 3: ⊢ #True; empty using 1 2 by auto
  from 3 show ?thesis by (simp add: sometimes-d-def)
qed

```

```

lemma NextEqvNext:
  assumes ⊢ f = g
  shows ⊢ ○ f = ○ g
proof -
  have 1: ⊢ f = g using assms by auto
  hence 2: ⊢ skip;f = skip;g by (rule RightChopEqvChop)
  from 1 show ?thesis by (metis 2 next-d-def)
qed

```

```

lemma NextAndNextImpNextRule:
  assumes ⊢ (f ∧ g) ⟶ h
  shows ⊢ (○ f ∧ ○ g) ⟶ ○ h
using assms by (auto simp: itl-defs)

```

```

lemma NextAndNextEqvNextRule:
  assumes ⊢ (f ∧ g) = h
  shows ⊢ (○ f ∧ ○ g) = ○ h
using assms by (metis NextAndNextImpNextRule Prop11 Prop12 int-eq int-simps(20))

```

```

lemma WeakNextEqvWeakNext:
  assumes ⊢ f = g
  shows ⊢ wnext f = wnext g
using assms using inteq-reflection by force

```

```

lemma DiamondImpDiamond:
  assumes ⊢ f ⟶ g
  shows ⊢ ◇ f ⟶ ◇ g
using assms by (simp add: RightChopImpChop sometimes-d-def)

```

```

lemma DiamondEqvDiamond:
  assumes ⊢ f = g
  shows ⊢ ◇ f = ◇ g
using assms using int-eq by force

```

```

lemma BoxEqvBox:
  assumes ⊢ f = g
  shows ⊢ □ f = □ g
using assms using inteq-reflection by force

```

```

lemma BoxAndBoxImpBoxRule:
  assumes ⊢ f ∧ g ⟶ h
  shows ⊢ □ f ∧ □ g ⟶ □ h

```

using *assms* **by** (*auto simp: itl-defs Valid-def*)

lemma *BoxAndBoxEqvBoxRule*:

assumes $\vdash (f \wedge g) = h$

shows $\vdash (\Box f \wedge \Box g) = \Box h$

using *assms BoxAndBoxImpBoxRule BoxImpBoxRule* **by** (*metis int-iffD1 int-iffD2 int-iffI Prop12*)

lemma *ImpBoxRule*:

assumes $\vdash f \longrightarrow g$

shows $\vdash \Box f \longrightarrow \Box g$

using *assms* **by** (*simp add: BoxImpBoxRule*)

lemma *BoxIntro*:

assumes $\vdash f \longrightarrow g$

$\vdash \text{more} \wedge f \longrightarrow \Box f$

shows $\vdash f \longrightarrow \Box g$

proof –

have 1: $\vdash \text{more} \wedge f \longrightarrow \Box f$ **using** *assms* **by** *auto*

hence 2: $\vdash f \longrightarrow (\text{empty} \vee \Box f)$ **by** (*auto simp: itl-defs*)

hence 3: $\vdash f \longrightarrow \text{wnext } f$ **by** (*auto simp: itl-defs*)

hence 4: $\vdash \Box(f \longrightarrow \text{wnext } f)$ **by** (*rule BoxGen*)

have 5: $\vdash (\Box(f \longrightarrow \text{wnext } f)) \wedge f \longrightarrow \Box f$ **by** (*rule BoxInduct*)

hence 6: $\vdash (\Box(f \longrightarrow \text{wnext } f)) \longrightarrow (f \longrightarrow \Box f)$ **by** *fastforce*

have 7: $\vdash f \longrightarrow \Box f$ **using** 4 6 *MP* **by** *blast*

have 8: $\vdash \Box f \longrightarrow f$ **by** (*rule BoxElim*)

have 9: $\vdash f = \Box f$ **using** 7 8 **by** *fastforce*

have 10: $\vdash f \longrightarrow g$ **using** *assms* **by** *auto*

hence 11: $\vdash \Box f \longrightarrow \Box g$ **by** (*rule ImpBoxRule*)

from 7 9 11 **show** *?thesis* **using** *lift-imp-trans* **by** *blast*

qed

lemma *NextLoop*:

assumes $\vdash f \longrightarrow \Box f$

shows $\vdash \neg f$

proof –

have 1: $\vdash f \longrightarrow \Box f$ **using** *assms* **by** *auto*

hence 2: $\vdash f \longrightarrow (\text{more} \wedge \text{wnext } f)$ **by** (*auto simp: more-defs wnext-defs next-defs*)

hence 3: $\vdash f \longrightarrow \text{wnext } f$ **by** *auto*

hence 4: $\vdash \Box(f \longrightarrow \text{wnext } f)$ **by** (*rule BoxGen*)

have 5: $\vdash \Box(f \longrightarrow \text{wnext } f) \wedge f \longrightarrow \Box f$ **by** (*rule BoxInduct*)

hence 6: $\vdash \Box(f \longrightarrow \text{wnext } f) \longrightarrow (f \longrightarrow \Box f)$ **by** *fastforce*

have 7: $\vdash f \longrightarrow \Box f$ **using** 4 6 *MP* **by** *blast*

have 8: $\vdash \Box f \longrightarrow f$ **by** (*rule BoxElim*)

have 9: $\vdash f = \Box f$ **using** 7 8 **by** *fastforce*

have 10: $\vdash f \longrightarrow \text{more}$ **using** 2 **by** *auto*

hence 11: $\vdash \Box f \longrightarrow \Box \text{more}$ **by** (*rule ImpBoxRule*)

have 12: $\vdash \neg(\Box \text{more})$ **by** (*auto simp: itl-defs*)

from 7 9 11 12 **show** *?thesis* **by** *fastforce*

qed

lemma *WnextEqvEmptyOrNext*:

$\vdash wnext\ f = (empty \vee \bigcirc f)$

by (*auto simp: itl-defs*)

lemma *NotEmptyAndNext*:

$\vdash \neg(empty \wedge \bigcirc f)$

by (*auto simp: itl-defs*)

lemma *BoxEqvAndWnextBox*:

$\vdash \Box f = (f \wedge wnext\ (\Box f))$

proof –

have 1: $\vdash \Box f \longrightarrow f \wedge wnext\ (\Box f)$

using *BoxImpNowAndWeakNext* **by** *blast*

have 2: $\vdash f \wedge wnext\ (\Box f) \longrightarrow f$

by *auto*

have 3: $\vdash more \wedge (f \wedge wnext\ (\Box f)) \longrightarrow \bigcirc (f \wedge wnext\ (\Box f))$

using 1 *NextImpNext WnextEqvEmptyOrNext empty-d-def int-iffD1*

by (*metis Prop01 Prop05 Prop08*)

have 4: $\vdash f \wedge wnext\ (\Box f) \longrightarrow \Box f$

using 2 3 *BoxIntro* **by** *blast*

from 1 4 **show** *?thesis* **by** *fastforce*

qed

lemma *BoxEqvAndEmptyOrNextBox*:

$\vdash \Box f = (f \wedge (empty \vee \bigcirc(\Box f)))$

using *BoxEqvAndWnextBox WnextEqvEmptyOrNext* **by** (*metis int-eq*)

lemma *BoxEqvBoxBox*:

$\vdash \Box f = \Box (\Box f)$

using *BoxGen BoxInduct*

by (*metis BoxImpNowAndWeakNext MP int-iffI Prop09 Prop12*)

lemma *BoxBoxImpBox*:

$\vdash \Box(\Box h) \longrightarrow \Box h$

by (*simp add: BoxElim*)

lemma *BoxImpBoxBox*:

$\vdash \Box h \longrightarrow \Box(\Box h)$

by (*auto simp: itl-defs*)

lemma *DiamondIntro*:

assumes $\vdash (f \wedge \neg g) \longrightarrow \bigcirc f$

shows $\vdash f \longrightarrow \Diamond g$

proof –

have 1: $\vdash f \wedge \neg g \longrightarrow \bigcirc f$

using *assms* **by** *auto*

hence 2: $\vdash f \wedge \neg g \wedge (\Box (\neg g)) \longrightarrow (\bigcirc f) \wedge (\Box (\neg g))$

by *auto*

have 3: $\vdash (\Box (\neg g)) \longrightarrow \neg g$

by (*rule BoxElim*)

hence 4: $\vdash \Box (\neg g) = ((\Box (\neg g)) \wedge \neg g)$
using *BoxImpBoxBox BoxBoxImpBox* **by** *fastforce*
have 5: $\vdash f \wedge (\Box (\neg g)) \longrightarrow \bigcirc f \wedge \Box (\neg g)$
using 2 4 **by** *fastforce*
have 6: $\vdash \Box (\neg g) = ((\neg g) \wedge \text{wnext}(\Box (\neg g)))$
using *BoxEqvAndWnextBox* **by** *metis*
have 7: $\vdash \bigcirc f \wedge \Box (\neg g) \longrightarrow \bigcirc f \wedge \text{wnext}(\Box (\neg g))$
using 6 **by** *auto*
have 8: $\vdash f \wedge (\Box (\neg g)) \longrightarrow \bigcirc f \wedge \text{wnext}(\Box (\neg g))$
using 5 7 **using** *lift-imp-trans* **by** *blast*
hence 9: $\vdash f \wedge (\Box (\neg g)) \longrightarrow \text{more} \wedge \text{wnext } f \wedge \text{wnext}(\Box (\neg g))$
by (*auto simp: itl-defs*)
hence 10: $\vdash f \wedge (\Box (\neg g)) \longrightarrow \text{wnext } f \wedge \text{wnext}(\Box (\neg g))$
by *auto*
hence 11: $\vdash f \wedge (\Box (\neg g)) \longrightarrow \text{wnext } (f \wedge \Box (\neg g))$
by (*auto simp: itl-defs*)
hence 12: $\vdash \Box (f \wedge (\Box (\neg g))) \longrightarrow \text{wnext } (f \wedge \Box (\neg g))$
by (*rule BoxGen*)
have 13: $\vdash \Box (f \wedge (\Box (\neg g))) \longrightarrow \text{wnext } (f \wedge \Box (\neg g)) \wedge f \wedge (\Box (\neg g)) \longrightarrow \Box (f \wedge (\Box (\neg g)))$
by (*rule BoxInduct*)
hence 14: $\vdash \Box (f \wedge (\Box (\neg g))) \longrightarrow \text{wnext } (f \wedge \Box (\neg g)) \longrightarrow ((f \wedge (\Box (\neg g))) \longrightarrow \Box (f \wedge (\Box (\neg g))))$
by *fastforce*
have 15: $\vdash ((f \wedge (\Box (\neg g))) \longrightarrow \Box (f \wedge (\Box (\neg g))))$
using 12 14 *MP* **by** *blast*
have 16: $\vdash \Box (f \wedge (\Box (\neg g))) \longrightarrow (f \wedge (\Box (\neg g)))$
by (*rule BoxElim*)
have 17: $\vdash \Box (f \wedge (\Box (\neg g))) = (f \wedge (\Box (\neg g)))$
using 16 15 **by** *fastforce*
have 18: $\vdash (f \wedge (\Box (\neg g))) \longrightarrow \text{more}$
using 9 **by** *auto*
hence 19: $\vdash \Box (f \wedge (\Box (\neg g))) \longrightarrow \Box \text{more}$
by (*rule ImpBoxRule*)
have 20: $\vdash \neg(\Box \text{more})$
by (*auto simp: itl-defs*)
have 21: $\vdash \neg(f \wedge (\Box (\neg g)))$
using 17 19 20 **by** *fastforce*
hence 22: $\vdash \neg f \vee \neg(\Box (\neg g))$
by *auto*
have 23: $\vdash (\neg(\Box (\neg g))) = \Diamond g$
by (*auto simp: always-d-def*)
from 22 23 **show** *?thesis* **by** *fastforce*
qed

lemma *DiamondIntroB*:

assumes $\vdash (f \wedge \neg g) \longrightarrow \bigcirc (f \wedge \neg g)$

shows $\vdash f \longrightarrow \Diamond g$

proof –

have 1: $\vdash (f \wedge \neg g) \longrightarrow \bigcirc (f \wedge \neg g)$ **using** *assms* **by** *auto*

hence 2: $\vdash \neg(f \wedge \neg g)$ **by** (*rule NextLoop*)

hence 3: $\vdash f \longrightarrow g$ **by** *auto*
 have 4: $\vdash g \longrightarrow \Diamond g$ **by** (rule *NowImpDiamond*)
 from 3 4 **show** *?thesis* **using** *lift-imp-trans* **by** *blast*
qed

lemma *NextContra* :

assumes $\vdash (f \wedge \neg g) \longrightarrow (\bigcirc f \wedge \neg(\bigcirc g))$
 shows $\vdash f \longrightarrow g$

proof –

have 1: $\vdash (f \wedge \neg g) \longrightarrow (\bigcirc f \wedge \neg(\bigcirc g))$ **using** *assms* **by** *auto*
 hence 2: $\vdash \neg(f \longrightarrow g) \longrightarrow \bigcirc(\neg(f \longrightarrow g))$ **by** (*auto simp: itl-defs Valid-def*)
 hence 3: $\vdash \neg\neg(f \longrightarrow g)$ **by** (rule *NextLoop*)
 from 3 **show** *?thesis* **by** *auto*

qed

lemma *DiamondDiamondEqvDiamond*:

$\vdash \Diamond(\Diamond f) = \Diamond f$

proof –

have 1: $\vdash \#True; \#True = \#True$ **by** (*auto simp: itl-defs*)
 hence 2: $\vdash (\#True; \#True); f = \#True; f$ **using** *LeftChopEqvChop* **by** *blast*
 have 3: $\vdash (\#True; \#True); f = \#True; (\#True; f)$ **using** *ChopAssoc* **by** *fastforce*
 from 2 3 **show** *?thesis* **by** (*metis inteq-reflection sometimes-d-def*)

qed

lemma *WeakNextDiamondInduct*:

assumes $\vdash wnext(\Diamond f) \longrightarrow f$
 shows $\vdash f$

proof –

have 1: $\vdash wnext(\Diamond f) \longrightarrow f$ **using** *assms* **by** *blast*
 hence 2: $\vdash \neg f \longrightarrow \neg(wnext(\Diamond f))$ **by** *fastforce*
 hence 3: $\vdash \neg f \longrightarrow \bigcirc(\neg(\Diamond f))$ **by** (*simp add: wnext-d-def*)
 have 4: $\vdash f \longrightarrow \Diamond f$ **by** (rule *NowImpDiamond*)
 hence 5: $\vdash \neg(\Diamond f) \longrightarrow \neg f$ **by** *auto*
 have 6: $\vdash \neg f \longrightarrow \bigcirc(\neg f)$ **using** 3 5 **using** *NextImpNext lift-imp-trans* **by** *blast*
 hence 7: $\vdash \neg\neg f$ **by** (rule *NextLoop*)
 from 7 **show** *?thesis* **by** *auto*

qed

lemma *EmptyNextInducta*:

assumes $\vdash empty \longrightarrow f$
 $\vdash \bigcirc f \longrightarrow f$

shows $\vdash f$

proof –

have 1: $\vdash empty \longrightarrow f$ **using** *assms* **by** *auto*
 have 2: $\vdash \bigcirc f \longrightarrow f$ **using** *assms* **by** *blast*
 have 3: $\vdash (empty \vee \bigcirc f) \longrightarrow f$ **using** 1 2 **by** *fastforce*
 have 4: $\vdash wnext f = (empty \vee \bigcirc f)$ **by** (rule *WnextEqvEmptyOrNext*)
 hence 5: $\vdash wnext f \longrightarrow f$ **using** 3 **by** *fastforce*
 hence 6: $\vdash \neg f \longrightarrow \neg(wnext f)$ **by** *auto*
 hence 7: $\vdash \neg f \longrightarrow \bigcirc(\neg f)$ **by** (*auto simp: wnext-d-def*)

hence 8: $\vdash \neg \neg f$ **by** (rule NextLoop)
 from 8 **show** ?thesis **by** auto
qed

lemma EmptyNextInductb:

assumes $\vdash \text{empty} \wedge f \longrightarrow g$
 $\vdash \bigcirc(f \longrightarrow g) \wedge f \longrightarrow g$
shows $\vdash f \longrightarrow g$

proof –

have 1: $\vdash \text{empty} \wedge f \longrightarrow g$ **using** assms **by** auto
have 2: $\vdash \bigcirc(f \longrightarrow g) \wedge f \longrightarrow g$ **using** assms **by** blast
have 3: $\vdash (\text{empty} \vee \bigcirc(f \longrightarrow g)) \wedge f \longrightarrow g$ **using** 1 2 **by** fastforce
hence 4: $\vdash \text{wnext } (f \longrightarrow g) \wedge f \longrightarrow g$ **using** WnextEqvEmptyOrNext **by** fastforce
hence 5: $\vdash \text{wnext } (f \longrightarrow g) \longrightarrow (f \longrightarrow g)$ **by** fastforce
hence 6: $\vdash \neg (f \longrightarrow g) \longrightarrow \neg (\text{wnext } (f \longrightarrow g))$ **by** fastforce
hence 7: $\vdash \neg (f \longrightarrow g) \longrightarrow \bigcirc (\neg(f \longrightarrow g))$ **by** (simp add: wnext-d-def)
hence 8: $\vdash \neg \neg (f \longrightarrow g)$ **by** (rule NextLoop)
 from 8 **show** ?thesis **by** auto
qed

lemma FinImpFin:

assumes $\vdash f \longrightarrow g$
shows $\vdash \text{fin } f \longrightarrow \text{fin } g$
using ImpBoxRule[of LIFT (empty \longrightarrow f) LIFT (empty \longrightarrow g)] assms
 fin-d-def[of f] fin-d-def[of g] **by** fastforce

lemma FinEqvFin:

assumes $\vdash f = g$
shows $\vdash \text{fin } f = \text{fin } g$
using assms **by** (simp add: FinImpFin Prop11)

lemma FinAndFinImpFinRule:

assumes $\vdash f \wedge g \longrightarrow h$
shows $\vdash \text{fin } f \wedge \text{fin } g \longrightarrow \text{fin } h$

proof –

have $\vdash f \wedge g \longrightarrow h$ **using** assms **by** auto
then show ?thesis **by** (simp add: fin-defs Valid-def)

qed

lemma FinAndFinEqvFinRule:

assumes $\vdash (f \wedge g) = h$
shows $\vdash (\text{fin } f \wedge \text{fin } g) = \text{fin } h$

using assms

by (simp add: FinAndFinImpFinRule FinImpFin Prop11 Prop12)

lemma HaltEqvHalt:

assumes $\vdash f = g$
shows $\vdash \text{halt } f = \text{halt } g$

proof –

have 1: $\vdash f = g$ **using** *assms* **by** *auto*
hence 2: $\vdash (\text{empty} = f) = (\text{empty} = g)$ **by** *auto*
hence 3: $\vdash \square(\text{empty} = f) = \square(\text{empty} = g)$ **by** (*rule BoxEqvBox*)
from 3 **show** ?thesis **by** (*simp add: halt-d-def*)
qed

lemma *BiImpDiImpDi*:

$\vdash bi (f \longrightarrow g) \longrightarrow di f \longrightarrow di g$
proof –
have 1: $\vdash bi (f \longrightarrow g) \longrightarrow (f; \#True) \longrightarrow (g; \#True)$ **by** (*rule BiChopImpChop*)
from 1 **show** ?thesis **by** (*simp add: di-d-def*)
qed

lemma *DiImpDi*:

assumes $\vdash f \longrightarrow g$
shows $\vdash di f \longrightarrow di g$
proof –
have 1: $\vdash f \longrightarrow g$ **using** *assms* **by** *auto*
hence 2: $\vdash f; \#True \longrightarrow g; \#True$ **by** (*rule LeftChopImpChop*)
from 2 **show** ?thesis **by** (*simp add: di-d-def*)
qed

lemma *BiImpBiRule*:

assumes $\vdash f \longrightarrow g$
shows $\vdash bi f \longrightarrow bi g$
proof –
have 1: $\vdash f \longrightarrow g$ **using** *assms* **by** *auto*
hence 2: $\vdash \neg g \longrightarrow \neg f$ **by** *auto*
hence 3: $\vdash di (\neg g) \longrightarrow di (\neg f)$ **by** (*rule DiImpDi*)
hence 4: $\vdash \neg (di (\neg f)) \longrightarrow \neg (di (\neg g))$ **by** *auto*
from 4 **show** ?thesis **by** (*simp add: bi-d-def*)
qed

lemma *DiEqvDi*:

assumes $\vdash f = g$
shows $\vdash di f = di g$
proof –
have 1: $\vdash f = g$ **using** *assms* **by** *auto*
hence 2: $\vdash f; \#True = g; \#True$ **by** (*rule LeftChopEqvChop*)
from 2 **show** ?thesis **by** (*simp add: di-d-def*)
qed

lemma *BiEqvBi*:

assumes $\vdash f = g$
shows $\vdash bi f = bi g$
proof –
have 1: $\vdash f = g$ **using** *assms* **by** *auto*
hence 2: $\vdash (\neg f) = (\neg g)$ **by** *auto*
hence 3: $\vdash di (\neg f) = di (\neg g)$ **by** (*rule DiEqvDi*)
hence 4: $\vdash \neg (di (\neg f)) = \neg (di (\neg g))$ **by** *auto*

from 4 show ?thesis by (simp add: bi-d-def)
qed

lemma *LeftChopChopImpChopRule*:

assumes $\vdash (f; g) \longrightarrow g$

shows $\vdash (f; g); h \longrightarrow (g; h)$

proof –

have 1: $\vdash (f; g) \longrightarrow g$ **using** *assms* **by** *blast*

hence 2: $\vdash (f; g); h \longrightarrow g; h$ **by** (rule *LeftChopImpChop*)

have 3: $\vdash f; (g; h) = (f; g); h$ **by** (rule *ChopAssoc*)

from 2 3 **show** ?thesis **by** *auto*

qed

lemma *AndChopCommute* :

$\vdash (f \wedge f1); g = (f1 \wedge f); g$

proof –

have 1: $\vdash (f \wedge f1) = (f1 \wedge f)$ **by** *auto*

from 1 **show** ?thesis **by** (rule *LeftChopEqvChop*)

qed

lemma *BiAndChopImport*:

$\vdash \text{bi } f \wedge (f1; g) \longrightarrow (f \wedge f1); g$

proof –

have 1: $\vdash f \longrightarrow (f1 \longrightarrow f \wedge f1)$ **by** *auto*

hence 2: $\vdash \text{bi } f \longrightarrow \text{bi } (f1 \longrightarrow f \wedge f1)$ **by** (rule *BiImpBiRule*)

have 3: $\vdash \text{bi } (f1 \longrightarrow (f \wedge f1)) \longrightarrow f1; g \longrightarrow (f \wedge f1); g$ **by** (rule *BiChopImpChop*)

from 2 3 **show** ?thesis **using** *MP* **by** *fastforce*

qed

lemma *StateAndChopImport*:

$\vdash (\text{init } w) \wedge (f; g) \longrightarrow ((\text{init } w) \wedge f); g$

proof –

have 1: $\vdash (\text{init } w) \longrightarrow \text{bi } (\text{init } w)$ **by** (rule *StateImpBi*)

hence 2: $\vdash (\text{init } w) \wedge (f; g) \longrightarrow \text{bi } (\text{init } w) \wedge (f; g)$ **by** *auto*

have 3: $\vdash \text{bi } (\text{init } w) \wedge (f; g) \longrightarrow ((\text{init } w) \wedge f); g$ **by** (rule *BiAndChopImport*)

from 2 3 **show** ?thesis **using** *MP* **by** *fastforce*

qed

5.4 Further Properties Di and Bi

lemma *ImpDi*:

$\vdash f \longrightarrow \text{di } f$

proof –

have 1: $\vdash f; \text{empty} = f$ **by** (rule *ChopEmpty*)

have 2: $\vdash \text{empty} \longrightarrow \#True$ **by** *auto*

hence 3: $\vdash f; \text{empty} \longrightarrow f; \#True$ **by** (rule *RightChopImpChop*)

have 4: $\vdash f \longrightarrow f; \#True$ **using** 1 3 **by** *fastforce*

from 4 **show** ?thesis **by** (simp add: di-d-def)

qed

lemma DiState:

$\vdash di (init w) = (init w)$

proof –

have 0: $\vdash (init (\neg w)) \longrightarrow bi (init (\neg w))$ **using** *StateImpBi* **by** *fastforce*

hence 1: $\vdash \neg(init w) \longrightarrow bi (\neg (init w))$ **using** *Initprop(2)* **by** (*metis inteq-reflection*)

hence 2: $\vdash (\neg (init w)) \longrightarrow \neg (di (\neg \neg (init w)))$ **by** (*simp add: bi-d-def*)

have 3: $\vdash (\neg (init w) \longrightarrow \neg (di (\neg \neg (init w)))) \longrightarrow (di (\neg \neg (init w)) \longrightarrow (init w))$ **by** *auto*

have 4: $\vdash di (\neg \neg (init w)) \longrightarrow (init w)$ **using** 2 3 *MP* **by** *blast*

have 5: $\vdash (init w) \longrightarrow \neg \neg (init w)$ **by** *auto*

hence 6: $\vdash di (init w) \longrightarrow di (\neg \neg (init w))$ **by** (*rule DiImpDi*)

have 7: $\vdash di (init w) \longrightarrow (init w)$ **using** 6 4 **using** *lift-imp-trans* **by** *metis*

have 8: $\vdash (init w) \longrightarrow di (init w)$ **by** (*rule ImpDi*)

from 7 8 **show** *?thesis* **by** *fastforce*

qed

lemma StateChop:

$\vdash (init w); f \longrightarrow (init w)$

using *DiState* **by** (*auto simp: itl-defs*)

lemma StateChopExportA:

$\vdash ((init w) \wedge f); g \longrightarrow (init w)$

using *DiState* **by** (*auto simp: itl-defs*)

lemma StateAndChop:

$\vdash ((init w) \wedge f); g = ((init w) \wedge (f; g))$

by (*simp add: AndChopB StateAndChopImport StateChopExportA Prop11 Prop12*)

lemma StateAndChopImpChopRule:

assumes $\vdash (init w) \wedge f \longrightarrow f1$

shows $\vdash (init w) \wedge (f; g) \longrightarrow (f1; g)$

proof –

have 1: $\vdash (init w) \wedge f \longrightarrow f1$ **using** *assms* **by** *auto*

hence 2: $\vdash ((init w) \wedge f); g \longrightarrow f1; g$ **by** (*rule LeftChopImpChop*)

have 3: $\vdash ((init w) \wedge f); g = ((init w) \wedge (f; g))$ **by** (*rule StateAndChop*)

from 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma StateImpChopEqvChop :

assumes $\vdash (init w) \longrightarrow (f = f1)$

shows $\vdash (init w) \longrightarrow ((f; g) = (f1; g))$

proof –

have 1: $\vdash (init w) \longrightarrow (f = f1)$ **using** *assms* **by** *auto*

hence 2: $\vdash (init w) \wedge f \longrightarrow f1$ **by** *auto*

hence 3: $\vdash (init w) \wedge (f; g) \longrightarrow (f1; g)$ **by** (*rule StateAndChopImpChopRule*)

have 4: $\vdash (init w) \wedge f1 \longrightarrow f$ **using** 1 **by** *auto*

hence 5: $\vdash (init w) \wedge (f1; g) \longrightarrow (f; g)$ **by** (*rule StateAndChopImpChopRule*)

from 3 5 **show** *?thesis* **by** *fastforce*

qed

lemma ChopEqvStateAndChop:

assumes $\vdash f = (\text{init } w) \wedge f1$
shows $\vdash (f; g) = ((\text{init } w) \wedge (f1; g))$
proof –
have 1: $\vdash f = ((\text{init } w) \wedge f1)$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g = (((\text{init } w) \wedge f1); g)$ **by** (*rule LeftChopEqvChop*)
have 3: $\vdash ((\text{init } w) \wedge f1); g = ((\text{init } w) \wedge (f1; g))$ **by** (*rule StateAndChop*)
from 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *DiIntro*:

$\vdash f \longrightarrow \text{di } f$
proof –
have 1: $\vdash f; \text{empty} = f$ **by** (*rule ChopEmpty*)
have 2: $\vdash \text{empty} \longrightarrow \# \text{True}$ **by** *auto*
hence 3: $\vdash \Box(\text{empty} \longrightarrow \# \text{True})$ **by** (*rule BoxGen*)
have 4: $\vdash \Box(\text{empty} \longrightarrow \# \text{True}) \longrightarrow (f; \text{empty} \longrightarrow f; \# \text{True})$ **by** (*rule BoxChopImpChop*)
have 5: $\vdash f; \text{empty} \longrightarrow f; \# \text{True}$ **using** 3 4 *MP* **by** *fastforce*
hence 6: $\vdash f; \text{empty} \longrightarrow \text{di } f$ **by** (*simp add: di-d-def*)
from 1 6 **show** *?thesis* **by** *fastforce*
qed

lemma *BiElim*:

$\vdash \text{bi } f \longrightarrow f$
proof –
have 1: $\vdash \neg f \longrightarrow \text{di } (\neg f)$ **by** (*rule DiIntro*)
have 2: $\vdash (\neg f \longrightarrow \text{di } (\neg f)) \longrightarrow (\neg (\text{di } (\neg f)) \longrightarrow f)$ **by** *auto*
have 3: $\vdash \neg (\text{di } (\neg f)) \longrightarrow f$ **using** 1 2 *MP* **by** *blast*
from 3 **show** *?thesis* **by** (*metis bi-d-def*)
qed

lemma *BiContraPosImpDist*:

$\vdash \text{bi } (\neg g \longrightarrow \neg f) \longrightarrow (\text{bi } f) \longrightarrow (\text{bi } g)$
proof –
have 1: $\vdash \text{bi } (\neg g \longrightarrow \neg f) \longrightarrow (\text{di } (\neg g)) \longrightarrow (\text{di } (\neg f))$ **by** (*rule BiImpDiImpDi*)
hence 2: $\vdash \text{bi } (\neg g \longrightarrow \neg f) \longrightarrow (\neg (\text{di } (\neg f))) \longrightarrow (\neg (\text{di } (\neg g)))$ **by** *auto*
from 2 **show** *?thesis* **by** (*metis bi-d-def*)
qed

lemma *BiImpDist*:

$\vdash \text{bi } (f \longrightarrow g) \longrightarrow (\text{bi } f) \longrightarrow (\text{bi } g)$
proof –
have 1: $\vdash (f \longrightarrow g) \longrightarrow (\neg g \longrightarrow \neg f)$ **by** *auto*
hence 2: $\vdash \neg (\neg g \longrightarrow \neg f) \longrightarrow \neg (f \longrightarrow g)$ **by** *auto*
hence 3: $\vdash \text{bi } (\neg (\neg g \longrightarrow \neg f) \longrightarrow \neg (f \longrightarrow g))$ **by** (*rule BiGen*)
have 4: $\vdash \text{bi } (\neg (\neg g \longrightarrow \neg f) \longrightarrow \neg (f \longrightarrow g))$
 \longrightarrow
 $\text{bi } (f \longrightarrow g) \longrightarrow \text{bi } (\neg g \longrightarrow \neg f)$ **by** (*rule BiContraPosImpDist*)
have 5: $\vdash \text{bi } (f \longrightarrow g) \longrightarrow \text{bi } (\neg g \longrightarrow \neg f)$ **using** 3 4 *MP* **by** *blast*
have 6: $\vdash \text{bi } (\neg g \longrightarrow \neg f) \longrightarrow (\text{bi } f) \longrightarrow (\text{bi } g)$ **by** (*rule BiContraPosImpDist*)
from 5 6 **show** *?thesis* **using** *lift-imp-trans* **by** *blast*

qed

lemma *IfChopEqvRule*:

assumes $\vdash f = \text{if}_i \text{ (init } w) \text{ then } f1 \text{ else } f2$

shows $\vdash f; g = \text{if}_i \text{ (init } w) \text{ then } (f1; g) \text{ else } (f2; g)$

proof –

have 1: $\vdash f = \text{if}_i \text{ (init } w) \text{ then } f1 \text{ else } f2$

using *assms* **by** *auto*

hence 2: $\vdash f = (((\text{init } w) \wedge f1) \vee ((\text{init } (\neg w)) \wedge f2))$

by (*simp add: itl-defs Valid-def*)

hence 3: $\vdash f; g = (((\text{init } w) \wedge f1); g \vee ((\text{init } (\neg w)) \wedge f2); g)$

by (*rule OrChopEqvRule*)

have 4: $\vdash ((\text{init } w) \wedge f1); g = ((\text{init } w) \wedge (f1; g))$

by (*rule StateAndChop*)

have 5: $\vdash ((\text{init } (\neg w)) \wedge f2); g = ((\text{init } (\neg w)) \wedge (f2; g))$

by (*rule StateAndChop*)

have 6: $\vdash f; g = (((\text{init } w) \wedge f1; g) \vee ((\text{init } (\neg w)) \wedge f2; g))$

using 3 4 5 **by** *fastforce*

from 6 **show** *?thesis* **by** (*simp add: itl-defs Valid-def*)

qed

lemma *ChopOrEqvRule*:

assumes $\vdash g = (g1 \vee g2)$

shows $\vdash f; g = (f; g1) \vee (f; g2)$

proof –

have 1: $\vdash g = (g1 \vee g2)$ **using** *assms* **by** *auto*

hence 2: $\vdash f; g = (f; (g1 \vee g2))$ **by** (*rule RightChopEqvChop*)

have 3: $\vdash f; (g1 \vee g2) = (f; g1 \vee f; g2)$ **by** (*rule ChopOrEqv*)

from 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *EmptyOrChopEqv*:

$\vdash (\text{empty} \vee f); g = (g \vee (f; g))$

proof –

have 1: $\vdash (\text{empty} \vee f); g = ((\text{empty}; g) \vee (f; g))$ **by** (*rule OrChopEqv*)

have 2: $\vdash \text{empty}; g = g$ **by** (*rule EmptyChop*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *EmptyOrNextChopEqv*:

$\vdash (\text{empty} \vee \circ f); g = (g \vee \circ(f; g))$

proof –

have 1: $\vdash (\text{empty} \vee \circ f); g = (g \vee ((\circ f); g))$ **by** (*rule EmptyOrChopEqv*)

have 2: $\vdash (\circ f); g = \circ(f; g)$ **by** (*rule NextChop*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *EmptyOrChopImpRule*:

assumes $\vdash f \longrightarrow \text{empty} \vee f1$

shows $\vdash f; g \longrightarrow g \vee (f1; g)$

proof –
have 1: $\vdash f \longrightarrow \text{empty} \vee f1$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g \longrightarrow (\text{empty} \vee f1); g$ **by** (rule *LeftChopImpChop*)
have 3: $\vdash (\text{empty} \vee f1); g = (g \vee (f1; g))$ **by** (rule *EmptyOrChopEqv*)
from 2 3 **show** ?thesis **by** *fastforce*
qed

lemma *EmptyOrChopEqvRule*:
assumes $\vdash f = (\text{empty} \vee f1)$
shows $\vdash f; g = (g \vee (f1; g))$
proof –
have 1: $\vdash f = (\text{empty} \vee f1)$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g = ((\text{empty} \vee f1); g)$ **by** (rule *LeftChopEqvChop*)
have 3: $\vdash (\text{empty} \vee f1); g = (g \vee (f1; g))$ **by** (rule *EmptyOrChopEqv*)
from 2 3 **show** ?thesis **by** *fastforce*
qed

lemma *EmptyOrNextChopImpRule*:
assumes $\vdash f \longrightarrow \text{empty} \vee \circ f1$
shows $\vdash f; g \longrightarrow g \vee \circ(f1; g)$
proof –
have 1: $\vdash f \longrightarrow \text{empty} \vee \circ f1$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g \longrightarrow (\text{empty} \vee \circ f1); g$ **by** (rule *LeftChopImpChop*)
have 3: $\vdash (\text{empty} \vee \circ f1); g = (g \vee \circ(f1; g))$ **by** (rule *EmptyOrNextChopEqv*)
from 2 3 **show** ?thesis **by** *fastforce*
qed

lemma *EmptyOrNextChopEqvRule*:
assumes $\vdash f = (\text{empty} \vee \circ f1)$
shows $\vdash f; g = (g \vee \circ(f1; g))$
proof –
have 1: $\vdash f = (\text{empty} \vee \circ f1)$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g = ((\text{empty} \vee \circ f1); g)$ **by** (rule *LeftChopEqvChop*)
have 3: $\vdash (\text{empty} \vee \circ f1); g = (g \vee \circ(f1; g))$ **by** (rule *EmptyOrNextChopEqv*)
from 2 3 **show** ?thesis **by** *fastforce*
qed

lemma *ChopEmptyOrImpRule*:
assumes $\vdash g \longrightarrow \text{empty} \vee g1$
shows $\vdash f; g \longrightarrow f \vee (f; g1)$
proof –
have 1: $\vdash g \longrightarrow \text{empty} \vee g1$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g \longrightarrow (f; \text{empty}) \vee (f; g1)$ **by** (rule *ChopOrImpRule*)
have 3: $\vdash f; \text{empty} = f$ **by** (rule *ChopEmpty*)
from 2 3 **show** ?thesis **by** *fastforce*
qed

lemma *StateAndEmptyImpBoxState*:
 $\vdash (\text{init } w) \wedge \text{empty} \longrightarrow \Box (\text{init } w)$
by (*simp add: itl-defs Valid-def*)

lemma *BoxEqvAndBox*:

$\vdash \Box f = (f \wedge \Box f)$

by (*simp add: itl-defs Valid-def*) *fastforce*

lemma *NotBoxImpNotOrNotNextBox*:

$\vdash \neg(\Box f) \longrightarrow \neg f \vee \neg(\bigcirc(\Box f))$

proof –

have 1: $\vdash f \wedge (\bigcirc(\Box f)) \longrightarrow \Box f$

using *BoxEqvAndEmptyOrNextBox* **by** *fastforce*

hence 2: $\vdash \neg(\Box f) \longrightarrow \neg(f \wedge (\bigcirc(\Box f)))$ **by** *fastforce*

have 3: $\vdash (\neg(f \wedge (\bigcirc(\Box f)))) = (\neg f \vee \neg(\bigcirc(\Box f)))$ **by** *auto*

from 2 3 **show** *?thesis* **by** *auto*

qed

lemma *BoxStateChopBoxEqvBox*:

$\vdash \Box (init\ w); \Box (init\ w) = \Box (init\ w)$

proof –

have 1: $\vdash (\Box (init\ w)) = ((init\ w) \wedge (\text{empty} \vee \bigcirc(\Box (init\ w))))$
by (*rule BoxEqvAndEmptyOrNextBox*)

hence 2: $\vdash (\Box (init\ w); \Box (init\ w)) =$
 $((init\ w) \wedge (\text{empty} \vee \bigcirc(\Box (init\ w))); \Box (init\ w))$
by (*metis StateAndChop integ-reflection*)

have 3: $\vdash ((\text{empty} \vee \bigcirc(\Box (init\ w))); \Box (init\ w)) =$
 $(\Box (init\ w) \vee \bigcirc(\Box (init\ w); \Box (init\ w)))$
by (*rule EmptyOrNextChopEqv*)

have 4: $\vdash (\Box (init\ w); \Box (init\ w)) =$
 $((init\ w) \wedge (\Box (init\ w) \vee \bigcirc(\Box (init\ w); \Box (init\ w))))$
using 2 3 **by** *fastforce*

have 5: $\vdash \neg(\Box (init\ w)) \longrightarrow \neg (init\ w) \vee \neg(\bigcirc(\Box (init\ w)))$
by (*rule NotBoxImpNotOrNotNextBox*)

have 6: $\vdash (\Box (init\ w); \Box (init\ w)) \wedge \neg(\Box (init\ w)) \longrightarrow$
 $\bigcirc(\Box (init\ w); \Box (init\ w)) \wedge \neg(\bigcirc(\Box (init\ w)))$
using 4 5 **by** *fastforce*

hence 7: $\vdash \Box (init\ w); \Box (init\ w) \longrightarrow \Box (init\ w)$
by (*rule NextContra*)

have 11: $\vdash \Box (init\ w) = ((init\ w) \wedge \Box (init\ w))$
by (*rule BoxEqvAndBox*)

have 12: $\vdash \text{empty}; \Box (init\ w) = \Box (init\ w)$
by (*rule EmptyChop*)

have 13: $\vdash ((init\ w) \wedge \text{empty}); \Box (init\ w) = ((init\ w) \wedge (\text{empty}; \Box (init\ w)))$
by (*rule StateAndChop*)

have 14: $\vdash \Box (init\ w) = ((init\ w) \wedge \text{empty}); \Box (init\ w)$
using 11 12 13 **by** *fastforce*

have 15: $\vdash (init\ w) \wedge \text{empty} \longrightarrow \Box (init\ w)$
by (*rule StateAndEmptyImpBoxState*)

hence 16: $\vdash ((init\ w) \wedge \text{empty}); \Box (init\ w) \longrightarrow \Box (init\ w); \Box (init\ w)$
by (*rule LeftChopImpChop*)

have 17: $\vdash \Box (init\ w) \longrightarrow \Box (init\ w); \Box (init\ w)$
using 14 16 **by** *fastforce*

from 7 17 show ?thesis by fastforce
qed

lemma NotBoxStateImpBoxYieldsNotBox:

$\vdash \neg(\Box(\text{init } w)) \longrightarrow (\Box(\text{init } w) \text{ yields } (\neg(\Box(\text{init } w))))$

proof –

have 1: $\vdash \Box(\text{init } w); \Box(\text{init } w) = \Box(\text{init } w)$ by (rule BoxStateChopBoxEqvBox)

have 2: $\vdash \Box(\text{init } w) = (\neg \neg(\Box(\text{init } w)))$ by auto

hence 3: $\vdash \Box(\text{init } w); \Box(\text{init } w) = \Box(\text{init } w); (\neg \neg(\Box(\text{init } w)))$ by (rule RightChopEqvChop)

have 4: $\vdash \neg(\Box(\text{init } w)) \longrightarrow \neg(\Box(\text{init } w); (\neg \neg(\Box(\text{init } w))))$ using 1 3 by auto

from 4 show ?thesis by (simp add: yields-d-def)

qed

lemma StateEqvBi:

$\vdash (\text{init } w) = \text{bi } (\text{init } w)$

proof –

have 1: $\vdash (\text{init } w) \longrightarrow \text{bi } (\text{init } w)$ by (rule StateImpBi)

have 2: $\vdash \text{bi } (\text{init } w) \longrightarrow (\text{init } w)$ by (rule BiElim)

from 1 2 show ?thesis by fastforce

qed

lemma TrueChopEqvDiamond:

$\vdash \#True; f = \Diamond f$

by (simp add: sometimes-d-def)

5.5 Properties of Da and Ba

lemma DaEqvDtDi:

$\vdash \text{da } f = \Diamond(\text{di } f)$

proof –

have 1: $\vdash \#True; (f; \#True) = \#True; (f; \#True)$ by auto

hence 2: $\vdash \#True; (f; \#True) = \#True; \text{di } f$ by (simp add: di-d-def)

have 3: $\vdash \#True; \text{di } f = \Diamond(\text{di } f)$ by (rule TrueChopEqvDiamond)

have 4: $\vdash \#True; (f; \#True) = \Diamond(\text{di } f)$ using 2 3 by fastforce

from 4 show ?thesis by (simp add: da-d-def)

qed

lemma DaEqvDiDt:

$\vdash \text{da } f = \text{di } (\Diamond f)$

proof –

have 1: $\vdash \#True; f = \Diamond f$ by (rule TrueChopEqvDiamond)

hence 2: $\vdash (\#True; f); \#True = (\Diamond f); \#True$ by (rule LeftChopEqvChop)

hence 3: $\vdash (\#True; f); \#True = \text{di } (\Diamond f)$ by (simp add: di-d-def)

have 4: $\vdash \#True; (f; \#True) = (\#True; f); \#True$ by (rule ChopAssoc)

have 5: $\vdash \#True; (f; \#True) = \text{di } (\Diamond f)$ using 3 4 by fastforce

from 5 show ?thesis by (simp add: da-d-def)

qed

lemma DtDiEqvDiDt:

$\vdash \Diamond (di\ f) = di\ (\Diamond\ f)$
by (*metis ChopAssoc di-d-def sometimes-d-def*)

lemma *DiamondNotEqvNotBox*:

$\vdash \Diamond (\neg\ f) = (\neg\ (\Box\ f))$

by (*simp add: always-d-def*)

lemma *BaEqvBiBt*:

$\vdash\ ba\ f = bi\ (\Box\ f)$

proof –

have 1: $\vdash\ da\ (\neg\ f) = di\ (\Diamond\ (\neg\ f))$ **by** (*rule DaEqvDiDt*)

have 2: $\vdash\ \Diamond\ (\neg\ f) = (\neg\ (\Box\ f))$ **by** (*rule DiamondNotEqvNotBox*)

hence 3: $\vdash\ di\ (\Diamond\ (\neg\ f)) = di\ (\neg\ (\Box\ f))$ **by** (*rule DiEqvDi*)

have 4: $\vdash\ da\ (\neg\ f) = di\ (\neg\ (\Box\ f))$ **using** 1 3 **by** *fastforce*

hence 5: $\vdash\ (\neg\ (da\ (\neg\ f))) = (\neg\ (di\ (\neg\ (\Box\ f))))$ **by** *auto*

hence 6: $\vdash\ (\neg\ (da\ (\neg\ f))) = bi\ (\Box\ f)$ **by** (*simp add: bi-d-def*)

from 6 **show** *?thesis* **by** (*simp add: ba-d-def*)

qed

lemma *DiNotEqvNotBi*:

$\vdash\ di\ (\neg\ f) = (\neg\ (bi\ f))$

proof –

have 1: $\vdash\ bi\ f = (\neg\ (di\ (\neg\ f)))$ **by** (*simp add: bi-d-def*)

from 1 **show** *?thesis* **by** *auto*

qed

lemma *NotDiamondNotEqvBox*:

$\vdash\ (\neg\ (\Diamond\ (\neg\ f))) = \Box\ f$

by (*simp add: always-d-def*)

lemma *BaEqvBtBi*:

$\vdash\ ba\ f = \Box\ (bi\ f)$

proof –

have 1: $\vdash\ da\ (\neg\ f) = \Diamond\ (di\ (\neg\ f))$ **by** (*rule DaEqvDtDi*)

have 2: $\vdash\ di\ (\neg\ f) = (\neg\ (bi\ f))$ **by** (*rule DiNotEqvNotBi*)

hence 3: $\vdash\ \Diamond\ (di\ (\neg\ f)) = \Diamond\ (\neg\ (bi\ f))$ **by** (*rule DiamondEqvDiamond*)

have 4: $\vdash\ (\neg\ (\Diamond\ (\neg\ (bi\ f)))) = \Box\ (bi\ f)$ **by** (*rule NotDiamondNotEqvBox*)

have 5: $\vdash\ (\neg\ (da\ (\neg\ f))) = \Box\ (bi\ f)$ **using** 1 2 3 4 **by** *fastforce*

from 5 **show** *?thesis* **by** (*simp add: ba-d-def*)

qed

lemma *BtBiEqvBiBt*:

$\vdash\ \Box\ (bi\ f) = bi\ (\Box\ f)$

proof –

have 1: $\vdash\ ba\ f = \Box\ (bi\ f)$ **by** (*rule BaEqvBtBi*)

have 2: $\vdash\ ba\ f = bi\ (\Box\ f)$ **by** (*rule BaEqvBiBt*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *BoxStateEqvBaBoxState*:

$\vdash \Box (init\ w) = ba\ (\Box (init\ w))$

proof –

have 1: $\vdash (init\ w) = bi\ (init\ w)$ **by** (rule *StateEqvBi*)

hence 2: $\vdash \Box (init\ w) = \Box (bi\ (init\ w))$ **by** (rule *BoxEqvBox*)

have 3: $\vdash \Box (bi\ (init\ w)) = bi\ (\Box (init\ w))$ **by** (rule *BtBiEqvBiBt*)

have 4: $\vdash \Box (init\ w) = \Box(\Box (init\ w))$ **by** (rule *BoxEqvBoxBox*)

hence 5: $\vdash bi\ (\Box (init\ w)) = bi\ (\Box(\Box (init\ w)))$ **by** (rule *BiEqvBi*)

have 6: $\vdash ba\ (\Box (init\ w)) = bi\ (\Box(\Box (init\ w)))$ **by** (rule *BaEqvBiBt*)

from 2 3 5 6 **show** *?thesis* **by** *fastforce*

qed

lemma *BaImpBi*:

$\vdash ba\ f \longrightarrow bi\ f$

proof –

have 1: $\vdash ba\ f = \Box(bi\ f)$ **by** (rule *BaEqvBtBi*)

have 2: $\vdash \Box(bi\ f) \longrightarrow bi\ f$ **by** (rule *BoxElim*)

from 1 2 **show** *?thesis* **using** *lift-imp-trans* **by** *fastforce*

qed

lemma *BaImpBt*:

$\vdash ba\ f \longrightarrow \Box f$

proof –

have 1: $\vdash ba\ f = bi\ (\Box f)$ **by** (rule *BaEqvBiBt*)

have 2: $\vdash bi\ (\Box f) \longrightarrow \Box f$ **by** (rule *BiElim*)

from 1 2 **show** *?thesis* **using** *lift-imp-trans* **by** *fastforce*

qed

lemma *DiamondImpDa*:

$\vdash \Diamond f \longrightarrow da\ f$

by (metis *DiIntro DiamondImpDiamond da-d-def di-d-def sometimes-d-def*)

lemma *DiImpDa*:

$\vdash di\ f \longrightarrow da\ f$

by (metis *NowImpDiamond da-d-def di-d-def sometimes-d-def*)

lemma *BoxAndChopImport*:

$\vdash \Box h \wedge f; g \longrightarrow f; (h \wedge g)$

proof –

have 1: $\vdash h \longrightarrow g \longrightarrow (h \wedge g)$ **by** *auto*

hence 2: $\vdash \Box h \longrightarrow \Box(g \longrightarrow (h \wedge g))$ **by** (rule *ImpBoxRule*)

have 3: $\vdash \Box(g \longrightarrow (h \wedge g)) \longrightarrow f; g \longrightarrow f; (h \wedge g)$ **by** (rule *BoxChopImpChop*)

from 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *BaAndChopImport*:

$\vdash ba\ f \wedge (g; g1) \longrightarrow (f \wedge g); (f \wedge g1)$

proof –

have 1: $\vdash ba\ f \longrightarrow bi\ f$ **by** (rule *BaImpBi*)

have 2: $\vdash bi\ f \wedge (g; g1) \longrightarrow (f \wedge g); g1$ **by** (rule *BiAndChopImport*)

have 3: $\vdash ba\ f \longrightarrow \Box f$ **by** (rule *BaImpBt*)

have 4: $\vdash \Box f \wedge (f \wedge g); g1 \longrightarrow (f \wedge g); (f \wedge g1)$ **by** (rule *BoxAndChopImport*)
from 1 2 3 4 **show** ?thesis **by** fastforce
qed

lemma *ChopAndCommute*:
 $\vdash f; (g \wedge g1) = f; (g1 \wedge g)$
proof –
have 1: $\vdash (g \wedge g1) = (g1 \wedge g)$ **by** auto
from 1 **show** ?thesis **by** (rule *RightChopEqvChop*)
qed

lemma *ChopAndA*:
 $\vdash f; (g \wedge g1) \longrightarrow f; g$
proof –
have 1: $\vdash (g \wedge g1) \longrightarrow g$ **by** auto
from 1 **show** ?thesis **by** (rule *RightChopImpChop*)
qed

lemma *ChopAndB*:
 $\vdash f; (g \wedge g1) \longrightarrow f; g1$
proof –
have 1: $\vdash (g \wedge g1) \longrightarrow g1$ **by** auto
from 1 **show** ?thesis **by** (rule *RightChopImpChop*)
qed

lemma *BoxStateAndChopEqvChop*:
 $\vdash (\Box (init\ w) \wedge (f; g)) = ((\Box (init\ w) \wedge f); (\Box (init\ w) \wedge g))$
proof –
have 1: $\vdash \Box (init\ w) = ba(\Box (init\ w))$
by (rule *BoxStateEqvBaBoxState*)
have 2: $\vdash ba(\Box (init\ w)) \wedge (f; g) \longrightarrow (\Box (init\ w) \wedge f); (\Box (init\ w) \wedge g)$
by (rule *BaAndChopImport*)
have 3: $\vdash \Box (init\ w) \wedge (f; g) \longrightarrow (\Box (init\ w) \wedge f); (\Box (init\ w) \wedge g)$
using 1 2 **by** fastforce
have 11: $\vdash (\Box (init\ w) \wedge f); (\Box (init\ w) \wedge g) \longrightarrow (\Box (init\ w)); (\Box (init\ w) \wedge g)$
by (rule *AndChopA*)
have 12: $\vdash (\Box (init\ w)); (\Box (init\ w) \wedge g) \longrightarrow (\Box (init\ w)); (\Box (init\ w))$
by (rule *ChopAndA*)
have 13: $\vdash (\Box (init\ w)); (\Box (init\ w)) = \Box (init\ w)$
by (rule *BoxStateChopBoxEqvBox*)
have 14: $\vdash (\Box (init\ w) \wedge f); (\Box (init\ w) \wedge g) \longrightarrow f; (\Box (init\ w) \wedge g)$
by (rule *AndChopB*)
have 15: $\vdash f; (\Box (init\ w) \wedge g) \longrightarrow f; g$
by (rule *ChopAndB*)
have 16: $\vdash (\Box (init\ w) \wedge f); (\Box (init\ w) \wedge g) \longrightarrow \Box (init\ w) \wedge (f; g)$
using 11 12 13 14 15 **by** fastforce
from 3 16 **show** ?thesis **by** fastforce
qed

lemma *DiEqvNotBiNot*:

$\vdash \text{di } f = (\neg(\text{bi } (\neg f)))$
proof –
have 1: $\vdash \text{bi } (\neg f) = (\neg(\text{di } (\neg \neg f)))$ **by** (*simp add: bi-d-def*)
hence 2: $\vdash \text{di } (\neg \neg f) = (\neg(\text{bi } (\neg f)))$ **by** *auto*
have 3: $\vdash f = (\neg \neg f)$ **by** *auto*
hence 4: $\vdash \text{di } f = \text{di } (\neg \neg f)$ **by** (*rule DiEqvDi*)
from 2 4 **show** *?thesis* **by** *auto*
qed

lemma *ChopAndBoxImport*:
 $\vdash f; g \wedge \Box h \longrightarrow f; (g \wedge h)$
proof –
have 1: $\vdash \Box h \wedge f; g \longrightarrow f; (h \wedge g)$ **by** (*rule BoxAndChopImport*)
have 2: $\vdash f; (h \wedge g) = f; (g \wedge h)$ **by** (*rule ChopAndCommute*)
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *AndChopAndCommute*:
 $\vdash (f \wedge g); (f1 \wedge g1) = (g \wedge f); (g1 \wedge f1)$
proof –
have 1: $\vdash (f \wedge g); (f1 \wedge g1) = (g \wedge f); (f1 \wedge g1)$ **by** (*rule AndChopCommute*)
have 2: $\vdash (g \wedge f); (f1 \wedge g1) = (g \wedge f); (g1 \wedge f1)$ **by** (*rule ChopAndCommute*)
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *ChopImpChop*:
assumes $\vdash f \longrightarrow f1 \vdash g \longrightarrow g1$
shows $\vdash f; g \longrightarrow f1; g1$
proof –
have 1: $\vdash f \longrightarrow f1$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g \longrightarrow f1; g$ **by** (*rule LeftChopImpChop*)
have 3: $\vdash g \longrightarrow g1$ **using** *assms* **by** *auto*
hence 4: $\vdash f1; g \longrightarrow f1; g1$ **by** (*rule RightChopImpChop*)
from 2 4 **show** *?thesis* **by** *fastforce*
qed

lemma *ChopEqvChop*:
assumes $\vdash f = f1 \vdash g = g1$
shows $\vdash f; g = f1; g1$
proof –
have 1: $\vdash f = f1$ **using** *assms* **by** *auto*
hence 2: $\vdash f; g = f1; g$ **by** (*rule LeftChopEqvChop*)
have 3: $\vdash g = g1$ **using** *assms* **by** *auto*
hence 4: $\vdash f1; g = f1; g1$ **by** (*rule RightChopEqvChop*)
from 2 4 **show** *?thesis* **by** *fastforce*
qed

lemma *BoxImpBoxImpBox*:
 $\vdash \Box h \longrightarrow \Box(g \longrightarrow \Box h \wedge g)$
proof –

have 1: $\vdash \Box h \longrightarrow (g \longrightarrow \Box h \wedge g)$ **by** *auto*
hence 2: $\vdash \Box(\Box h) \longrightarrow \Box(g \longrightarrow \Box h \wedge g)$ **by** (*rule ImpBoxRule*)
have 3: $\vdash \Box h = \Box(\Box h)$ **by** (*rule BoxEqvBoxBox*)
from 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *BoxChopImpChopBox*:
 $\vdash \Box h \longrightarrow f; g \longrightarrow f; (\Box h \wedge g)$
proof –
have 1: $\vdash \Box h \longrightarrow \Box(g \longrightarrow \Box h \wedge g)$ **by** (*rule BoxImpBoxImpBox*)
have 2: $\vdash \Box(g \longrightarrow \Box h \wedge g) \longrightarrow f; g \longrightarrow f; (\Box h \wedge g)$ **by** (*rule BoxChopImpChop*)
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *NotChopEqvYieldsNot*:
 $\vdash (\neg(f; g)) = f \text{ yields } (\neg g)$
proof –
have 1: $\vdash g = (\neg \neg g)$ **by** *auto*
hence 2: $\vdash f; g = f; (\neg \neg g)$ **by** (*rule RightChopEqvChop*)
hence 3: $\vdash (\neg(f; g)) = (\neg(f; (\neg \neg g)))$ **by** *auto*
from 3 **show** *?thesis* **by** (*simp add: yields-d-def*)
qed

lemma *NotDiFalse*:
 $\vdash \neg(di \# False)$
proof –
have 1: $\vdash (init \# True) \longrightarrow bi (init \# True)$ **by** (*rule StateImpBi*)
hence 2: $\vdash \#True \longrightarrow bi \#True$ **by** (*auto simp: itl-defs*)
have 3: $\vdash \#True$ **by** *auto*
have 4: $\vdash bi \#True$ **using** 2 3 *MP* **by** *auto*
hence 5: $\vdash \neg(di (\neg \#True))$ **by** (*simp add: bi-d-def*)
have 6: $\vdash (\neg \#True) = \#False$ **by** *auto*
hence 7: $\vdash di (\neg \#True) = di \#False$ **by** (*rule DiEqvDi*)
from 5 7 **show** *?thesis* **by** *auto*
qed

lemma *StateAndEmptyChop*:
 $\vdash ((init\ w) \wedge empty); f = ((init\ w) \wedge f)$
proof –
have 1: $\vdash ((init\ w) \wedge empty); f = ((init\ w) \wedge empty; f)$ **by** (*rule StateAndChop*)
have 2: $\vdash empty; f = f$ **by** (*rule EmptyChop*)
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *StateAndNextChop*:
 $\vdash ((init\ w) \wedge \bigcirc f); g = ((init\ w) \wedge \bigcirc(f; g))$
proof –
have 1: $\vdash ((init\ w) \wedge \bigcirc f); g = ((init\ w) \wedge (\bigcirc f); g)$ **by** (*rule StateAndChop*)
have 2: $\vdash (\bigcirc f); g = \bigcirc(f; g)$ **by** (*rule NextChop*)
from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *NextAndEqvNextAndNext*:

$\vdash \bigcirc (f \wedge g) = (\bigcirc f \wedge \bigcirc g)$

by (*auto simp: itl-defs*)

lemma *NextStateAndChop*:

$\vdash \bigcirc(((init\ w) \wedge f); g) = (\bigcirc (init\ w) \wedge \bigcirc(f; g))$

proof –

have 1: $\vdash ((init\ w) \wedge f); g = ((init\ w) \wedge f; g)$ **by** (*rule StateAndChop*)

hence 2: $\vdash \bigcirc(((init\ w) \wedge f); g) = \bigcirc((init\ w) \wedge f; g)$ **by** (*rule NextEqvNext*)

have 3: $\vdash \bigcirc((init\ w) \wedge f; g) = (\bigcirc (init\ w) \wedge \bigcirc(f; g))$ **by** (*rule NextAndEqvNextAndNext*)

from 2 3 **show** ?thesis **by** *fastforce*

qed

lemma *StateYieldsEqv*:

$\vdash ((init\ w) \longrightarrow (f\ yields\ g)) = ((init\ w) \wedge f)\ yields\ g$

proof –

have 1: $\vdash ((init\ w) \wedge f); (\neg g) = ((init\ w) \wedge f; (\neg g))$ **by** (*rule StateAndChop*)

hence 2: $\vdash ((init\ w) \longrightarrow \neg(f; (\neg g))) = (\neg(((init\ w) \wedge f); (\neg g)))$ **by** *auto*

from 2 **show** ?thesis **by** (*simp add: yields-d-def*)

qed

lemma *StateAndDi*:

$\vdash ((init\ w) \wedge di\ f) = di\ ((init\ w) \wedge f)$

proof –

have 1: $\vdash ((init\ w) \wedge f); \#True = ((init\ w) \wedge f; \#True)$ **by** (*rule StateAndChop*)

from 1 **show** ?thesis **by** (*metis di-d-def inteq-reflection*)

qed

lemma *DiNext*:

$\vdash di(\bigcirc f) = \bigcirc(di\ f)$

proof –

have 1: $\vdash (\bigcirc f); \#True = \bigcirc(f; \#True)$ **by** (*rule NextChop*)

from 1 **show** ?thesis **by** (*simp add: di-d-def*)

qed

lemma *DiNextState*:

$\vdash di(\bigcirc (init\ w)) = \bigcirc (init\ w)$

proof –

have 1: $\vdash di(\bigcirc (init\ w)) = \bigcirc(di\ (init\ w))$ **by** (*rule DiNext*)

have 2: $\vdash di\ (init\ w) = (init\ w)$ **by** (*rule DiState*)

hence 3: $\vdash \bigcirc(di\ (init\ w)) = \bigcirc (init\ w)$ **by** (*rule NextEqvNext*)

from 1 3 **show** ?thesis **by** *fastforce*

qed

lemma *StateImpBiGen*:

assumes $\vdash (init\ w) \longrightarrow f$

shows $\vdash (init\ w) \longrightarrow bi\ f$

proof –

have 1: $\vdash (\text{init } w) \longrightarrow f$ **using** *assms* **by** *auto*
hence 2: $\vdash \neg f \longrightarrow \neg (\text{init } w)$ **by** *auto*
hence 3: $\vdash \text{di } (\neg f) \longrightarrow \text{di } (\neg (\text{init } w))$ **by** (*rule DiImpDi*)
hence 4: $\vdash \text{di } (\neg f) \longrightarrow \text{di } (\text{init } (\neg w))$ **by** (*metis Initprop(2) inteq-reflection*)
have 5: $\vdash \text{di } (\text{init } (\neg w)) = (\text{init } (\neg w))$ **by** (*rule DiState*)
have 6: $\vdash \text{di } (\neg f) \longrightarrow \neg (\text{init } w)$ **using** 4 5 **using** *Initprop(2)* **by** *fastforce*
hence 7: $\vdash (\text{init } w) \longrightarrow \neg (\text{di } (\neg f))$ **by** *auto*
from 7 **show** *?thesis* **by** (*simp add: bi-d-def*)
qed

lemma *ChopAndNotChopImp:*

$\vdash f; g \wedge \neg (f; g1) \longrightarrow f; (g \wedge \neg g1)$

proof –

have 1: $\vdash g \longrightarrow (g \wedge \neg g1) \vee g1$ **by** *auto*
hence 2: $\vdash f; g \longrightarrow f; ((g \wedge \neg g1) \vee g1)$ **by** (*rule RightChopImpChop*)
have 3: $\vdash f; ((g \wedge \neg g1) \vee g1) \longrightarrow (f; (g \wedge \neg g1)) \vee (f; g1)$ **by** (*rule ChopOrImp*)
have 4: $\vdash f; g \longrightarrow f; (g \wedge \neg g1) \vee f; g1$ **using** 2 3 *MP* **by** *fastforce*
from 4 **show** *?thesis* **by** *auto*
qed

lemma *ChopAndYieldsImp:*

$\vdash f; g \wedge f \text{ yields } g1 \longrightarrow f; (g \wedge g1)$

proof –

have 1: $\vdash g \longrightarrow (g \wedge g1) \vee \neg g1$ **by** *auto*
hence 2: $\vdash f; g \longrightarrow f; ((g \wedge g1) \vee \neg g1)$ **by** (*rule RightChopImpChop*)
have 3: $\vdash f; ((g \wedge g1) \vee \neg g1) \longrightarrow (f; (g \wedge g1)) \vee (f; (\neg g1))$ **by** (*rule ChopOrImp*)
have 4: $\vdash f; g \longrightarrow f; (g \wedge g1) \vee f; (\neg g1)$ **using** 2 3 *MP* **by** *fastforce*
hence 5: $\vdash f; g \wedge \neg (f; (\neg g1)) \longrightarrow f; (g \wedge g1)$ **by** *auto*
from 5 **show** *?thesis* **by** (*simp add: yields-d-def*)
qed

lemma *ChopAndYieldsMP:*

$\vdash f; g \wedge f \text{ yields } (g \longrightarrow g1) \longrightarrow f; g1$

proof –

have 1: $\vdash f; g \wedge f \text{ yields } (g \longrightarrow g1) \longrightarrow f; (g \wedge (g \longrightarrow g1))$ **by** (*rule ChopAndYieldsImp*)
have 2: $\vdash g \wedge (g \longrightarrow g1) \longrightarrow g1$ **by** *auto*
hence 3: $\vdash f; (g \wedge (g \longrightarrow g1)) \longrightarrow f; g1$ **by** (*rule RightChopImpChop*)
from 1 3 **show** *?thesis* **by** *fastforce*
qed

lemma *OrYieldsImp:*

$\vdash (f \vee f1) \text{ yields } g = ((f \text{ yields } g) \wedge (f1 \text{ yields } g))$

proof –

have 1: $\vdash ((f \vee f1); (\neg g)) = ((f; (\neg g)) \vee (f1; (\neg g)))$ **by** (*rule OrChopEqv*)
hence 2: $\vdash (\neg ((f \vee f1); (\neg g))) = (\neg (f; (\neg g)) \wedge \neg (f1; (\neg g)))$ **by** *auto*
from 2 **show** *?thesis* **by** (*simp add: yields-d-def*)
qed

lemma *LeftYieldsImpYields:*

assumes $\vdash f \longrightarrow f1$

shows $\vdash (f1 \text{ yields } g) \longrightarrow (f \text{ yields } g)$
proof –
have 1: $\vdash f \longrightarrow f1$ **using** *assms* **by** *auto*
hence 2: $\vdash f; (\neg g) \longrightarrow f1; (\neg g)$ **by** (*rule LeftChopImpChop*)
hence 3: $\vdash \neg (f1; (\neg g)) \longrightarrow \neg (f; (\neg g))$ **by** *auto*
from 3 **show** *?thesis* **by** (*simp add: yields-d-def*)
qed

lemma *LeftYieldsEqvYields*:
assumes $\vdash f = f1$
shows $\vdash (f \text{ yields } g) = (f1 \text{ yields } g)$
proof –
have 1: $\vdash f = f1$ **using** *assms* **by** *auto*
hence 2: $\vdash f; (\neg g) = f1; (\neg g)$ **by** (*rule LeftChopEqvChop*)
hence 3: $\vdash (\neg (f; (\neg g))) = (\neg (f1; (\neg g)))$ **by** *auto*
from 3 **show** *?thesis* **by** (*simp add: yields-d-def*)
qed

5.6 Properties of Fin

lemma *FinEqvTrueChopAndEmpty*:
 $\vdash \text{fin } f = \# \text{True}; (f \wedge \text{empty})$
proof –
have 1: $\vdash \text{fin } f = \Box(\text{empty} \longrightarrow f)$
by (*simp add: fin-d-def*)
have 2: $\vdash \Box(\text{empty} \longrightarrow f) = (\neg(\Diamond(\neg(\text{empty} \longrightarrow f))))$
by (*simp add: always-d-def*)
have 3: $\vdash (\neg(\text{empty} \longrightarrow f)) = (\neg f \wedge \text{empty})$
by *auto*
hence 4: $\vdash \Diamond(\neg(\text{empty} \longrightarrow f)) = \Diamond(\neg f \wedge \text{empty})$
using *DiamondEqvDiamond* **by** *blast*
hence 5: $\vdash \neg(\Diamond(\neg(\text{empty} \longrightarrow f))) = (\neg(\Diamond(\neg f \wedge \text{empty})))$
by *auto*
have 6: $\vdash (\neg(\Diamond(\neg f \wedge \text{empty}))) = \# \text{True}; (f \wedge \text{empty})$
using *Finprop(4) sometimes-d-def* **by** (*metis int-eq int-simps(4)*)
from 1 2 5 6 **show** *?thesis* **by** *fastforce*
qed

lemma *DiamondFin*:
 $\vdash \Diamond(\text{fin } w) = \text{fin } w$
by (*metis DiamondDiamondEqvDiamond FinEqvTrueChopAndEmpty TrueChopEqvDiamond inteq-reflection*)

lemma *ChopFinExportA*:
 $\vdash f; (g \wedge \text{fin } w) \longrightarrow \text{fin } w$
using *DiamondFin*
by (*metis ChopAndB ChopImpDiamond inteq-reflection lift-imp-trans*)

lemma *FinImpBox*:
 $\vdash \text{fin } w \longrightarrow \Box(\text{fin } w)$

by (*simp add: BoxImpBoxBox fin-d-def*)

lemma *FinAndChopImport*:

$\vdash (fin\ w) \wedge (f;g) \longrightarrow f;((fin\ w) \wedge g)$

proof –

have 1: $\vdash fin\ w \longrightarrow \Box(fin\ w)$ **by** (*rule FinImpBox*)

hence 2: $\vdash fin\ w \wedge f;g \longrightarrow \Box(fin\ w) \wedge (f;g)$ **by** *auto*

have 3: $\vdash \Box(fin\ w) \wedge (f;g) \longrightarrow f;((fin\ w) \wedge g)$ **using** *BoxAndChopImport* **by** *blast*

from 2 3 **show** *?thesis* **using** *MP* **by** *fastforce*

qed

lemma *FinAndChop*:

$\vdash (f;(g \wedge fin\ w)) = (fin\ w \wedge f;g)$

using *FinAndChopImport ChopFinExportA ChopAndA ChopAndCommute* **by** *fastforce*

lemma *ChopAndEmptyEqvEmptyChopEmpty*:

$\vdash ((f;g) \wedge empty) = (f \wedge empty);(g \wedge empty)$

by (*auto simp: itl-defs*)

lemma *FinAndEmpty*:

$\vdash ((fin\ w) \wedge empty) = (w \wedge empty)$

proof –

have 1: $\vdash ((fin\ w) \wedge empty) = (\#True;(w \wedge empty) \wedge empty)$

using *FinEqvTrueChopAndEmpty* **by** *fastforce*

have 2: $\vdash (\#True;(w \wedge empty) \wedge empty) = ((\#True \wedge empty);(w \wedge empty))$

using *ChopAndEmptyEqvEmptyChopEmpty* [of *LIFT*($\#True$) *LIFT*($w \wedge empty$)]

by (*metis AndChopA ChopAndA ChopAndEmptyEqvEmptyChopEmpty Prop11 Prop12 int-eq*)

have 3: $\vdash (\#True \wedge empty);(w \wedge empty) = (empty;(w \wedge empty))$

using *LeftChopEqvChop* **by** *fastforce*

have 4: $\vdash (empty;(w \wedge empty)) = (w \wedge empty)$

using *EmptyChop* **by** *blast*

from 1 2 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *AndFinEqvChopAndEmpty*:

$\vdash (f \wedge fin\ g) = f;(g \wedge empty)$

proof –

have 1: $\vdash (f \wedge fin\ g) = (f;empty \wedge fin\ g)$

using *ChopEmpty* **by** (*metis int-eq*)

have 2: $\vdash (fin\ g \wedge f;empty) = (f;(empty \wedge fin\ g))$

using *FinAndChop* **by** *fastforce*

have 3: $\vdash (empty \wedge fin\ g) = (fin\ g \wedge empty)$

by *auto*

have 4: $\vdash (fin\ g \wedge empty) = (g \wedge empty)$

using *FinAndEmpty* **by** *metis*

have 5: $\vdash (empty \wedge fin\ g) = (g \wedge empty)$

using 3 4 **by** *auto*

hence 6: $\vdash f;(empty \wedge fin\ g) = f;(g \wedge empty)$

using *RightChopEqvChop* **by** *blast*

from 1 2 5 show ?thesis by (metis integ-reflection lift-and-com)
qed

lemma AndFinEqvChopStateAndEmpty:

$\vdash (f \wedge \text{fin } (\text{init } w)) = f; ((\text{init } w) \wedge \text{empty})$

using AndFinEqvChopAndEmpty by blast

lemma FinStateEqvStateAndEmptyOrNextFinState:

$\vdash \text{fin } (\text{init } w) = (((\text{init } w) \wedge \text{empty}) \vee \bigcirc (\text{fin } (\text{init } w)))$

proof –

have 1: $\vdash \text{fin } (\text{init } w) = \Box(\text{empty} \longrightarrow \text{init } w)$

by (simp add: fin-d-def)

have 2: $\vdash \Box(\text{empty} \longrightarrow \text{init } w) =$

$((\text{empty} \longrightarrow \text{init } w) \wedge \text{wnext } (\Box(\text{empty} \longrightarrow \text{init } w)))$

by (rule BoxEqvAndWnextBox)

have 3: $\vdash \text{fin } (\text{init } w) = ((\text{empty} \longrightarrow \text{init } w) \wedge \text{wnext } (\text{fin } (\text{init } w)))$

using 1 2 by (simp add: fin-d-def)

have 4: $\vdash \text{wnext } (\text{fin } (\text{init } w)) = (\text{empty} \vee \bigcirc (\text{fin } (\text{init } w)))$

by (rule WnextEqvEmptyOrNext)

have 5: $\vdash \text{fin } (\text{init } w) = ((\text{empty} \longrightarrow \text{init } w) \wedge (\text{empty} \vee \bigcirc (\text{fin } (\text{init } w))))$

using 3 4 by fastforce

have 6: $\vdash ((\text{empty} \longrightarrow \text{init } w) \wedge (\text{empty} \vee \bigcirc (\text{fin } (\text{init } w)))) =$

$((\text{empty} \longrightarrow \text{init } w) \wedge \text{empty}) \vee ((\text{empty} \longrightarrow \text{init } w) \wedge \bigcirc (\text{fin } (\text{init } w)))$

by auto

have 7: $\vdash ((\text{empty} \longrightarrow \text{init } w) \wedge \text{empty}) = ((\text{init } w) \wedge \text{empty})$

by auto

have 8: $\vdash ((\text{empty} \longrightarrow \text{init } w) \wedge \bigcirc (\text{fin } (\text{init } w))) = \bigcirc (\text{fin } (\text{init } w))$

by (metis 1 BoxElim DiamondFin NextDiamondImpDiamond int-eq lift-and-com lift-imp-trans Prop10)

have 9: $\vdash (((\text{empty} \longrightarrow \text{init } w) \wedge \text{empty}) \vee ((\text{empty} \longrightarrow \text{init } w) \wedge \bigcirc (\text{fin } (\text{init } w)))) =$

$((\text{init } w) \wedge \text{empty}) \vee \bigcirc (\text{fin } (\text{init } w))$

using 7 8 by auto

from 5 6 8 9 show ?thesis by fastforce

qed

lemma FinChopEqvOr:

$\vdash (\text{fin } (\text{init } w)); f = (((\text{init } w) \wedge f) \vee \bigcirc ((\text{fin } (\text{init } w)); f))$

proof –

have 1: $\vdash \text{fin } (\text{init } w) = (((\text{init } w) \wedge \text{empty}) \vee \bigcirc (\text{fin } (\text{init } w)))$

by (rule FinStateEqvStateAndEmptyOrNextFinState)

hence 2: $\vdash (\text{fin } (\text{init } w)); f = (((\text{init } w) \wedge \text{empty}) \vee \bigcirc (\text{fin } (\text{init } w))); f$

by (rule LeftChopEqvChop)

have 3: $\vdash (((\text{init } w) \wedge \text{empty}) \vee \bigcirc (\text{fin } (\text{init } w))); f$

$= (((\text{init } w) \wedge \text{empty}); f \vee \bigcirc (\text{fin } (\text{init } w))); f$

by (rule OrChopEqv)

have 4: $\vdash ((\text{init } w) \wedge \text{empty}); f = ((\text{init } w) \wedge f)$

by (rule StateAndEmptyChop)

have 5: $\vdash \bigcirc (\text{fin } (\text{init } w)); f = \bigcirc ((\text{fin } (\text{init } w)); f)$

by (rule NextChop)

from 2 3 4 5 show ?thesis by fastforce

qed

lemma *FinChopEqvDiamond*:

$\vdash (\text{fin } (\text{init } w)); f = \Diamond ((\text{init } w) \wedge f)$

proof –

have 1: $\vdash (\text{fin } (\text{init } w)) = (\# \text{True}; ((\text{init } w) \wedge \text{empty}))$

by (*rule FinEqvTrueChopAndEmpty*)

hence 2: $\vdash (\text{fin } (\text{init } w)); f = (\# \text{True}; ((\text{init } w) \wedge \text{empty}); f)$

by (*rule LeftChopEqvChop*)

have 3: $\vdash \# \text{True}; ((\text{init } w) \wedge \text{empty}); f = (\# \text{True}; ((\text{init } w) \wedge \text{empty}); f)$

by (*rule ChopAssoc*)

have 4: $\vdash \# \text{True}; ((\text{init } w) \wedge \text{empty}); f = \Diamond (((\text{init } w) \wedge \text{empty}); f)$

by (*simp add: sometimes-d-def*)

have 5: $\vdash ((\text{init } w) \wedge \text{empty}); f = ((\text{init } w) \wedge f)$

using *StateAndEmptyChop* **by** *blast*

hence 6: $\vdash \Diamond (((\text{init } w) \wedge \text{empty}); f) = \Diamond ((\text{init } w) \wedge f)$

by (*rule DiamondEqvDiamond*)

from 2 3 4 6 **show** *?thesis* **by** *fastforce*

qed

lemma *NotDiamondAndNot*:

$\vdash \neg (\Diamond (f \wedge \neg f))$

proof –

have 1: $\vdash (\neg (\Diamond (f \wedge \neg f))) = \Box (\neg (f \wedge \neg f))$ **using** *NotDiamondNotEqvBox* **by** *fastforce*

have 2: $\vdash \neg (f \wedge \neg f)$ **by** *simp*

have 3: $\vdash \Box (\neg (f \wedge \neg f))$ **using** 2 **by** (*simp add: BoxGen*)

from 1 3 **show** *?thesis* **by** *fastforce*

qed

lemma *FinYields*:

$\vdash (\text{fin } (\text{init } w)) \text{ yields } (\text{init } w)$

proof –

have 1: $\vdash (\text{fin } (\text{init } w)); (\neg (\text{init } w)) = \Diamond ((\text{init } w) \wedge \neg (\text{init } w))$ **by** (*rule FinChopEqvDiamond*)

have 2: $\vdash \neg (\Diamond ((\text{init } w) \wedge \neg (\text{init } w)))$ **by** (*rule NotDiamondAndNot*)

have 3: $\vdash \neg ((\text{fin } (\text{init } w)); (\neg (\text{init } w)))$ **using** 1 2 **by** *fastforce*

from 3 **show** *?thesis* **by** (*simp add: yields-d-def*)

qed

lemma *ImpAndFinStateOrFinNotState*:

$\vdash f \longrightarrow (f \wedge \text{fin } (\text{init } w)) \vee \text{fin } (\neg (\text{init } w))$

by (*simp add: itl-defs Valid-def*)

lemma *AndFinChopEqvStateAndChop*:

$\vdash (f \wedge \text{fin } (\text{init } w)); g = f; ((\text{init } w) \wedge g)$

proof –

have 1: $\vdash (\text{fin } (\text{init } w)) \text{ yields } (\text{init } w)$

by (*rule FinYields*)

have 2: $\vdash f \wedge \text{fin } (\text{init } w) \longrightarrow \text{fin } (\text{init } w)$

by *auto*

hence 3: $\vdash (\text{fin } (\text{init } w)) \text{ yields } (\text{init } w) \longrightarrow (f \wedge \text{fin } (\text{init } w)) \text{ yields } (\text{init } w)$

by (rule *LeftYieldsImpYields*)
 have 4: $\vdash (f \wedge \text{fin } (\text{init } w)) \text{ yields } (\text{init } w)$
 using 1 3 MP by fastforce
 have 5: $\vdash (f \wedge \text{fin } (\text{init } w)); g \wedge (f \wedge \text{fin } (\text{init } w)) \text{ yields } (\text{init } w)$
 $\longrightarrow (f \wedge \text{fin } (\text{init } w)); (g \wedge (\text{init } w))$
 by (rule *ChopAndYieldsImp*)
 have 6: $\vdash (f \wedge \text{fin } (\text{init } w)); g \longrightarrow (f \wedge \text{fin } (\text{init } w)); (g \wedge (\text{init } w))$
 using 4 5 by fastforce
 have 7: $\vdash (f \wedge \text{fin } (\text{init } w)); (g \wedge (\text{init } w)) \longrightarrow f; (g \wedge (\text{init } w))$
 by (rule *AndChopA*)
 have 8: $\vdash g \wedge (\text{init } w) \longrightarrow (\text{init } w) \wedge g$
 by auto
 hence 9: $\vdash f; (g \wedge (\text{init } w)) \longrightarrow f; ((\text{init } w) \wedge g)$
 by (rule *RightChopImpChop*)
 have 10: $\vdash (f \wedge \text{fin } (\text{init } w)); g \longrightarrow f; ((\text{init } w) \wedge g)$
 using 6 7 9 by fastforce
 have 11: $\vdash f \longrightarrow (f \wedge \text{fin } (\text{init } w)) \vee \text{fin } (\neg (\text{init } w))$
 by (rule *ImpAndFinStateOrFinNotState*)
 hence 12: $\vdash f; ((\text{init } w) \wedge g) \longrightarrow$
 $((f \wedge \text{fin } (\text{init } w)) \vee \text{fin } (\neg (\text{init } w))); ((\text{init } w) \wedge g)$
 by (rule *LeftChopImpChop*)
 have 13: $\vdash ((f \wedge \text{fin } (\text{init } w)) \vee \text{fin } (\neg (\text{init } w))); ((\text{init } w) \wedge g)$
 $=$
 $((f \wedge \text{fin } (\text{init } w)); ((\text{init } w) \wedge g) \vee (\text{fin } (\neg (\text{init } w))); ((\text{init } w) \wedge g))$
 by (rule *OrChopEqv*)
 have 14: $\vdash (\text{fin } (\text{init } (\neg w))); ((\text{init } w) \wedge g) \longrightarrow \Diamond (\text{init } (\neg w)) \wedge ((\text{init } w) \wedge g)$
 using *FinChopEqvDiamond* by fastforce
 have 141: $\vdash \neg (\Diamond (\text{init } (\neg w)) \wedge ((\text{init } w) \wedge g)) \longrightarrow$
 $\neg (\text{fin } (\text{init } (\neg w))); ((\text{init } w) \wedge g)$
 using 14 by fastforce
 have 15: $\vdash \neg (\Diamond (\text{init } (\neg w)) \wedge ((\text{init } w) \wedge g))$
 using *NotDiamondAndNot Initprop(2)* by (auto simp: *itl-defs*)
 have 151: $\vdash \neg (\text{fin } (\text{init } (\neg w))); ((\text{init } w) \wedge g)$
 using 15 141 by fastforce
 have 1511: $\vdash (\text{fin } (\neg (\text{init } w))); ((\text{init } w) \wedge g) \longrightarrow \#False$
 using 151 by (metis *Initprop(2)* *int-simps(14)* *inteq-reflection*)
 have 152: $\vdash (f \wedge \text{fin } (\text{init } w)); ((\text{init } w) \wedge g) \vee (\text{fin } (\neg (\text{init } w))); ((\text{init } w) \wedge g) \longrightarrow$
 $(f \wedge \text{fin } (\text{init } w)); ((\text{init } w) \wedge g)$
 using 1511 by fastforce
 have 16: $\vdash f; ((\text{init } w) \wedge g) \longrightarrow (f \wedge \text{fin } (\text{init } w)); ((\text{init } w) \wedge g)$
 using 12 13 152 by fastforce
 have 17: $\vdash (f \wedge \text{fin } (\text{init } w)); ((\text{init } w) \wedge g) \longrightarrow (f \wedge \text{fin } (\text{init } w)); g$
 by (rule *ChopAndB*)
 have 18: $\vdash f; ((\text{init } w) \wedge g) \longrightarrow (f \wedge \text{fin } (\text{init } w)); g$
 using 16 17 by fastforce
 from 10 18 show ?thesis by fastforce
 qed

lemma *DiAndFinEqvChopState*:

$\vdash \text{di } (f \wedge \text{fin } (\text{init } w)) = f; (\text{init } w)$

proof –

have 1: $\vdash (f \wedge \text{fin}(\text{init } w)); \# \text{True} = f; ((\text{init } w) \wedge \# \text{True})$ **by** (rule *AndFinChopEqvStateAndChop*)
have 2: $\vdash ((\text{init } w) \wedge \# \text{True}) = (\text{init } w)$ **by** *auto*
hence 3: $\vdash (f; ((\text{init } w) \wedge \# \text{True})) = (f; (\text{init } w))$ **by** (rule *RightChopEqvChop*)
have 4: $\vdash (f \wedge \text{fin}(\text{init } w)); \# \text{True} = f; (\text{init } w)$ **using** 1 3 **by** *auto*
from 4 **show** ?thesis **by** (simp add: di-d-def)

qed

lemma *FinNotStateEqvNotFinState*:

$\vdash \text{fin}(\text{init } (\neg w)) = (\neg (\text{fin}(\text{init } w)))$
using *FinEqvTrueChopAndEmpty*
by (metis *Finprop*(4) *Initprop*(2) *int-eq*)

lemma *BiImpFinEqvYieldsState*:

$\vdash \text{bi}(f \longrightarrow \text{fin}(\text{init } w)) = f \text{ yields } (\text{init } w)$

proof –

have 1: $\vdash \text{di}(f \wedge \text{fin}(\text{init } (\neg w))) = f; (\text{init } (\neg w))$
by (rule *DiAndFinEqvChopState*)
have 2: $\vdash (f \wedge \text{fin}(\text{init } (\neg w))) = (f \wedge \neg(\text{fin}(\text{init } w)))$
using *FinNotStateEqvNotFinState* **by** *fastforce*
have 3: $\vdash (f \wedge \neg(\text{fin}(\text{init } w))) = (\neg(f \longrightarrow \text{fin}(\text{init } w)))$
by *auto*
have 4: $\vdash (f \wedge \text{fin}(\text{init } (\neg w))) = (\neg(f \longrightarrow \text{fin}(\text{init } w)))$
using 2 3 **by** *fastforce*
hence 5: $\vdash \text{di}(f \wedge \text{fin}(\text{init } (\neg w))) = \text{di}(\neg(f \longrightarrow \text{fin}(\text{init } w)))$
by (rule *DiEqvDi*)
have 6: $\vdash \text{di}(\neg(f \longrightarrow \text{fin}(\text{init } w))) = (\neg(\text{bi}(f \longrightarrow \text{fin}(\text{init } w))))$
by (rule *DiNotEqvNotBi*)
have 7: $\vdash \neg(\text{bi}(f \longrightarrow \text{fin}(\text{init } w))) = f; (\text{init } (\neg w))$
using 1 5 6 *Initprop* **by** *fastforce*
hence 8: $\vdash \text{bi}(f \longrightarrow \text{fin}(\text{init } w)) = (\neg(f; (\neg(\text{init } w))))$
by (metis *Initprop*(2) *int-eq* *int-simps*(7))
from 8 **show** ?thesis **by** (simp add: yields-d-def)

qed

lemma *StateImpYields*:

assumes $\vdash (\text{init } w) \wedge f \longrightarrow \text{fin}(\text{init } w1)$
shows $\vdash (\text{init } w) \longrightarrow (f \text{ yields } (\text{init } w1))$

proof –

have 1: $\vdash (\text{init } w) \wedge f \longrightarrow \text{fin}(\text{init } w1)$ **using** *assms* **by** *auto*
hence 2: $\vdash (\text{init } w) \longrightarrow (f \longrightarrow \text{fin}(\text{init } w1))$ **by** *auto*
hence 3: $\vdash (\text{init } w) \longrightarrow \text{bi}(f \longrightarrow \text{fin}(\text{init } w1))$ **by** (rule *StateImpBiGen*)
have 4: $\vdash \text{bi}(f \longrightarrow \text{fin}(\text{init } w1)) = f \text{ yields } (\text{init } w1)$ **by** (rule *BiImpFinEqvYieldsState*)
from 3 4 **show** ?thesis **by** *fastforce*

qed

lemma *StateAndYieldsImpYields*:

assumes $\vdash (\text{init } w) \wedge f \longrightarrow f1$
shows $\vdash (\text{init } w) \wedge (f1 \text{ yields } g) \longrightarrow (f \text{ yields } g)$

proof –

have 1: $\vdash (\text{init } w) \wedge f \longrightarrow f1$ **using** *assms* **by** *auto*
hence 2: $\vdash (\text{init } w) \wedge (f; (\neg g)) \longrightarrow f1; (\neg g)$ **by** (*rule StateAndChopImpChopRule*)
hence 3: $\vdash (\text{init } w) \wedge \neg (f1; (\neg g)) \longrightarrow \neg (f; (\neg g))$ **by** *auto*
from 3 **show** ?thesis **by** (*simp add: yields-d-def*)
qed

lemma *AndYieldsA*:

$\vdash f \text{ yields } g \longrightarrow (f \wedge f1) \text{ yields } g$
proof –
have 1: $\vdash f \wedge f1 \longrightarrow f$ **by** *auto*
from 1 **show** ?thesis **by** (*rule LeftYieldsImpYields*)
qed

lemma *AndYieldsB*:

$\vdash f1 \text{ yields } g \longrightarrow (f \wedge f1) \text{ yields } g$
proof –
have 1: $\vdash f \wedge f1 \longrightarrow f1$ **by** *auto*
from 1 **show** ?thesis **by** (*rule LeftYieldsImpYields*)
qed

lemma *RightYieldsImpYields*:

assumes $\vdash g \longrightarrow g1$
shows $\vdash (f \text{ yields } g) \longrightarrow (f \text{ yields } g1)$
proof –
have 1: $\vdash g \longrightarrow g1$ **using** *assms* **by** *auto*
hence 2: $\vdash \neg g1 \longrightarrow \neg g$ **by** *auto*
hence 3: $\vdash f; (\neg g1) \longrightarrow f; (\neg g)$ **by** (*rule RightChopImpChop*)
hence 4: $\vdash \neg (f; (\neg g)) \longrightarrow \neg (f; (\neg g1))$ **by** *auto*
from 4 **show** ?thesis **by** (*simp add: yields-d-def*)
qed

lemma *RightYieldsEqvYields*:

assumes $\vdash g = g1$
shows $\vdash (f \text{ yields } g) = (f \text{ yields } g1)$
proof –
have 1: $\vdash g = g1$ **using** *assms* **by** *auto*
hence 2: $\vdash (\neg g) = (\neg g1)$ **by** *auto*
hence 3: $\vdash f; (\neg g) = f; (\neg g1)$ **by** (*rule RightChopEqvChop*)
hence 4: $\vdash (\neg (f; (\neg g))) = (\neg (f; (\neg g1)))$ **by** *auto*
from 4 **show** ?thesis **by** (*simp add: yields-d-def*)
qed

lemma *BoxImpYields*:

$\vdash \Box g \longrightarrow f \text{ yields } g$
proof –
have 1: $\vdash f; (\neg g) \longrightarrow \Diamond(\neg g)$ **by** (*rule ChopImpDiamond*)
hence 2: $\vdash \neg (\Diamond(\neg g)) \longrightarrow \neg (f; (\neg g))$ **by** *auto*
from 2 **show** ?thesis **by** (*simp add: yields-d-def always-d-def*)
qed

lemma *BoxEqvTrueYields*:

$\vdash \Box f = \#True \text{ yields } f$

proof –

have 1: $\vdash \#True; (\neg f) = \Diamond (\neg f)$ **by** (rule *TrueChopEqvDiamond*)

hence 2: $\vdash (\neg (\#True; (\neg f))) = (\neg (\Diamond (\neg f)))$ **by** *auto*

have 3: $\vdash \Box f = (\neg (\Diamond (\neg f)))$ **by** (simp add: *always-d-def*)

have 4: $\vdash \Box f = (\neg (\#True; (\neg f)))$ **using** 2 3 **by** *fastforce*

from 4 **show** ?thesis **by** (simp add: *yields-d-def*)

qed

lemma *YieldsGen*:

assumes $\vdash g$

shows $\vdash f \text{ yields } g$

proof –

have 1: $\vdash g$ **using** *assms* **by** *auto*

hence 2: $\vdash \Box g$ **by** (rule *BoxGen*)

have 3: $\vdash \Box g \longrightarrow f \text{ yields } g$ **by** (rule *BoxImpYields*)

from 2 3 **show** ?thesis **using** *MP* **by** *fastforce*

qed

lemma *YieldsAndYieldsEqvYieldsAnd*:

$\vdash ((f \text{ yields } g) \wedge (f \text{ yields } g1)) = f \text{ yields } (g \wedge g1)$

proof –

have 1: $\vdash f; (\neg g \vee \neg g1) = ((f; (\neg g)) \vee (f; (\neg g1)))$ **by** (rule *ChopOrEqv*)

hence 2: $\vdash ((f; (\neg g)) \vee (f; (\neg g1))) = f; (\neg g \vee \neg g1)$ **by** *auto*

have 3: $\vdash (\neg g \vee \neg g1) = (\neg (g \wedge g1))$ **by** *auto*

hence 4: $\vdash f; (\neg g \vee \neg g1) = f; (\neg (g \wedge g1))$ **by** (rule *RightChopEqvChop*)

have 5: $\vdash (f; (\neg g)) \vee (f; (\neg g1)) = f; (\neg (g \wedge g1))$ **using** 2 4 **by** *fastforce*

hence 6: $\vdash (\neg (f; (\neg g)) \wedge \neg (f; (\neg g1))) = (\neg (f; (\neg (g \wedge g1))))$ **by** (auto simp: *itl-defs*)

from 6 **show** ?thesis **by** (simp add: *yields-d-def*)

qed

lemma *YieldsAndYieldsImpAndYieldsAnd*:

$\vdash (f \text{ yields } g) \wedge (f1 \text{ yields } g1) \longrightarrow (f \wedge f1) \text{ yields } (g \wedge g1)$

proof –

have 1: $\vdash f \text{ yields } g \longrightarrow (f \wedge f1) \text{ yields } g$

by (rule *AndYieldsA*)

have 2: $\vdash f1 \text{ yields } g1 \longrightarrow (f \wedge f1) \text{ yields } g1$

by (rule *AndYieldsB*)

have 3: $\vdash ((f \wedge f1) \text{ yields } g \wedge (f \wedge f1) \text{ yields } g1) = (f \wedge f1) \text{ yields } (g \wedge g1)$

by (rule *YieldsAndYieldsEqvYieldsAnd*)

from 1 2 3 **show** ?thesis **by** *fastforce*

qed

lemma *YieldsYieldsEqvChopYields*:

$\vdash f \text{ yields } (g \text{ yields } h) = (f; g) \text{ yields } h$

proof –

have 1: $\vdash f; (g; (\neg h)) = (f; g); (\neg h)$ **by** (rule *ChopAssoc*)

hence 2: $\vdash f; (g; (\neg h)) = (f; g); (\neg h)$ **by** *auto*

have 3: $\vdash g; (\neg h) = (\neg \neg (g; (\neg h)))$ **by** *auto*

hence 4: $\vdash f; (g; (\neg h)) = f; (\neg \neg (g; (\neg h)))$ **by** (rule *RightChopEqvChop*)
have 5: $\vdash f; (\neg \neg (g; (\neg h))) = (f; g); (\neg h)$ **using** 2 4 **by** *auto*
hence 6: $\vdash f; (\neg (g \text{ yields } h)) = (f; g); (\neg h)$ **by** (simp add: *yields-d-def*)
hence 7: $\vdash (\neg (f; (\neg (g \text{ yields } h)))) = (\neg ((f; g); (\neg h)))$ **by** *auto*
from 7 **show** ?thesis **by** (simp add: *yields-d-def*)
qed

lemma *EmptyYields*:

$\vdash \text{empty} \text{ yields } f = f$

proof –

have 1: $\vdash \text{empty}; (\neg f) = (\neg f)$ **by** (rule *EmptyChop*)

hence 2: $\vdash (\neg (\text{empty}; (\neg f))) = f$ **by** *auto*

from 2 **show** ?thesis **by** (simp add: *yields-d-def*)

qed

lemma *NextYields*:

$\vdash (\bigcirc f) \text{ yields } g = \text{wnext } (f \text{ yields } g)$

proof –

have 1: $\vdash (\bigcirc f); (\neg g) = \bigcirc(f; (\neg g))$ **by** (rule *NextChop*)

hence 2: $\vdash (\neg ((\bigcirc f); (\neg g))) = (\neg (\bigcirc(f; (\neg g))))$ **by** *auto*

hence 3: $\vdash (\bigcirc f) \text{ yields } g = (\neg (\bigcirc(f; (\neg g))))$ **by** (simp add: *yields-d-def*)

have 4: $\vdash (\neg (\bigcirc(f; (\neg g)))) = \text{wnext } (\neg (f; (\neg g)))$ **by** (auto simp: *wnext-d-def*)

have 5: $\vdash (\bigcirc f) \text{ yields } g = \text{wnext } (\neg (f; (\neg g)))$ **using** 3 4 **by** *fastforce*

from 5 **show** ?thesis **by** (simp add: *yields-d-def*)

qed

lemma *SkipChopEqvNext*:

$\vdash \text{skip}; f = \bigcirc f$

by (simp add: *next-d-def*)

lemma *SkipYieldsEqvWeakNext*:

$\vdash \text{skip} \text{ yields } f = \text{wnext } f$

proof –

have 1: $\vdash \text{skip}; (\neg f) = \bigcirc(\neg f)$ **by** (rule *SkipChopEqvNext*)

hence 2: $\vdash (\neg (\text{skip}; (\neg f))) = (\neg (\bigcirc(\neg f)))$ **by** *auto*

have 3: $\vdash (\neg (\bigcirc(\neg f))) = \text{wnext } f$ **by** (auto simp: *wnext-d-def*)

have 4: $\vdash (\neg (\text{skip}; (\neg f))) = \text{wnext } f$ **using** 2 3 **by** *fastforce*

from 4 **show** ?thesis **by** (simp add: *yields-d-def*)

qed

lemma *NextImpSkipYields*:

$\vdash \bigcirc f \longrightarrow \text{skip} \text{ yields } f$

proof –

have 1: $\vdash \bigcirc f \longrightarrow \text{wnext } f$ **using** *WnextEqvEmptyOrNext* **by** *fastforce*

have 2: $\vdash \text{skip} \text{ yields } f = \text{wnext } f$ **by** (rule *SkipYieldsEqvWeakNext*)

from 1 2 **show** ?thesis **by** *fastforce*

qed

lemma *MoreEqvSkipChopTrue*:

$\vdash \text{more} = \text{skip}; \# \text{True}$

proof –
have 1: $\vdash \text{skip} ; \# \text{True} = \bigcirc \# \text{True}$ **by** (rule *SkipChopEqvNext*)
hence 2: $\vdash \bigcirc \# \text{True} = \text{skip} ; \# \text{True}$ **by** *auto*
from 2 **show** ?thesis **by** (simp add: more-d-def)
qed

lemma *MoreChopImpMore*:

$\vdash \text{more} ; f \longrightarrow \text{more}$

proof –
have 1: $\vdash (\bigcirc \# \text{True}); f = \bigcirc(\# \text{True}; f)$ **by** (rule *NextChop*)
have 2: $\vdash \bigcirc(\# \text{True}; f) \longrightarrow \text{more}$ **by** (auto simp: itl-defs)
have 3: $\vdash (\bigcirc \# \text{True}; f) \longrightarrow \text{more}$ **using** 1 2 **by** *fastforce*
from 3 **show** ?thesis **by** (metis more-d-def)
qed

lemma *ChopMoreImpMore*:

$\vdash f ; \text{more} \longrightarrow \text{more}$

proof –
have 1: $\vdash f ; \text{more} \longrightarrow \Diamond \text{more}$ **by** (rule *ChopImpDiamond*)
have 2: $\vdash \Diamond \text{more} \longrightarrow \text{more}$ **by** (auto simp: itl-defs)
from 1 2 **show** ?thesis **by** *fastforce*
qed

lemma *MoreChopEqvNextDiamond*:

$\vdash \text{more} ; f = \bigcirc(\Diamond f)$

proof –
have 1: $\vdash \text{more} ; f = (\bigcirc \# \text{True}); f$ **by** (simp add: more-d-def)
have 2: $\vdash (\bigcirc \# \text{True}); f = \bigcirc(\# \text{True}; f)$ **by** (rule *NextChop*)
have 3: $\vdash \text{more} ; f = \bigcirc(\# \text{True}; f)$ **using** 1 2 **by** *fastforce*
from 3 **show** ?thesis **by** (simp add: sometimes-d-def)
qed

lemma *WeakNextBoxImpMoreYields*:

$\vdash \text{more} \text{ yields } f = \text{wnext}(\Box f)$

proof –
have 1: $\vdash \text{more} ; (\neg f) = \bigcirc(\Diamond (\neg f))$ **by** (rule *MoreChopEqvNextDiamond*)
have 2: $\vdash \bigcirc(\Diamond (\neg f)) = \bigcirc(\neg(\Box f))$ **by** (auto simp: always-d-def)
have 3: $\vdash \bigcirc(\neg(\Box f)) = (\neg (\text{wnext}(\Box f)))$ **by** (auto simp: wnext-d-def)
have 4: $\vdash \text{more} ; (\neg f) = (\neg(\text{more} \text{ yields } f))$ **by** (simp add: yields-d-def)
from 1 2 3 4 **show** ?thesis **by** *fastforce*
qed

lemma *NotEqvYieldsMore*:

$\vdash (\neg f) = f \text{ yields more}$

proof –
have 1: $\vdash f ; \text{empty} = f$ **by** (rule *ChopEmpty*)
hence 2: $\vdash (\neg (f ; \text{empty})) = (\neg f)$ **by** *auto*
have 3: $\vdash \text{empty} = (\neg \text{more})$ **by** (auto simp: empty-d-def)
hence 4: $\vdash f ; \text{empty} = f ; (\neg \text{more})$ **by** (rule *RightChopEqvChop*)
hence 5: $\vdash (\neg (f ; \text{empty})) = (\neg (f ; (\neg \text{more})))$ **by** *auto*

have 6: $\vdash (\neg f) = (\neg (f; (\neg \text{more})))$ **using** 2 5 **by** *fastforce*
from 6 **show** *?thesis* **by** (*metis yields-d-def*)
qed

lemma *LeftChopImpMoreRule*:

assumes $\vdash f \longrightarrow \text{more}$
shows $\vdash f; g \longrightarrow \text{more}$
proof –
 have 1: $\vdash f \longrightarrow \text{more}$ **using** *assms* **by** *auto*
 hence 2: $\vdash f; g \longrightarrow \text{more}; g$ **by** (*rule LeftChopImpChop*)
 have 3: $\vdash \text{more}; g \longrightarrow \text{more}$ **by** (*rule MoreChopImpMore*)
from 2 3 **show** *?thesis* **using** *lift-imp-trans* **by** *blast*

qed

lemma *RightChopImpMoreRule*:

assumes $\vdash g \longrightarrow \text{more}$
shows $\vdash f; g \longrightarrow \text{more}$
proof –
 have 1: $\vdash g \longrightarrow \text{more}$ **using** *assms* **by** *auto*
 hence 2: $\vdash f; g \longrightarrow f; \text{more}$ **by** (*rule RightChopImpChop*)
 have 3: $\vdash f; \text{more} \longrightarrow \text{more}$ **by** (*rule ChopMoreImpMore*)
from 2 3 **show** *?thesis* **using** *lift-imp-trans* **by** *blast*

qed

lemma *NotDiEqvBiNot*:

$\vdash (\neg (di\ f)) = bi\ (\neg f)$
proof –
 have 1: $\vdash f = (\neg \neg f)$ **by** *auto*
 hence 2: $\vdash di\ f = di\ (\neg \neg f)$ **by** (*rule DiEqvDi*)
 hence 3: $\vdash (\neg (di\ f)) = (\neg (di\ (\neg \neg f)))$ **by** *auto*
from 3 **show** *?thesis* **by** (*simp add: bi-d-def*)

qed

lemma *ChopImpDi*:

$\vdash f; g \longrightarrow di\ f$
proof –
 have 1: $\vdash g \longrightarrow \#True$ **by** *auto*
 hence 2: $\vdash f; g \longrightarrow f; \#True$ **by** (*rule RightChopImpChop*)
from 2 **show** *?thesis* **by** (*simp add: di-d-def*)

qed

lemma *TrueEqvTrueChopTrue*:

$\vdash \#True = \#True; \#True$
proof –
 have 1: $\vdash \#True; \#True \longrightarrow \#True$ **by** *auto*
 have 2: $\vdash \#True \longrightarrow di\ \#True$ **by** (*rule DiIntro*)
 hence 3: $\vdash \#True \longrightarrow \#True; \#True$ **by** (*simp add: di-d-def*)
from 1 3 **show** *?thesis* **by** *auto*

qed

lemma *DiEqvDiDi*:

$\vdash \text{di } f = \text{di } (\text{di } f)$

proof –

have 1: $\vdash \#True = \#True; \#True$ **by** (rule *TrueEqvTrueChopTrue*)

hence 2: $\vdash f; \#True = f; (\#True; \#True)$ **by** (rule *RightChopEqvChop*)

have 3: $\vdash f; (\#True; \#True) = (f; \#True); \#True$ **by** (rule *ChopAssoc*)

have 4: $\vdash f; \#True = (f; \#True); \#True$ **using** 2 3 **by** *fastforce*

from 4 **show** *?thesis* **by** (metis *di-d-def*)

qed

lemma *BiEqvBiBi*:

$\vdash \text{bi } f = \text{bi } (\text{bi } f)$

proof –

have 1: $\vdash \text{di } (\neg f) = \text{di } (\text{di } (\neg f))$ **by** (rule *DiEqvDiDi*)

have 2: $\vdash \text{di } (\neg f) = (\neg (\text{bi } f))$ **by** (rule *DiNotEqvNotBi*)

hence 3: $\vdash \text{di } (\text{di } (\neg f)) = \text{di } (\neg (\text{bi } f))$ **by** (rule *DiEqvDi*)

have 4: $\vdash \text{di } (\neg f) = \text{di } (\neg (\text{bi } f))$ **using** 1 3 **by** *fastforce*

hence 5: $\vdash (\neg (\text{di } (\neg f))) = (\neg (\text{di } (\neg (\text{bi } f))))$ **by** *fastforce*

from 5 **show** *?thesis* **by** (metis *bi-d-def*)

qed

lemma *DiOrEqv*:

$\vdash \text{di } (f \vee g) = (\text{di } f \vee \text{di } g)$

proof –

have 1: $\vdash (f \vee g); \#True = (f; \#True \vee g; \#True)$ **by** (rule *OrChopEqv*)

from 1 **show** *?thesis* **by** (simp add: *di-d-def*)

qed

lemma *DiAndA*:

$\vdash \text{di } (f \wedge g) \longrightarrow \text{di } f$

proof –

have 1: $\vdash (f \wedge g); \#True \longrightarrow f; \#True$ **by** (rule *AndChopA*)

from 1 **show** *?thesis* **by** (simp add: *di-d-def*)

qed

lemma *DiAndB*:

$\vdash \text{di } (f \wedge g) \longrightarrow \text{di } g$

proof –

have 1: $\vdash (f \wedge g); \#True \longrightarrow g; \#True$ **by** (rule *AndChopB*)

from 1 **show** *?thesis* **by** (simp add: *di-d-def*)

qed

lemma *DiAndImpAnd*:

$\vdash \text{di } (f \wedge g) \longrightarrow \text{di } f \wedge \text{di } g$

proof –

have 1: $\vdash \text{di } (f \wedge g) \longrightarrow \text{di } f$ **by** (rule *DiAndA*)

have 2: $\vdash \text{di } (f \wedge g) \longrightarrow \text{di } g$ **by** (rule *DiAndB*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *DiSkipEqvMore*:

$\vdash \text{di skip} = \text{more}$

proof –

have 1: $\vdash \text{skip} ; \# \text{True} = \circ \# \text{True}$ **by** (rule *SkipChopEqvNext*)

have 2: $\vdash \circ \# \text{True} = \text{more}$ **by** (auto simp: *more-d-def*)

have 3: $\vdash \text{skip} ; \# \text{True} = \text{more}$ **using** 1 2 **by** fastforce

from 3 **show** ?thesis **by** (simp add: *di-d-def*)

qed

lemma *DiMoreEqvMore*:

$\vdash \text{di more} = \text{more}$

proof –

have 1: $\vdash \text{di} (\circ \# \text{True}) = \circ (\text{di} \# \text{True})$ **by** (rule *DiNext*)

have 2: $\vdash \circ (\text{di} \# \text{True}) \longrightarrow \text{more}$ **by** (auto simp: *itl-defs*)

have 3: $\vdash \text{di} (\circ \# \text{True}) \longrightarrow \text{more}$ **using** 1 2 **by** fastforce

hence 4: $\vdash \text{di more} \longrightarrow \text{more}$ **by** (simp add: *more-d-def*)

have 5: $\vdash \text{more} \longrightarrow \text{di more}$ **by** (rule *ImpDi*)

from 4 5 **show** ?thesis **by** fastforce

qed

lemma *DiIfEqvRule*:

assumes $\vdash f = \text{if}_i (\text{init } w) \text{ then } g \text{ else } h$

shows $\vdash \text{di } f = \text{if}_i (\text{init } w) \text{ then } (\text{di } g) \text{ else } (\text{di } h)$

proof –

have 1: $\vdash f = \text{if}_i (\text{init } w) \text{ then } g \text{ else } h$ **using** assms **by** auto

hence 2: $\vdash f ; \# \text{True} = \text{if}_i (\text{init } w) \text{ then } (g ; \# \text{True}) \text{ else } (h ; \# \text{True})$ **by** (rule *IfChopEqvRule*)

from 2 **show** ?thesis **by** (simp add: *di-d-def*)

qed

lemma *DiEmpty*:

$\vdash \text{di empty}$

proof –

have 1: $\vdash \# \text{True}$ **by** auto

have 2: $\vdash \text{empty} ; \# \text{True} = \# \text{True}$ **by** (rule *EmptyChop*)

have 3: $\vdash \text{empty} ; \# \text{True}$ **using** 1 2 **by** auto

from 3 **show** ?thesis **by** (simp add: *di-d-def*)

qed

lemma *DaNotEqvNotBa*:

$\vdash \text{da} (\neg f) = (\neg (\text{ba } f))$

proof –

have 1: $\vdash \text{ba } f = (\neg (\text{da} (\neg f)))$ **by** (simp add: *ba-d-def*)

from 1 **show** ?thesis **by** fastforce

qed

lemma *DaEqvDa*:

assumes $\vdash f = g$

shows $\vdash \text{da } f = \text{da } g$

using assms **using** *int-eq* **by** force

lemma *DaEqvNotBaNot*:

$\vdash da\ f = (\neg (ba\ (\neg\ f)))$

proof –

have 1: $\vdash ba\ (\neg\ f) = (\neg (da\ (\neg\ \neg\ f)))$ **by** (*simp add: ba-d-def*)

hence 2: $\vdash da\ (\neg\ \neg\ f) = (\neg (ba\ (\neg\ f)))$ **by** *fastforce*

have 3: $\vdash f = (\neg\ \neg\ f)$ **by** *simp*

hence 4: $\vdash da\ f = da\ (\neg\ \neg\ f)$ **by** (*rule DaEqvDa*)

from 2 4 **show** *?thesis* **by** *simp*

qed

lemma *BaElim*:

$\vdash ba\ f \longrightarrow f$

proof –

have 1: $\vdash ba\ f = \Box(bi\ f)$ **by** (*rule BaEqvBtBi*)

have 2: $\vdash bi\ f \longrightarrow f$ **by** (*rule BiElim*)

hence 3: $\vdash \Box(bi\ f \longrightarrow f)$ **by** (*rule BoxGen*)

have 4: $\vdash \Box(bi\ f \longrightarrow f) \longrightarrow \Box(bi\ f) \longrightarrow \Box f$ **by** (*rule BoxImpDist*)

have 5: $\vdash \Box(bi\ f) \longrightarrow \Box f$ **using** 3 4 *MP* **by** *fastforce*

have 6: $\vdash \Box f \longrightarrow f$ **by** (*rule BoxElim*)

from 1 5 6 **show** *?thesis* **using** *BaImpBt lift-imp-trans* **by** *metis*

qed

lemma *DaIntro*:

$\vdash f \longrightarrow da\ f$

proof –

have 1: $\vdash ba\ (\neg\ f) \longrightarrow (\neg\ f)$ **by** (*rule BaElim*)

hence 2: $\vdash \neg\ \neg\ f \longrightarrow \neg (ba\ (\neg\ f))$ **by** *fastforce*

have 3: $\vdash f = (\neg\ \neg\ f)$ **by** *simp*

have 4: $\vdash da\ f = (\neg (ba\ (\neg\ f)))$ **by** (*rule DaEqvNotBaNot*)

from 2 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *BaGen*:

assumes $\vdash f$

shows $\vdash ba\ f$

proof –

have 1: $\vdash f$ **using** *assms* **by** *auto*

hence 2: $\vdash \Box f$ **by** (*rule BoxGen*)

hence 3: $\vdash bi(\Box f)$ **by** (*rule BiGen*)

have 4: $\vdash ba\ f = bi(\Box f)$ **by** (*rule BaEqvBiBt*)

from 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *BaImpDist*:

$\vdash ba\ (f \longrightarrow g) \longrightarrow ba\ f \longrightarrow ba\ g$

proof –

have 1: $\vdash bi\ (f \longrightarrow g) \longrightarrow (bi\ f \longrightarrow bi\ g)$ **by** (*rule BiImpDist*)

hence 2: $\vdash \Box(bi\ (f \longrightarrow g) \longrightarrow (bi\ f \longrightarrow bi\ g))$ **by** (*rule BoxGen*)

have 3: $\vdash \Box(bi\ (f \longrightarrow g) \longrightarrow (bi\ f \longrightarrow bi\ g))$

\longrightarrow
 $(\Box (bi (f \longrightarrow g)) \longrightarrow (\Box (bi f) \longrightarrow \Box (bi g)))$
by (*meson 2 BoxImpDist MP lift-imp-trans Prop01 Prop05 Prop09*)
have 4: $\vdash \Box (bi (f \longrightarrow g)) \longrightarrow (\Box (bi f) \longrightarrow \Box (bi g))$ **using** 2 3 *MP* **by** *fastforce*
have 5: $\vdash ba (f \longrightarrow g) = \Box (bi (f \longrightarrow g))$ **by** (*rule BaEqvBtBi*)
have 6: $\vdash ba f = \Box (bi f)$ **by** (*rule BaEqvBtBi*)
have 7: $\vdash ba g = \Box (bi g)$ **by** (*rule BaEqvBtBi*)
from 4 5 6 7 **show** *?thesis* **by** *fastforce*
qed

lemma *BaAndEqv*:

$\vdash ba (f \wedge g) = (ba f \wedge ba g)$
proof –
have 1: $\vdash ba (f \wedge g) = \Box (bi (f \wedge g))$
by (*rule BaEqvBtBi*)
have 2: $\vdash bi (f \wedge g) = (bi f \wedge bi g)$
by (*auto simp: itl-defs*)
hence 3: $\vdash \Box (bi (f \wedge g)) = \Box (bi f \wedge bi g)$
using *BoxEqvBox* **by** *blast*
have 4: $\vdash \Box (bi f \wedge bi g) = (\Box (bi f) \wedge \Box (bi g))$
by (*metis 2 BoxAndBoxEqvBoxRule integ-reflection*)
have 5: $\vdash ba f = \Box (bi f)$
by (*rule BaEqvBtBi*)
have 6: $\vdash ba g = \Box (bi g)$
by (*rule BaEqvBtBi*)
from 1 3 4 5 6 **show** *?thesis* **by** *fastforce*
qed

lemma *BaImpBaEqvBa*:

$\vdash ba (f = g) \longrightarrow (ba f = ba g)$
proof –
have 1: $\vdash ba (f \longrightarrow g) \longrightarrow ba f \longrightarrow ba g$ **by** (*rule BaImpDist*)
have 2: $\vdash ba (g \longrightarrow f) \longrightarrow ba g \longrightarrow ba f$ **by** (*rule BaImpDist*)
have 3: $\vdash ba (f = g) = ba ((f \longrightarrow g) \wedge (g \longrightarrow f))$ **by** (*auto simp: itl-defs*)
have 4: $\vdash ba ((f \longrightarrow g) \wedge (g \longrightarrow f)) = (ba((f \longrightarrow g)) \wedge ba((g \longrightarrow f)))$ **by** (*rule BaAndEqv*)
have 5: $\vdash ((ba f \longrightarrow ba g) \wedge (ba g \longrightarrow ba f)) = (ba f = ba g)$ **by** *auto*
from 1 2 3 4 5 **show** *?thesis* **by** *fastforce*
qed

lemma *BaImpBa*:

assumes $\vdash f \longrightarrow g$
shows $\vdash ba f \longrightarrow ba g$
using *BaGen BaImpDist MP assms* **by** *metis*

lemma *BaEqvBa*:

assumes $\vdash f = g$
shows $\vdash ba f = ba g$
using *BaGen BaImpBaEqvBa MP assms* **by** *metis*

lemma *DaImpDa*:

assumes $\vdash f \longrightarrow g$
shows $\vdash da\ f \longrightarrow da\ g$
using *assms* **by** (*metis DaEqvDtDi DiAndB DiamondImpDiamond inteq-reflection Prop10*)

lemma *DiamondEqvDiamondDiamond*:

$\vdash \Diamond f = \Diamond (\Diamond f)$

proof –

have 1: $\vdash \Diamond (\Diamond f) = \#True;(\#True;f)$
 by (*simp add: sometimes-d-def*)
have 2: $\vdash \#True;(\#True;f) = (\#True;\#True);f$
 by (*rule ChopAssoc*)
have 3: $\vdash (\#True;\#True);f = \#True;f$
 using *LeftChopEqvChop TrueEqvTrueChopTrue* **by** (*metis int-eq*)
have 4: $\vdash \#True;f = \Diamond f$
 by (*simp add: sometimes-d-def*)
from 1 2 3 4 **show** ?thesis **by** *fastforce*
qed

lemma *DaEqvDaDa*:

$\vdash da\ f = da\ (da\ f)$

proof –

have 1: $\vdash da\ f = \Diamond (di\ f)$
 by (*rule DaEqvDtDi*)
have 2: $\vdash di\ f = (di\ (di\ f))$
 by (*rule DiEqvDiDi*)
hence 3: $\vdash \Diamond (di\ f) = \Diamond (di\ (di\ f))$
 by (*rule DiamondEqvDiamond*)
have 4: $\vdash \Diamond (di\ f) = \Diamond (\Diamond (di\ (di\ f)))$
 using *DiamondEqvDiamondDiamond DiEqvDiDi* **using** 3 **by** *fastforce*
have 5: $\vdash \Diamond (di\ (di\ f)) = di\ (\Diamond (di\ f))$
 by (*rule DtDiEqvDiDt*)
hence 6: $\vdash \Diamond (\Diamond (di\ (di\ f))) = \Diamond (di\ (\Diamond (di\ f)))$
 by (*rule DiamondEqvDiamond*)
have 7: $\vdash da\ f = \Diamond (di\ (\Diamond (di\ f)))$
 using 1 3 4 6 **by** *fastforce*
have 8: $\vdash da\ (\Diamond (di\ f)) = \Diamond (di\ (\Diamond (di\ f)))$
 by (*rule DaEqvDtDi*)
have 9: $\vdash da\ (da\ f) = da\ (\Diamond (di\ f))$
 using 1 **by** (*rule DaEqvDa*)
from 7 8 9 **show** ?thesis **by** *fastforce*
qed

lemma *BaEqvBaBa*:

$\vdash ba\ f = ba\ (ba\ f)$

proof –

have 1: $\vdash da\ (\neg f) = da\ (da\ (\neg f))$ **by** (*rule DaEqvDaDa*)
have 2: $\vdash da\ (da\ (\neg f)) = (\neg (ba\ (\neg (da\ (\neg f)))))$ **by** (*rule DaEqvNotBaNot*)
have 3: $\vdash (\neg (da\ (da\ (\neg f)))) = ba\ (\neg (da\ (\neg f)))$ **by** (*auto simp: ba-d-def*)
have 4: $\vdash (\neg (da\ (\neg f))) = ba\ (\neg (da\ (\neg f)))$ **using** 1 2 3 **by** *fastforce*

from 4 show ?thesis by (metis ba-d-def)
qed

lemma *BaLeftChopImpChop*:

$\vdash \text{ba } (f \longrightarrow f1) \longrightarrow f; g \longrightarrow f1; g$

proof –

have 1: $\vdash \text{ba } (f \longrightarrow f1) \longrightarrow \text{bi } (f \longrightarrow f1)$ by (rule *BaImpBi*)

have 2: $\vdash \text{bi } (f \longrightarrow f1) \longrightarrow f; g \longrightarrow f1; g$ by (rule *BiChopImpChop*)

from 1 2 show ?thesis by fastforce

qed

lemma *BaRightChopImpChop*:

$\vdash \text{ba } (g \longrightarrow g1) \longrightarrow f; g \longrightarrow f; g1$

proof –

have 1: $\vdash \text{ba } (g \longrightarrow g1) \longrightarrow \Box(g \longrightarrow g1)$ by (rule *BaImpBt*)

have 2: $\vdash \Box(g \longrightarrow g1) \longrightarrow f; g \longrightarrow f; g1$ by (rule *BoxChopImpChop*)

from 1 2 show ?thesis by fastforce

qed

lemma *ChopAndBaImport*:

$\vdash (f; f1) \wedge \text{ba } g \longrightarrow (f \wedge g); (f1 \wedge g)$

proof –

have 1: $\vdash \text{ba } g \wedge (f; f1) \longrightarrow (g \wedge f); (g \wedge f1)$ by (rule *BaAndChopImport*)

have 2: $\vdash (g \wedge f); (g \wedge f1) = (f \wedge g); (f1 \wedge g)$ by (rule *AndChopAndCommute*)

from 1 2 show ?thesis by fastforce

qed

lemma *BaImpBaImpBaAnd*:

$\vdash \text{ba } h \longrightarrow \text{ba}(g \longrightarrow \text{ba } h \wedge g)$

proof –

have 1: $\vdash \text{ba } h \longrightarrow (g \longrightarrow \text{ba } h \wedge g)$ by fastforce

hence 2: $\vdash \text{ba}(\text{ba } h) \longrightarrow \text{ba}(g \longrightarrow \text{ba } h \wedge g)$ by (rule *BaImpBa*)

have 3: $\vdash \text{ba } h = \text{ba}(\text{ba } h)$ by (rule *BaEqvBaBa*)

from 2 3 show ?thesis by fastforce

qed

lemma *BaChopImpChopBa*:

$\vdash \text{ba } f \longrightarrow g; g1 \longrightarrow g; ((\text{ba } f) \wedge g1)$

proof –

have 1: $\vdash \text{ba } f \longrightarrow \text{ba } (g1 \longrightarrow (\text{ba } f) \wedge g1)$ by (rule *BaImpBaImpBaAnd*)

have 2: $\vdash \text{ba } (g1 \longrightarrow \text{ba } f \wedge g1) \longrightarrow g; g1 \longrightarrow g; (\text{ba } f \wedge g1)$ by (rule *BaRightChopImpChop*)

from 1 2 show ?thesis by fastforce

qed

lemma *DiNotBaImpNotBa*:

$\vdash \text{di } (\neg (\text{ba } f)) \longrightarrow \neg (\text{ba } f)$

proof –

have 1: $\vdash ba\ f = ba\ (ba\ f)$ **by** (rule BaEqvBaBa)
have 2: $\vdash ba\ (ba\ f) \longrightarrow bi\ (ba\ f)$ **by** (rule BaImpBi)
have 3: $\vdash ba\ f \longrightarrow bi\ (ba\ f)$ **using** 1 2 **by** fastforce
hence 4: $\vdash ba\ f \longrightarrow \neg\ (di\ (\neg\ (ba\ f)))$ **by** (simp add: bi-d-def)
from 4 **show** ?thesis **by** fastforce
qed

lemma NotBaChopImpNotBa:

$\vdash (\neg\ (ba\ f)); g \longrightarrow \neg\ (ba\ f)$
proof –
have 1: $\vdash (\neg\ (ba\ f)); g \longrightarrow di\ (\neg\ (ba\ f))$ **by** (rule ChopImpDi)
have 2: $\vdash di\ (\neg\ (ba\ f)) \longrightarrow \neg\ (ba\ f)$ **by** (rule DiNotBaImpNotBa)
from 1 2 **show** ?thesis **using** lift-imp-trans **by** blast
qed

lemma DiamondFinImpFin:

$\vdash \Diamond\ (fin\ f) \longrightarrow fin\ f$
proof –
have 1: $\vdash fin\ f = \#True;(f \wedge empty)$
by (rule FinEqvTrueChopAndEmpty)
hence 2: $\vdash \Diamond\ (fin\ f) = \#True;(\#True;(f \wedge empty))$
by (metis ChopEqvChop TrueEqvTrueChopTrue inteq-reflection sometimes-d-def)
have 3: $\vdash \#True;(\#True;(f \wedge empty)) = (\#True;\#True);(f \wedge empty)$
by (rule ChopAssoc)
have 4: $\vdash (\#True;\#True);(f \wedge empty) = \#True;(f \wedge empty)$
using TrueEqvTrueChopTrue **using** LeftChopEqvChop **by** (metis int-eq)
from 1 2 3 4 **show** ?thesis **by** fastforce
qed

lemma ChopFinImpFin:

$\vdash f; fin\ (init\ w) \longrightarrow fin\ (init\ w)$
proof –
have 1: $\vdash f; fin\ (init\ w) \longrightarrow \Diamond\ (fin\ (init\ w))$ **by** (rule ChopImpDiamond)
have 2: $\vdash \Diamond\ (fin\ (init\ w)) \longrightarrow fin\ (init\ w)$ **by** (rule DiamondFinImpFin)
from 1 2 **show** ?thesis **using** lift-imp-trans **by** blast
qed

lemma FinImpYieldsFin:

$\vdash fin\ (init\ w) \longrightarrow f\ yields\ (fin\ (init\ w))$
proof –
have 1: $\vdash f; fin\ (init\ (\neg\ w)) \longrightarrow fin\ (init\ (\neg\ w))$
by (rule ChopFinImpFin)
have 2: $\vdash fin\ (init\ (\neg\ w)) = (\neg\ (fin\ (init\ w)))$
using FinNotStateEqvNotFinState **by** blast
hence 3: $\vdash f; fin\ (init\ (\neg\ w)) = f; (\neg\ (fin\ (init\ w)))$
by (rule RightChopEqvChop)
have 4: $\vdash f; (\neg\ (fin\ (init\ w))) \longrightarrow \neg\ (fin\ (init\ w))$
using 1 2 3 **by** fastforce
hence 5: $\vdash fin\ (init\ w) \longrightarrow \neg\ (f; (\neg\ (fin\ (init\ w))))$

by fastforce
 from 5 show ?thesis by (simp add: yields-d-def)
 qed

lemma ChopAndFin:

$\vdash ((f; g) \wedge \text{fin } (\text{init } w)) = f; (g \wedge \text{fin } (\text{init } w))$
 proof –
 have 1: $\vdash \text{fin } (\text{init } w) \longrightarrow f \text{ yields } (\text{fin } (\text{init } w))$
 by (rule FinImpYieldsFin)
 hence 2: $\vdash (f; g) \wedge \text{fin } (\text{init } w) \longrightarrow (f; g) \wedge f \text{ yields } (\text{fin } (\text{init } w))$
 by auto
 have 3: $\vdash (f; g) \wedge f \text{ yields } (\text{fin } (\text{init } w)) \longrightarrow f; (g \wedge \text{fin } (\text{init } w))$
 by (rule ChopAndYieldsImp)
 have 4: $\vdash (f; g) \wedge \text{fin } (\text{init } w) \longrightarrow f; (g \wedge \text{fin } (\text{init } w))$
 using 2 3 by fastforce
 have 11: $\vdash f; (g \wedge \text{fin } (\text{init } w)) \longrightarrow f; g$
 by (rule ChopAndA)
 have 12: $\vdash f; (g \wedge \text{fin } (\text{init } w)) \longrightarrow f; \text{fin } (\text{init } w)$
 by (rule ChopAndB)
 have 13: $\vdash f; \text{fin } (\text{init } w) \longrightarrow \Diamond (\text{fin } (\text{init } w))$
 by (rule ChopImpDiamond)
 have 14: $\vdash \Diamond (\text{fin } (\text{init } w)) \longrightarrow \text{fin } (\text{init } w)$
 by (rule DiamondFinImpFin)
 have 15: $\vdash f; (g \wedge \text{fin } (\text{init } w)) \longrightarrow (f; g) \wedge \text{fin } (\text{init } w)$
 using 11 12 13 14 by fastforce
 from 4 15 show ?thesis by fastforce
 qed

lemma ChopAndNotFin:

$\vdash (f; g \wedge \neg (\text{fin } (\text{init } w))) = f; (g \wedge \neg (\text{fin } (\text{init } w)))$
 proof –
 have 1: $\vdash (f; g \wedge \text{fin } (\text{init } (\neg w))) = f; (g \wedge \text{fin } (\text{init } (\neg w)))$
 by (rule ChopAndFin)
 have 2: $\vdash \text{fin } (\text{init } (\neg w)) = (\neg (\text{fin } (\text{init } w)))$
 using FinNotStateEqvNotFinState by blast
 hence 3: $\vdash (g \wedge \text{fin } (\text{init } (\neg w))) = (g \wedge \neg (\text{fin } (\text{init } w)))$
 by auto
 hence 4: $\vdash f; (g \wedge \text{fin } (\text{init } (\neg w))) = f; (g \wedge \neg (\text{fin } (\text{init } w)))$
 by (rule RightChopEqvChop)
 from 1 2 4 show ?thesis by fastforce
 qed

lemma FinChopChain:

$\vdash ((\text{init } w) \longrightarrow \text{fin } (\text{init } w1)); ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))$
 $\longrightarrow ((\text{init } w) \longrightarrow \text{fin } (\text{init } w2))$
 proof –
 have 1: $\vdash (\text{init } w) \wedge ((\text{init } w) \longrightarrow \text{fin } (\text{init } w1)); ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))$
 \longrightarrow
 $(\text{init } w) \wedge ((\text{init } w) \longrightarrow \text{fin } (\text{init } w1)); ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))$

by (*rule StateAndChopImport*)
have 2: $\vdash (\text{init } w) \wedge ((\text{init } w) \longrightarrow \text{fin } (\text{init } w1)) \longrightarrow \text{fin } (\text{init } w1)$
by *auto*
have 3: $\vdash ((\text{init } w) \wedge ((\text{init } w) \longrightarrow \text{fin } (\text{init } w1))); ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))$
 \longrightarrow
 $(\text{fin } (\text{init } w1)); ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))$
using 2 **by** (*rule LeftChopImpChop*)
have 4: $\vdash (\text{fin } (\text{init } w1)); ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2)) =$
 $\Diamond((\text{init } w1) \wedge ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2)))$
by (*rule FinChopEqvDiamond*)
have 41: $\vdash ((\text{init } w1) \wedge ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))) \longrightarrow \text{fin } (\text{init } w2)$
by *auto*
have 42: $\vdash \Diamond((\text{init } w1) \wedge ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))) \longrightarrow \Diamond(\text{fin } (\text{init } w2))$
using 41 *DiamondImpDiamond* **by** *blast*
have 5: $\vdash \Diamond(\text{fin } (\text{init } w2)) \longrightarrow \text{fin } (\text{init } w2)$
using *DiamondFinImpFin* **by** *blast*
have 6: $\vdash (\text{init } w) \wedge ((\text{init } w) \longrightarrow \text{fin } (\text{init } w1)); ((\text{init } w1) \longrightarrow \text{fin } (\text{init } w2))$
 $\longrightarrow \text{fin } (\text{init } w2)$
using 1 3 4 5 42 **by** *fastforce*
from 6 **show** ?thesis **by** *fastforce*
qed

lemma *ChopRule*:

assumes $\vdash (\text{init } w) \wedge f \longrightarrow \text{fin } (\text{init } w1)$
 $\vdash (\text{init } w1) \wedge f1 \longrightarrow \text{fin } (\text{init } w2)$
shows $\vdash (\text{init } w) \wedge (f; f1) \longrightarrow \text{fin } (\text{init } w2)$
proof –
have 1: $\vdash (\text{init } w) \wedge (f; f1) \longrightarrow ((\text{init } w) \wedge f); f1$ **by** (*rule StateAndChopImport*)
have 2: $\vdash (\text{init } w) \wedge f \longrightarrow \text{fin } (\text{init } w1)$ **using** *assms* **by** *auto*
hence 3: $\vdash ((\text{init } w) \wedge f); f1 \longrightarrow (\text{fin } (\text{init } w1)); f1$ **by** (*rule LeftChopImpChop*)
have 4: $\vdash (\text{fin } (\text{init } w1)); f1 = \Diamond((\text{init } w1) \wedge f1)$ **by** (*rule FinChopEqvDiamond*)
have 5: $\vdash (\text{init } w1) \wedge f1 \longrightarrow \text{fin } (\text{init } w2)$ **using** *assms* **by** *auto*
hence 6: $\vdash \Diamond((\text{init } w1) \wedge f1) \longrightarrow \Diamond(\text{fin } (\text{init } w2))$ **by** (*rule DiamondImpDiamond*)
have 7: $\vdash \Diamond(\text{fin } (\text{init } w2)) \longrightarrow \text{fin } (\text{init } w2)$ **using** *DiamondFinImpFin* **by** *blast*
from 1 3 4 6 7 **show** ?thesis **by** *fastforce*
qed

lemma *ChopRep*:

assumes $\vdash (\text{init } w) \wedge f \longrightarrow f1 \wedge \text{fin } (\text{init } w1)$
 $\vdash (\text{init } w1) \wedge g \longrightarrow g1$
shows $\vdash (\text{init } w) \wedge (f; g) \longrightarrow (f1; g1)$
proof –
have 1: $\vdash (\text{init } w) \wedge f \longrightarrow f1 \wedge \text{fin } (\text{init } w1)$ **using** *assms* **by** *auto*
hence 2: $\vdash (\text{init } w) \wedge (f; g) \longrightarrow (f1 \wedge \text{fin } (\text{init } w1)); g$ **by** (*rule StateAndChopImpChopRule*)
have 3: $\vdash (f1 \wedge \text{fin } (\text{init } w1)); g = f1; ((\text{init } w1) \wedge g)$ **by** (*rule AndFinChopEqvStateAndChop*)
have 4: $\vdash (\text{init } w1) \wedge g \longrightarrow g1$ **using** *assms* **by** *auto*
hence 5: $\vdash f1; ((\text{init } w1) \wedge g) \longrightarrow f1; g1$ **by** (*rule RightChopImpChop*)
from 2 3 5 **show** ?thesis **by** *fastforce*
qed

lemma *ChopRepAndFin*:

assumes $\vdash (\text{init } w) \wedge f \longrightarrow f1 \wedge \text{fin } (\text{init } w1)$
 $\vdash (\text{init } w1) \wedge g \longrightarrow g1 \wedge \text{fin } (\text{init } w2)$
shows $\vdash (\text{init } w) \wedge (f; g) \longrightarrow (f1; g1) \wedge \text{fin } (\text{init } w2)$

proof –

have 1: $\vdash (\text{init } w) \wedge f \longrightarrow f1 \wedge \text{fin } (\text{init } w1)$ **using** *assms* **by** *auto*
have 2: $\vdash (\text{init } w1) \wedge g \longrightarrow g1 \wedge \text{fin } (\text{init } w2)$ **using** *assms* **by** *auto*
have 3: $\vdash (\text{init } w) \wedge (f; g) \longrightarrow f1; (g1 \wedge \text{fin } (\text{init } w2))$ **using** 1 2 **by** (*rule ChopRep*)
have 4: $\vdash f1; (g1 \wedge \text{fin } (\text{init } w2)) \longrightarrow f1; g1$ **by** (*rule ChopAndA*)
have 5: $\vdash f1; (g1 \wedge \text{fin } (\text{init } w2)) \longrightarrow f1; \text{fin } (\text{init } w2)$ **by** (*rule ChopAndB*)
have 6: $\vdash f1; \text{fin } (\text{init } w2) \longrightarrow \text{fin } (\text{init } w2)$ **by** (*rule ChopFinImpFin*)
show ?thesis **by** (*meson* 1 2 3 4 *ChopRule Prop12 lift-imp-trans*)
qed

lemma *TrueChopMoreEqvMore*:

$\vdash \# \text{True} ; \text{more} = \text{more}$

by (*metis ChopMoreImpMore NowImpDiamond TrueChopEqvDiamond int-eq int-iffI*)

lemma *MoreChopLoop*:

assumes $\vdash f \longrightarrow \text{more} ; f$

shows $\vdash \neg f$

proof –

have 1: $\vdash f \longrightarrow \text{more} ; f$
using *assms* **by** *auto*
hence 11: $\vdash \Diamond f \longrightarrow \Diamond (\text{more};f)$
by (*rule DiamondImpDiamond*)
have 12: $\vdash \Diamond (\text{more};f) = \# \text{True};(\text{more};f)$
by (*simp add: sometimes-d-def*)
have 13: $\vdash \# \text{True};(\text{more};f) = (\# \text{True};\text{more});f$
by (*rule ChopAssoc*)
have 14: $\vdash \Diamond (\text{more};f) = \text{more};f$
using *TrueChopMoreEqvMore* 12 13 **by** (*metis int-eq*)
have 2: $\vdash \text{more} ; f = \bigcirc(\Diamond f)$
by (*rule MoreChopEqvNextDiamond*)
have 3: $\vdash \Diamond f \longrightarrow \bigcirc(\Diamond f)$
using 11 14 2 **by** *fastforce*
hence 4: $\vdash \neg(\Diamond f)$
by (*rule NextLoop*)
have 5: $\vdash \neg(\Diamond f) \longrightarrow \neg f$
using *NowImpDiamond* **by** *fastforce*
from 4 5 **show** ?thesis **using** *MP* **by** *blast*
qed

lemma *MoreChopContra*:

assumes $\vdash f \wedge \neg g \longrightarrow (\text{more} ; (f \wedge \neg g))$

shows $\vdash f \longrightarrow g$

proof –

have 1: $\vdash f \wedge \neg g \longrightarrow (\text{more} ; (f \wedge \neg g))$ **using** *assms* **by** *auto*

hence 2: $\vdash \neg (f \wedge \neg g)$ by (rule *MoreChopLoop*)
 from 2 show ?thesis by auto
 qed

lemma *ChopLoop*:
 assumes $\vdash f \longrightarrow g;f$
 $\vdash g \longrightarrow \text{more}$
 shows $\vdash \neg f$
proof –
 have 1: $\vdash f \longrightarrow g; f$ using *assms* by auto
 have 2: $\vdash g \longrightarrow \text{more}$ using *assms* by auto
 hence 3: $\vdash g; f \longrightarrow \text{more} ; f$ by (rule *LeftChopImpChop*)
 have 4: $\vdash f \longrightarrow \text{more} ; f$ using 1 3 by *fastforce*
 from 4 show ?thesis using *MoreChopLoop* by auto
 qed

lemma *ChopContra*:
 assumes $\vdash f \wedge \neg g \longrightarrow h; f \wedge \neg (h; g)$
 $\vdash h \longrightarrow \text{more}$
 shows $\vdash f \longrightarrow g$
proof –
 have 1: $\vdash f \wedge \neg g \longrightarrow h; f \wedge \neg (h; g)$ using *assms* by auto
 have 2: $\vdash h \longrightarrow \text{more}$ using *assms* by auto
 have 3: $\vdash h; f \wedge \neg (h; g) \longrightarrow h; (f \wedge \neg g)$ by (rule *ChopAndNotChopImp*)
 have 4: $\vdash h; (f \wedge \neg g) \longrightarrow \text{more} ; (f \wedge \neg g)$ using 2 by (rule *LeftChopImpChop*)
 have 5: $\vdash f \wedge \neg g \longrightarrow \text{more} ; (f \wedge \neg g)$ using 1 3 4 by *fastforce*
 from 5 show ?thesis using *MoreChopContra* by auto
 qed

5.7 Properties of Chopstar and Chopplus

lemma *Chopstardef*:
 $\vdash \text{chopstar } f = \text{powerstar } (f \wedge \text{more})$
 by (simp add: *chopstar-d-def*)

lemma *AndEmptyChopAndEmptyEqvAndEmpty*:
 $\vdash (f \wedge \text{empty}); (f \wedge \text{empty}) = (f \wedge \text{empty})$
 by (auto simp add: *Valid-def itl-defs*) (metis *INil-ilen*)

lemma *PowerCommute*:
 $\vdash f ; \text{power } f \ n = \text{power } f \ n ; f$
proof
 (induct *n*)
 case 0
 then show ?case
 by (metis *ChopEmpty EmptyChop inteq-reflection power-d.pow-0*)
 next
 case (Suc *n*)
 then show ?case
 by (metis *ChopAssoc inteq-reflection power-d.pow-Suc*)

qed

lemma *ChopInductL*:

assumes $\vdash g \vee f; h \longrightarrow h$

shows $\vdash (\text{power } f \ n); g \longrightarrow h$

proof

(*induct n*)

case 0

then show ?case **using** *EmptyChop assms*

by (*metis MP Prop12 int-eq int-iffD1 int-simps(33) pow-0*)

next

case (*Suc n*)

then show ?case **using** *assms*

by (*metis ChopAndA ChopAssoc Prop03 Prop10 Prop12 inteq-reflection lift-and-com pow-Suc*)

qed

lemma *ChopInductMoreL*:

assumes $\vdash g \vee ((f \wedge \text{more}); h) \longrightarrow h$

shows $\vdash (\text{power } f \ n); g \longrightarrow h$

proof

(*induct n*)

case 0

then show ?case **using** *assms* **by** (*metis ChopInductL pow-0*)

next

case (*Suc n*)

then show ?case

proof –

have 1: $\vdash \text{power } f \ (\text{Suc } n); g = (f; \text{power } f \ n); g$

by *simp*

have 2: $\vdash (f; \text{power } f \ n); g = f; ((\text{power } f \ n); g)$

by (*meson ChopAssoc Prop11*)

have 3: $\vdash f; ((\text{power } f \ n); g) \longrightarrow f; h$

by (*simp add: RightChopImpChop Suc.hyps*)

have 31: $\vdash f = ((f \wedge \text{more}) \vee (f \wedge \text{empty}))$

unfolding *empty-d-def* **by** *fastforce*

have 4: $\vdash f; h = ((f \wedge \text{more}); h \vee ((f \wedge \text{empty}); h))$

using 31 **by** (*simp add: OrChopEqvRule*)

have 5: $\vdash ((f \wedge \text{more}); h) \longrightarrow h$ **using** *assms* **by** *auto*

have 6: $\vdash ((f \wedge \text{empty}); h) \longrightarrow h$

by (*meson AndChopB EmptyChop Prop11 lift-imp-trans*)

from 5 6 4 3 2 1 **show** ?thesis **by** *fastforce*

qed

qed

lemma *ChopInductR*:

assumes $\vdash g \vee h; f \longrightarrow h$

shows $\vdash g; (\text{power } f \ n) \longrightarrow h$

proof

(*induct n*)

case 0

```

then show ?case using ChopEmpty assms
by (metis MP Prop12 int-iffD2 int-simps(33) inteq-reflection pow-0)
next
case (Suc n)
then show ?case using assms
by (metis AndChopA ChopAssoc PowerCommute Prop05 Prop10 inteq-reflection lift-and-com
      lift-imp-trans pow-Suc)
qed

```

```

lemma ChopExistPower:
   $\vdash (g; (\exists n. \text{power } f \ n)) = (\exists n. g; \text{power } f \ n)$ 
using ChopExist by fastforce

```

```

lemma ExistChopPower:
   $\vdash (\exists n. (\text{power } f \ n); g) = (\exists n. \text{power } f \ n); g$ 
using ExistChop by fastforce

```

```

lemma PowerStarCommute:
   $\vdash f; (\exists n. \text{power } f \ n) = (\exists n. \text{power } f \ n); f$ 
proof –
  have 1:  $\vdash f; (\exists n. \text{power } f \ n) =$ 
     $(\exists n. f; \text{power } f \ n)$ 
    using ChopExistPower by blast
  have 2:  $\vdash (\exists n. f; \text{power } f \ n) =$ 
     $(\exists n. (\text{power } f \ n); f)$ 
    using PowerCommute by fastforce
  have 3:  $\vdash (\exists n. (\text{power } f \ n); f) =$ 
     $(\exists n. (\text{power } f \ n)); f$ 
    using ExistChopPower by blast
  from 1 2 3 show ?thesis by fastforce
qed

```

```

lemma PowerSucAndEmptyEqvAndEmpty:
   $\vdash (\text{power } (f \wedge \text{empty}) \ (Suc \ n)) = (f \wedge \text{empty})$ 
proof
  (induct n)
  case 0
  then show ?case using ChopEmpty
  by (metis pow-0 pow-Suc)
next
  case (Suc n)
  then show ?case
  by (metis AndEmptyChopAndEmptyEqvAndEmpty inteq-reflection pow-Suc)
qed

```

```

lemma PowerOr:
   $\vdash (\text{power } (f \vee g) \ (Suc \ n)) = ( (f; \text{power } (f \vee g) \ n) \vee$ 
     $(g; \text{power } (f \vee g) \ n))$ 
  by (simp add: OrChopEqvRule)

```


lemma *PowerEmptyOrMore*:

$\vdash (\text{power } (f \wedge \text{empty}) \vee (f \wedge \text{more})) (\text{Suc } n) =$
 $((f \wedge \text{empty}); (\text{power } (f \wedge \text{empty}) \vee (f \wedge \text{more})) n) \vee$
 $(f \wedge \text{more}); (\text{power } (f \wedge \text{empty}) \vee (f \wedge \text{more})) n)$

using *PowerOr* **by** *auto*

lemma *PSEqvEmptyOrChopPS*:

$\vdash \text{powerstar } f = (\text{empty} \vee f; \text{powerstar } f)$

using *PowerstarEqvSem Valid-def* **by** *blast*

lemma *EmptyImpCS*:

$\vdash \text{empty} \longrightarrow f^*$

proof –

have 1: $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$ **by** (*rule ChopstarEqv*)

have 2: $\vdash \text{empty} \longrightarrow \text{empty} \vee (f \wedge \text{more}); f^*$ **by** *auto*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *CSEqvOrChopCS*:

$\vdash f^* = (\text{empty} \vee (f; f^*))$

proof –

have 1: $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$ **by** (*rule ChopstarEqv*)

have 2: $\vdash (f \wedge \text{more}); f^* \longrightarrow f; f^*$ **by** (*rule AndChopA*)

have 3: $\vdash f^* \longrightarrow \text{empty} \vee f; f^*$ **using** 1 2 **by** (*metis int-iffD1 Prop08*)

have 4: $\vdash \text{empty} \longrightarrow f^*$ **by** (*rule EmptyImpCS*)

have 5: $\vdash f \longrightarrow \text{empty} \vee (f \wedge \text{more})$ **by** (*auto simp: empty-d-def*)

have 6: $\vdash f; f^* \longrightarrow f^* \vee (f \wedge \text{more}); f^*$ **using** 5 **by** (*rule EmptyOrChopImpRule*)

have 7: $\vdash f^* \longrightarrow \text{empty} \vee (f \wedge \text{more}); f^*$ **using** 1 **by** *fastforce*

have 8: $\vdash f; f^* \longrightarrow \text{empty} \vee (f \wedge \text{more}); f^*$ **using** 6 7 **by** *fastforce*

hence 9: $\vdash f; f^* \longrightarrow f^*$ **using** 1 **by** *fastforce*

have 10: $\vdash \text{empty} \vee f; f^* \longrightarrow f^*$ **using** 9 4 **by** *fastforce*

from 3 10 **show** *?thesis* **by** *fastforce*

qed

lemma *PowerChopCommute*:

$\vdash ((f \wedge \text{more}); \text{power } (f \wedge \text{more}) n) = \text{power } (f \wedge \text{more}) n; ((f \wedge \text{more}))$

using *PowerCommute* **by** *auto*

lemma *ChopExist*:

$\vdash (g; (\exists n. \text{power } (f \wedge \text{more}) n)) = (\exists n. g; \text{power } (f \wedge \text{more}) n)$

using *ChopExistPower* **by** *auto*

lemma *ExistChop*:

$\vdash (\exists n. (\text{power } (f \wedge \text{more}) n); g) = (\exists n. \text{power } (f \wedge \text{more}) n); g$

using *ExistChopPower* **by** *auto*

lemma *PowerstarInductL*:

assumes $\vdash g \vee f; h \longrightarrow h$

shows $\vdash (\text{powerstar } f); g \longrightarrow h$

proof –
have 1: $\vdash (\text{powerstar } f);g = (\exists n. \text{power } f \ n);g$
by (*simp add: powerstar-d-def LeftChopEqvChop*)
have 2: $\vdash (\exists n. \text{power } f \ n);g =$
 $(\exists n. (\text{power } f \ n);g)$
using *ExistChopPower* **by** *fastforce*
have 3: $\bigwedge n. \vdash (\text{power } f \ n);g \longrightarrow h$
using *ChopInductL assms* **by** *blast*
have 4: $\vdash (\exists n. ((\text{power } f \ n));g) \longrightarrow h$
using 3 **by** (*auto simp add: Valid-def*)
from 1 2 4 **show** ?thesis **by** (*metis inteq-reflection*)
qed

lemma *PowerstarInductMoreL*:

assumes $\vdash g \vee ((f \wedge \text{more}));h \longrightarrow h$

shows $\vdash (\text{powerstar } f);g \longrightarrow h$

proof –
have 1: $\vdash (\text{powerstar } f);g = (\exists n. \text{power } f \ n);g$
by (*simp add: powerstar-d-def LeftChopEqvChop*)
have 2: $\vdash (\exists n. \text{power } f \ n);g =$
 $(\exists n. (\text{power } f \ n);g)$
using *ExistChopPower* **by** *fastforce*
have 3: $\bigwedge n. \vdash (\text{power } f \ n);g \longrightarrow h$
using *ChopInductMoreL assms* **by** *blast*
have 4: $\vdash (\exists n. ((\text{power } f \ n));g) \longrightarrow h$
using 3 **by** (*auto simp add: Valid-def*)
from 1 2 4 **show** ?thesis **by** (*metis inteq-reflection*)
qed

lemma *ChopstarInductL*:

assumes $\vdash g \vee f;h \longrightarrow h$

shows $\vdash (\text{chopstar } f);g \longrightarrow h$

proof –
have 1: $\vdash (\text{chopstar } f);g = ((\exists n. \text{power } (f \wedge \text{more}) \ n));g$
by (*simp add: chopstar-d-def powerstar-d-def LeftChopEqvChop*)
have 2: $\vdash (\exists n. \text{power } (f \wedge \text{more}) \ n);g =$
 $(\exists n. (\text{power } (f \wedge \text{more}) \ n);g)$
using *ExistChopPower* **by** *fastforce*
have 21: $\vdash g \vee (f \wedge \text{more});h \longrightarrow h$
using *AndChopA Prop03 Prop10 assms int-simps(33) inteq-reflection* **by** *fastforce*
have 3: $\bigwedge n. \vdash (\text{power } (f \wedge \text{more}) \ n);g \longrightarrow h$
using 21 *ChopInductL[of g LIFT(f \wedge more) h]* *assms* **by** *auto*
have 4: $\vdash (\exists n. (\text{power } (f \wedge \text{more}) \ n);g) \longrightarrow h$
using 3 **by** *fastforce*
from 1 2 4 **show** ?thesis **by** (*metis inteq-reflection*)
qed

lemma *ChopstarInductMoreL*:

assumes $\vdash g \vee (f \wedge \text{more});h \longrightarrow h$

shows $\vdash (\text{chopstar } f);g \longrightarrow h$

proof –
have 1: $\vdash (\text{chopstar } f);g = ((\exists n. \text{power } (f \wedge \text{more}) n));g$
by (*simp add: chopstar-d-def powerstar-d-def LeftChopEqvChop*)
have 2: $\vdash (\exists n. \text{power } (f \wedge \text{more}) n);g =$
 $(\exists n. (\text{power } (f \wedge \text{more}) n);g)$
using *ExistChopPower* **by** *fastforce*
have 3: $\bigwedge n. \vdash (\text{power } (f \wedge \text{more}) n);g \longrightarrow h$
using *ChopInductL assms* **by** (*metis*)
have 4: $\vdash (\exists n. (\text{power } (f \wedge \text{more}) n);g) \longrightarrow h$
using 3 **by** *fastforce*
from 1 2 4 **show** ?thesis **by** (*metis inteq-reflection*)
qed

lemma *PowerstarInductR*:
assumes $\vdash g \vee h;f \longrightarrow h$
shows $\vdash g;(\text{powerstar } f) \longrightarrow h$
proof –
have 1: $\vdash g;(\text{powerstar } f) = g;(\exists n. \text{power } f n)$
by (*simp add: powerstar-d-def*)
have 2: $\vdash (g;(\exists n. \text{power } f n)) = (\exists n. g;(\text{power } f n))$
using *ChopExistPower* **by** *blast*
have 3: $\bigwedge n. \vdash g;(\text{power } f n) \longrightarrow h$
using *ChopInductR assms* **by** *blast*
have 4: $\vdash (\exists n. g;(\text{power } f n)) \longrightarrow h$
using 3 **by** (*auto simp add: Valid-def*)
from 1 2 4 **show** ?thesis **by** (*metis inteq-reflection*)
qed

lemma *ChopstarInductR*:
assumes $\vdash g \vee h;f \longrightarrow h$
shows $\vdash g;(\text{chopstar } f) \longrightarrow h$
proof –
have 1: $\vdash g;(\text{chopstar } f) =$
 $g;((\exists n. \text{power } (f \wedge \text{more}) n))$
by (*simp add: chopstar-d-def powerstar-d-def*)
have 2: $\vdash (g;(\exists n. \text{power } (f \wedge \text{more}) n)) =$
 $((\exists n. g;\text{power } (f \wedge \text{more}) n))$
using *ChopExistPower LeftChopEqvChop* **by** *fastforce*
have 21: $\vdash g \vee h;(f \wedge \text{more}) \longrightarrow h$
using *ChopAndA assms* **by** *fastforce*
have 3: $\bigwedge n. \vdash g;(\text{power } (f \wedge \text{more}) n) \longrightarrow h$
using 21 *ChopInductR[of g h LIFT(f \wedge more)] assms* **by** *auto*
have 4: $\vdash (\exists n. g;((\text{power } (f \wedge \text{more}) n))) \longrightarrow h$
using 3 **by** (*auto simp add: Valid-def*)
from 1 2 4 **show** ?thesis **by** *fastforce*
qed

lemma *ChopstarInductMoreR*:
assumes $\vdash g \vee h;(f \wedge \text{more}) \longrightarrow h$
shows $\vdash g;(\text{chopstar } f) \longrightarrow h$

proof –
have 1: $\vdash g;(\text{chopstar } f) = g;(\exists n. \text{power } (f \wedge \text{more}) n)$
by (*simp add: chopstar-d-def powerstar-d-def*)
have 2: $\vdash (g;(\exists n. \text{power } (f \wedge \text{more}) n)) =$
 $(\exists n. g;\text{power } (f \wedge \text{more}) n)$
using *ChopExistPower LeftChopEqvChop* **by** *fastforce*
have 3: $\bigwedge n. \vdash g;(\text{power } (f \wedge \text{more}) n) \longrightarrow h$
using *ChopInductR assms* **by** (*metis*)
have 4: $\vdash (\exists n. g;(\text{power } (f \wedge \text{more}) n)) \longrightarrow h$
using 3 **by** (*auto simp add: Valid-def*)
from 1 2 4 **show** ?thesis **by** *fastforce*
qed

lemma *PSAndMoreImpPS*:

$\vdash \text{powerstar } (f \wedge \text{more}) \longrightarrow \text{powerstar } f$

proof –

have 2: $\vdash \text{empty} \vee ((f \wedge \text{more}); \text{powerstar } f \longrightarrow \text{powerstar } f)$
using *AndChopA PSEqvEmptyOrChopPS* **by** *fastforce*
have 3: $\vdash \text{powerstar } (f \wedge \text{more}); \text{empty} \longrightarrow \text{powerstar } f$
using 2 *PowerstarInductL* **by** *blast*
from 2 3 **show** ?thesis **by** (*metis ChopEmpty int-eq*)

qed

lemma *PSImpAndMorePS*:

$\vdash \text{powerstar } f \longrightarrow \text{powerstar } (f \wedge \text{more})$

by (*meson ChopEmpty PSEqvEmptyOrChopPS PowerstarInductMoreL int-iffD2 lift-imp-trans*)

lemma *FPSAndMoreEqvFPS*:

$\vdash \text{powerstar } (f \wedge \text{more}) = \text{powerstar } f$

using *PSAndMoreImpPS PSImpAndMorePS* **by** *fastforce*

lemma *ChopstarImpPowerstar*:

$\vdash f^* \longrightarrow \text{powerstar } f$

by (*metis ChopEmpty ChopstarInductL PSEqvEmptyOrChopPS int-eq int-iffD2*)

lemma *PowerstarImpChopstar*:

$\vdash \text{powerstar } f \longrightarrow f^*$

by (*metis CSEqvOrChopCS ChopEmpty PowerstarInductL int-iffD2 inteq-reflection*)

lemma *ChopstarEqvPowerstar*:

$\vdash f^* = \text{powerstar } f$

using *ChopstarImpPowerstar PowerstarImpChopstar* **by** *fastforce*

lemma *PowerchopAndMore*:

$\vdash ((\text{power } (f \wedge \text{more}) (\text{Suc } n)) \wedge \text{more}) = (\text{power } (f \wedge \text{more}) (\text{Suc } n))$

proof

(*induct n*)

case 0

then show ?case

by (*metis (no-types, lifting) AndChopB DiamondEmpty MoreChopEqvNextDiamond Prop10 int-eq-true*)

```

    inteq-reflection more-d-def pow-0 pow-Suc)
next
case (Suc n)
then show ?case
by (metis Prop10 Prop11 Prop12 RightChopImpMoreRule pow-Suc)
qed

lemma ExistPowerAndMoreExpand:
   $\vdash (\exists n. \text{power } (f \wedge \text{more}) n) = (\text{empty} \vee (\exists n. (\text{power } (f \wedge \text{more}) (\text{Suc } n))))$ 
using powersem1[of LIFT( $f \wedge \text{more}$ )] by auto

```

```

lemma CSAndMoreEqvAndMoreChop:
   $\vdash (f^* \wedge \text{more}) = (f \wedge \text{more}); f^*$ 
proof -
  have 1:  $\vdash (\text{empty} \vee (f \wedge \text{more}); f^*) \wedge \text{more} \longrightarrow (f \wedge \text{more}); f^*$ 
    by (auto simp: empty-d-def)
  have 2:  $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$ 
    by (rule ChopstarEqv)
  have 3:  $\vdash f^* \wedge \text{more} \longrightarrow (f \wedge \text{more}); f^*$ 
    using 1 2 by fastforce
  have 4:  $\vdash (f \wedge \text{more}); f^* \longrightarrow f^*$ 
    using 2 by fastforce
  have 5:  $\vdash (f \wedge \text{more}) \longrightarrow \text{more}$ 
    by auto
  hence 6:  $\vdash (f \wedge \text{more}); f^* \longrightarrow \text{more}$ 
    by (rule LeftChopImpMoreRule)
  have 7:  $\vdash (f \wedge \text{more}); f^* \longrightarrow f^* \wedge \text{more}$ 
    using 4 6 by fastforce
  from 3 7 show ?thesis by fastforce
qed

```

```

lemma CSAndMoreImpChopCS:
   $\vdash f^* \wedge \text{more} \longrightarrow f; f^*$ 
proof -
  have 1:  $\vdash (f^* \wedge \text{more}) = (f \wedge \text{more}); f^*$  by (rule CSAndMoreEqvAndMoreChop)
  have 2:  $\vdash (f \wedge \text{more}); f^* \longrightarrow f; f^*$  by (rule AndChopA)
  from 1 2 show ?thesis by fastforce
qed

```

```

lemma NotAndMoreEqvEmptyOr:
   $\vdash \neg (f \wedge \text{more}) = (\text{empty} \vee \neg f)$ 
by (auto simp: empty-d-def)

```

```

lemma MoreAndEmptyOrEqvMoreAnd:
   $\vdash (\text{more} \wedge (\text{empty} \vee \neg f)) = (\text{more} \wedge \neg f)$ 
by (auto simp: empty-d-def)

```

lemma *CSMoreNotImpChopCSAndMore*:

$\vdash f^* \wedge \text{more} \wedge \neg f \longrightarrow (f \wedge \text{more}); (f^* \wedge \text{more})$

proof –

have 1: $\vdash (f^* \wedge \text{more}) = (f \wedge \text{more}); f^*$

by (*rule CSAndMoreEqvAndMoreChop*)

have 2: $\vdash \text{empty} \vee \text{more}$

by (*auto simp: empty-d-def*)

hence 3: $\vdash f^* \longrightarrow \text{empty} \vee (f^* \wedge \text{more})$

by *auto*

hence 4: $\vdash (f \wedge \text{more}); f^* \longrightarrow (f \wedge \text{more}) \vee ((f \wedge \text{more}); (f^* \wedge \text{more}))$

by (*rule ChopEmptyOrImpRule*)

hence 5: $\vdash (f \wedge \text{more}); f^* \wedge \neg(f \wedge \text{more}) \longrightarrow ((f \wedge \text{more}); (f^* \wedge \text{more}))$

by *fastforce*

have 6: $\vdash (f \wedge \text{more}); f^* = ((f \wedge \text{more}); f^* \wedge \text{more})$ **using** 1

by *auto*

have 7: $\vdash ((f \wedge \text{more}); f^* \wedge \neg(f \wedge \text{more})) = ((f \wedge \text{more}); f^* \wedge \text{more} \wedge \neg(f \wedge \text{more}))$

using 6 **by** *auto*

have 8: $\vdash (f \wedge \text{more}); f^* \wedge \text{more} \wedge \neg f \longrightarrow (f \wedge \text{more}); (f^* \wedge \text{more})$

using 5 7 **by** *auto*

have 9: $\vdash (f^* \wedge \text{more} \wedge \neg f) = ((f^* \wedge \text{more}) \wedge (\text{more} \wedge \neg f))$

by *auto*

have 10: $\vdash ((f^* \wedge \text{more}) \wedge (\text{more} \wedge \neg f)) = ((f \wedge \text{more}); f^* \wedge (\text{more} \wedge \neg f))$

using 1 **by** *fastforce*

from 1 8 9 10 **show** *?thesis* **by** *fastforce*

qed

lemma *ChopplusCommuteImpA*:

$\vdash f^*;f \longrightarrow f;f^*$

by (*metis CSEqvOrChopCS ChopAndB ChopEmpty ChopstarInductL EmptyImpCS Prop02 Prop03 Prop10 inteq-reflection*)

lemma *ChopplusCommuteImpB*:

$\vdash f;f^* \longrightarrow f^*;f$

by (*metis ChopstarEqvPowerstar PowerStarCommute int-iffD1 inteq-reflection powerstar-d-def*)

lemma *ChopplusCommute*:

$\vdash f;f^* = f^*;f$

using *ChopplusCommuteImpA ChopplusCommuteImpB* **by** *fastforce*

lemma *CSEqvOrChopCSB*:

$\vdash f^* = (\text{empty} \vee (f^*;f))$

by (*meson CSEqvOrChopCS ChopplusCommute Prop06*)

lemma *CSAndMoreImpCSChop*:

$\vdash f^* \wedge \text{more} \longrightarrow f^*;f$

proof –

have 1: $\vdash (f^* \wedge \text{more}) = (f \wedge \text{more}); f^*$

```

    by (rule CSAndMoreEqvAndMoreChop)
have 2:  $\vdash \text{empty} \vee \text{more}$ 
    by (auto simp: empty-d-def)
have 3:  $\vdash f^* \longrightarrow \text{empty} \vee (f^* \wedge \text{more})$ 
    by auto
have 4:  $\vdash (f \wedge \text{more}); f^* \longrightarrow$ 
       $(f \wedge \text{more}) \vee ((f \wedge \text{more}); (f^* \wedge \text{more}))$ 
    by (rule ChopEmptyOrImpRule)
have 5:  $\vdash f^* \wedge \text{more} \wedge \neg f \longrightarrow (f \wedge \text{more}); (f^* \wedge \text{more})$ 
    by (rule CSMoreNotImpChopCSAndMore)
have 6:  $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$ 
    by (rule ChopstarEqv)
have 7:  $\vdash f^*; f = (f \vee ((f \wedge \text{more}); f^*); f)$ 
    by (rule EmptyOrChopEqvRule)
have 8:  $\vdash (f \wedge \text{more}); (f^*; f) = ((f \wedge \text{more}); f^*); f$ 
    by (rule ChopAssoc)
have 9:  $\vdash (f^* \wedge \text{more}) \wedge \neg (f^*; f) \longrightarrow$ 
       $(f \wedge \text{more}); (f^* \wedge \text{more}) \wedge \neg ((f \wedge \text{more}); (f^*; f))$ 
    using 5 7 8 by fastforce
have 10:  $\vdash f \wedge \text{more} \longrightarrow \text{more}$ 
    by auto
from 9 10 show ?thesis by (rule ChopContra)
qed

```

lemma *PowerChopPower*:

$\vdash (\text{power } (f \wedge \text{more}) \ n); (\text{power } (f \wedge \text{more}) \ k) = (\text{power } (f \wedge \text{more}) \ (n+k))$

proof

(induct n arbitrary: k)

case 0

then show ?case **using** *EmptyChopSem* **by** auto

next

case (Suc n)

then show ?case

by (metis (no-types, lifting) ChopAssoc add-Suc inteq-reflection pow-Suc)

qed

lemma *CSChopCS*:

$\vdash f^*; f^* = f^*$

by (metis CSEqvOrChopCS ChopEmpty ChopstarInductL EmptyImpCS Prop02 Prop03 Prop11 RightChopImpChop
inteq-reflection)

lemma *NotEmptyEqvMore*:

$\vdash (\neg \text{empty}) = \text{more}$

by (simp add: empty-d-def)

lemma *NotCSImpMore*:

$\vdash \neg (f^*) \longrightarrow \text{more}$

proof –

have 1: $\vdash \text{empty} \longrightarrow (f^*)$ **using** *EmptyImpCS* **by** blast

hence 2: $\vdash \neg \text{empty} \vee (f^*)$ by *fastforce*
 from 2 show ?thesis using 1 *NotEmptyEqvMore* by *fastforce*
 qed

lemma *CSChopCSImpCS*:

$\vdash f^*; f^* \longrightarrow f^*$

proof –

have 1: $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$

by (rule *ChopstarEqv*)

hence 2: $\vdash f^*; f^* = (f^* \vee ((f \wedge \text{more}); f^*); f^*)$

by (rule *EmptyOrChopEqvRule*)

have 21: $\vdash f^*; f^* \wedge \neg (f^*) \longrightarrow ((f \wedge \text{more}); f^*); f^*$

using 2 by *auto*

have 22: $\vdash \neg (f^*) = (\neg \text{empty} \wedge \neg ((f \wedge \text{more}); f^*))$

using 1 by *fastforce*

have 23: $\vdash \neg (f^*) \longrightarrow \neg ((f \wedge \text{more}); f^*)$

using 2 22 by *fastforce*

have 24: $\vdash f^*; f^* \wedge \neg (f^*) \longrightarrow \neg (f^*)$

by *auto*

have 25: $\vdash f^*; f^* \wedge \neg (f^*) \longrightarrow \neg ((f \wedge \text{more}); f^*)$

using 23 24 *MP* by *auto*

have 3: $\vdash f^*; f^* \wedge \neg (f^*) \longrightarrow ((f \wedge \text{more}); f^*); f^* \wedge \neg ((f \wedge \text{more}); f^*)$

using 21 25 by *fastforce*

have 4: $\vdash (f \wedge \text{more}); (f^*; f^*) = ((f \wedge \text{more}); f^*); f^*$

by (rule *ChopAssoc*)

have 5: $\vdash f^*; f^* \wedge \neg (f^*) \longrightarrow (f \wedge \text{more}); (f^*; f^*) \wedge \neg ((f \wedge \text{more}); f^*)$

using 3 4 by *fastforce*

have 6: $\vdash f \wedge \text{more} \longrightarrow \text{more}$

by *auto*

from 5 6 show ?thesis using *ChopContra* by *blast*

qed

lemma *ImpChopPlus*:

$\vdash f \longrightarrow f; f^*$

proof –

have 1: $\vdash f^* = (\text{empty} \vee f; f^*)$ by (rule *CSEqvOrChopCS*)

hence 2: $\vdash f; f^* = (f; \text{empty} \vee f; (f; f^*))$ using *ChopOrEqvRule* by *blast*

have 3: $\vdash f; \text{empty} = f$ using *ChopEmpty* by *blast*

from 2 3 show ?thesis by *fastforce*

qed

lemma *ImpCS*:

$\vdash f \longrightarrow f^*$

proof –

have 1: $\vdash f \longrightarrow f; f^*$ by (rule *ImpChopPlus*)

hence 2: $\vdash f \longrightarrow \text{empty} \vee f; f^*$ by *auto*

from 2 show ?thesis using *CSEqvOrChopCS* by *fastforce*

qed

lemma *CSChopImpCS*:
 $\vdash f^*; f \longrightarrow f^*$
proof –
have 1: $\vdash f \longrightarrow f^*$ **by** (rule *ImpCS*)
hence 2: $\vdash f^*; f \longrightarrow f^*; f^*$ **by** (rule *RightChopImpChop*)
have 3: $\vdash f^*; f^* \longrightarrow f^*$ **by** (rule *CSChopCSImpCS*)
from 2 3 **show** ?thesis **using** lift-imp-trans **by** blast
qed

lemma *ChopPlusImpCS*:
 $\vdash f; f^* \longrightarrow f^*$
proof –
have 1: $\vdash f; f^* \longrightarrow \text{empty} \vee f; f^*$ **by** auto
from 1 **show** ?thesis **using** CSEqvOrChopCS **by** fastforce
qed

lemma *CSChopEqvOrChopPlusChop*:
 $\vdash f^*; g = (g \vee (f; f^*) ; g)$
proof –
have 1: $\vdash f^* = (\text{empty} \vee f; f^*)$ **by** (rule *CSEqvOrChopCS*)
from 1 **show** ?thesis **using** EmptyOrChopEqvRule **by** blast
qed

lemma *CSElim*:
assumes $\vdash \text{empty} \longrightarrow g$
 $\vdash (f \wedge \text{more}) ; g \longrightarrow g$
shows $\vdash f^* \longrightarrow g$
proof –
have 1: $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}) ; f^*)$
by (rule *ChopstarEqv*)
have 2: $\vdash \text{empty} \longrightarrow g$
using assms **by** blast
have 3: $\vdash (f \wedge \text{more}) ; g \longrightarrow g$
using assms **by** blast
have 31: $\vdash \neg g \longrightarrow \text{more}$
using 2 **by** (auto simp: empty-d-def)
have 32: $\vdash \neg g \longrightarrow \neg ((f \wedge \text{more}) ; g)$
using 3 **by** fastforce
have 33: $\vdash f^* \wedge \text{more} \longrightarrow (f \wedge \text{more}) ; f^*$
using 1 **using** CSAndMoreEqvAndMoreChop **by** fastforce
have 34: $\vdash f^* \wedge \neg g \longrightarrow f^* \wedge \text{more}$
using 31 **by** auto
have 35: $\vdash f^* \wedge \neg g \longrightarrow (f \wedge \text{more}) ; f^*$
using 33 34 **by** fastforce
have 36: $\vdash f^* \wedge \neg g \longrightarrow \neg ((f \wedge \text{more}) ; g)$
using 32 **by** auto
have 4: $\vdash f^* \wedge \neg g \longrightarrow (f \wedge \text{more}) ; f^* \wedge \neg ((f \wedge \text{more}) ; g)$
using 35 36 **by** fastforce
have 5: $\vdash f \wedge \text{more} \longrightarrow \text{more}$

```

    by auto
  from 4 5 show ?thesis using ChopContra by blast
qed

lemma ChopstarImp:
  assumes  $\vdash f;(\text{chopstar } g) \vee \text{empty} \longrightarrow (\text{chopstar } g)$ 
  shows  $\vdash (\text{chopstar } f) \longrightarrow (\text{chopstar } g)$ 
  using assms ChopstarInductL ChopEmpty
  by (metis int-eq int-simps(33) lift-and-com)

lemma CSCSImpCS:
   $\vdash (f^*)^* \longrightarrow f^*$ 
proof -
  have 1:  $\vdash \text{empty} \longrightarrow f^*$  by (rule EmptyImpCS)
  have 2:  $\vdash (f^* \wedge \text{more}); f^* \longrightarrow f^*; f^*$  by (rule AndChopA)
  have 3:  $\vdash f^*; f^* \longrightarrow f^*$  by (rule CSChopCSImpCS)
  have 4:  $\vdash (f^* \wedge \text{more}); f^* \longrightarrow f^*$  using 2 3 lift-imp-trans by blast
  from 1 4 show ?thesis using CSElim by blast
qed

lemma CSImpCSCS:
   $\vdash f^* \longrightarrow (f^*)^*$ 
  using ImpCS by auto

lemma CSCSEqvCS:
   $\vdash (f^*)^* = f^*$ 
  by (simp add: CSCSImpCS CSImpCSCS int-iffI)

lemma RightEmptyOrChopEqv:
   $\vdash g;(\text{empty} \vee f) = (g \vee (g;f))$ 
proof -
  have 1:  $\vdash g;(\text{empty} \vee f) = (g;\text{empty} \vee g;f)$  by (rule ChopOrEqv)
  have 2:  $\vdash g;\text{empty} = g$  by (rule ChopEmpty)
  from 1 2 show ?thesis by fastforce
qed

lemma RightEmptyOrChopEqvRule:
  assumes  $\vdash f = (\text{empty} \vee f1)$ 
  shows  $\vdash g;f = (g \vee (g;f1))$ 
proof -
  have 1:  $\vdash f = (\text{empty} \vee f1)$  using assms by auto
  hence 2:  $\vdash g;f = g;(\text{empty} \vee f1)$  by (rule RightChopEqvChop)
  have 3:  $\vdash g;(\text{empty} \vee f1) = (g \vee (g;f1))$  by (rule RightEmptyOrChopEqv)
  from 2 3 show ?thesis by fastforce
qed

lemma ChopPlusEqvOrChopChopPlus:
   $\vdash (f;f^*) = (f \vee f; (f;f^*))$ 
proof -

```

have 1: $\vdash f^* = (\text{empty} \vee f; f^*)$ **by** (rule *CSEqvOrChopCS*)
from 1 **show** ?thesis **by** (rule *RightEmptyOrChopEqvRule*)
qed

lemma *CSAndEmptyEqvEmpty*:
 $\vdash ((f^*) \wedge \text{empty}) = \text{empty}$
using *EmptyImpCS* **by** *fastforce*

lemma *NotAndMoreChopAndEmpty*:
 $\vdash \neg(((f \wedge \text{more}); g) \wedge \text{empty})$
by (metis *AndChopA ChopEmpty LeftChopImpMoreRule Prop01 empty-d-def int-simps(14)*
int-simps(25) int-simps(4) inteq-reflection lift-and-com)

lemma *NotChopAndMoreAndEmpty*:
 $\vdash \neg((f; (g \wedge \text{more})) \wedge \text{empty})$
by (metis (no-types, lifting) *ChopAndEmptyEqvEmptyChopEmpty ChopEmpty ChopImpDiamond DiamondFin*
Finprop(1) NotEmptyEqvMore Prop12 always-d-def empty-d-def fin-d-def int-simps(14) int-simps(2)
int-simps(21) inteq-reflection sometimes-d-def)

lemma *ChopCSAndEmptyEqvAndEmpty*:
 $\vdash ((f; f^*) \wedge \text{empty}) = (f \wedge \text{empty})$
proof –
have 1: $\vdash ((f; f^*) \wedge \text{empty}) = (f \wedge \text{empty}); (f^* \wedge \text{empty})$
using *ChopAndEmptyEqvEmptyChopEmpty* **by** *blast*
have 2: $\vdash (f \wedge \text{empty}); (f^* \wedge \text{empty}) = (f \wedge \text{empty}); \text{empty}$
using *CSAndEmptyEqvEmpty* **using** *RightChopEqvChop* **by** *blast*
have 3: $\vdash (f \wedge \text{empty}); \text{empty} = (f \wedge \text{empty})$
by (rule *ChopEmpty*)
from 1 2 3 **show** ?thesis **by** *fastforce*
qed

lemma *AndMoreChopAndMoreEqvAndMoreChop*:
 $\vdash ((f \wedge \text{more}); g \wedge \text{more}) = (f \wedge \text{more}); g$
using *ChopImpDi DiAndB DiMoreEqvMore* **by** *fastforce*

lemma *ChopPlusEqv*:
 $\vdash (f; f^*) = (f \vee (f \wedge \text{more}); (f; f^*))$
proof –
have 1: $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$
by (rule *ChopstarEqv*)
have 2: $\vdash f^* = (\text{empty} \vee f; f^*)$
by (rule *CSEqvOrChopCS*)
hence 3: $\vdash (\text{empty} \vee f; f^*) = (\text{empty} \vee (f \wedge \text{more}); f^*)$
using 1 2 **by** *fastforce*
have 4: $\vdash (f \wedge \text{more}); (f^*) = (f \wedge \text{more}); (\text{empty} \vee f; f^*)$
using 2 **using** *RightChopEqvChop* **by** *blast*
hence 5: $\vdash \text{empty} \vee f; f^* = \text{empty} \vee (f \wedge \text{more}); (\text{empty} \vee f; f^*)$
using 3 4 **by** *fastforce*
have 6: $\vdash (f \wedge \text{more}); (\text{empty} \vee f; f^*) =$

```

      ((f ∧ more); empty ∨ (f ∧ more); (f;f*))
    using ChopOrEqv by blast
  have 7: ⊢ (f ∧ more); empty = (f ∧ more)
    using ChopEmpty by blast
  have 8: ⊢ (empty ∨ f;f*) =
    (empty ∨ (f ∧ more) ∨ (f ∧ more); (f;f*))
    using 5 6 7 by (metis 2 3 inteq-reflection)
  have 9: ⊢ ((empty ∨ f;f*) ∧ more) = (f;f* ∧ more)
    by (auto simp: empty-d-def)
  have 10: ⊢ ((empty ∨ (f ∧ more) ∨ (f ∧ more); (f;f*)) ∧ more) =
    (((f ∧ more) ∨ (f ∧ more); (f;f*)) ∧ more)
    by (auto simp: empty-d-def)
  have 12: ⊢ (f;f* ∧ more) = ((f ∧ more) ∨ (f ∧ more); (f;f*))
    by (metis 1 3 6 7 9 CSAndMoreEqvAndMoreChop inteq-reflection)
  have 13: ⊢ (f;f* ∧ empty) = (f ∧ empty)
    by (rule ChopCSAndEmptyEqvAndEmpty)
  have 14: ⊢ ((f ∧ more) ∨ (f ∧ more); (f;f*) ∨ (f ∧ empty)) =
    (f ∨ (f ∧ more); (f;f*))
    by (auto simp: empty-d-def)
  have 15: ⊢ f;f* = ((f;f* ∧ empty) ∨ (f;f* ∧ more))
    by (auto simp: empty-d-def)
  from 12 13 14 15 show ?thesis by fastforce
qed

```

lemma *ChopPlusImpChopPlus*:

```

  assumes ⊢ f ⟶ g
  shows ⊢ f;f* ⟶ g;g*
proof -
  have 1: ⊢ f ⟶ g
    using assms by auto
  have 2: ⊢ f;f* = (f ∨ (f ∧ more); (f;f*))
    by (rule ChopPlusEqv)
  have 3: ⊢ g;g* = (g ∨ (g ∧ more); (g;g*))
    by (rule ChopPlusEqv)
  have 4: ⊢ f;f* ∧ ¬ (g;g*) ⟶ ((f ∧ more); (f;f*)) ∧ ¬ ((g ∧ more); (g;g*))
    using 1 2 3 by fastforce
  have 5: ⊢ f ∧ more ⟶ g ∧ more using 1
    by auto
  have 6: ⊢ (f ∧ more); (f;f*) ⟶ (g ∧ more); (f;f*)
    using 5 by (rule LeftChopImpChop)
  have 7: ⊢ f;f* ∧ ¬ (g;g*) ⟶
    ((g ∧ more); (f;f*)) ∧ ¬ ((g ∧ more); (g;g*))
    using 4 6 by fastforce
  have 8: ⊢ g ∧ more ⟶ more
    by auto
  from 7 8 show ?thesis using ChopContra by blast
qed

```

lemma *ChopChopPlusImpChopPlus*:

$\vdash f; (f;f^*) \longrightarrow f;f^*$
proof –
have 1: $\vdash \text{empty} \vee \text{more}$ **by** (auto simp: empty-d-def)
hence 2: $\vdash f \longrightarrow \text{empty} \vee (f \wedge \text{more})$ **by** auto
hence 3: $\vdash f; (f;f^*) \longrightarrow (f;f^*) \vee (f \wedge \text{more}); (f;f^*)$ **by** (rule EmptyOrChopImpRule)
have 4: $\vdash f;f^* = (f \vee (f \wedge \text{more}); (f;f^*))$ **by** (rule ChopPlusEqv)
hence 5: $\vdash (f \wedge \text{more}); (f;f^*) \longrightarrow f;f^*$ **by** auto
from 3 5 **show** ?thesis **using** ChopPlusImpCS RightChopImpChop **by** blast
qed

lemma CSImpCS:
assumes $\vdash f \longrightarrow g$
shows $\vdash f^* \longrightarrow g^*$
proof –
have 1: $\vdash f \longrightarrow g$ **using** assms **by** auto
hence 2: $\vdash f;f^* \longrightarrow g;g^*$ **by** (rule ChopPlusImpChopPlus)
hence 3: $\vdash \text{empty} \vee f;f^* \longrightarrow \text{empty} \vee g;g^*$ **by** auto
from 2 3 **show** ?thesis **using** CSEqvOrChopCS **by** (metis inteq-reflection)
qed

lemma ChopPlusIntro:
assumes $\vdash f \wedge \neg g \longrightarrow (g \wedge \text{more}); f$
shows $\vdash f \longrightarrow g;g^*$
proof –
have 1: $\vdash f \wedge \neg g \longrightarrow (g \wedge \text{more}); f$ **using** assms **by** auto
have 2: $\vdash g;g^* = (g \vee (g \wedge \text{more}); (g;g^*))$ **by** (rule ChopPlusEqv)
have 3: $\vdash f \wedge \neg (g;g^*) \longrightarrow$
 $(g \wedge \text{more}); f \wedge \neg ((g \wedge \text{more}); (g;g^*))$ **using** 1 2 **by** fastforce
have 4: $\vdash g \wedge \text{more} \longrightarrow \text{more}$ **by** auto
from 3 4 **show** ?thesis **using** ChopContra **by** blast
qed

lemma ChopPlusElim:
assumes $\vdash f \longrightarrow g$
 $\vdash (f \wedge \text{more}); g \longrightarrow g$
shows $\vdash f;f^* \longrightarrow g$
proof –
have 1: $\vdash f;f^* = (f \vee (f \wedge \text{more}); (f;f^*))$ **by** (rule ChopPlusEqv)
have 2: $\vdash f \longrightarrow g$ **using** assms **by** blast
hence 21: $\vdash \neg g \longrightarrow \neg f$ **by** auto
have 3: $\vdash (f \wedge \text{more}); g \longrightarrow g$ **using** assms **by** blast
hence 31: $\vdash \neg g \longrightarrow \neg ((f \wedge \text{more}); g)$ **by** fastforce
hence 32: $\vdash f;f^* \wedge \neg g \longrightarrow \neg ((f \wedge \text{more}); g)$ **by** auto
have 33: $\vdash f;f^* \wedge \neg g \longrightarrow (f \wedge \text{more}); (f;f^*)$ **using** 1 21 **by** fastforce
have 4: $\vdash f;f^* \wedge \neg g \longrightarrow$
 $(f \wedge \text{more}); (f;f^*) \wedge \neg ((f \wedge \text{more}); g)$ **using** 31 33 **by** fastforce
have 5: $\vdash f \wedge \text{more} \longrightarrow \text{more}$ **by** auto
from 4 5 **show** ?thesis **using** ChopContra **by** blast
qed

lemma *ChopPlusElimWithoutMore*:

assumes $\vdash f \longrightarrow g$

$\vdash f; g \longrightarrow g$

shows $\vdash f; f^* \longrightarrow g$

proof –

have 1: $\vdash f \longrightarrow g$ **using** *assms* **by** *blast*

have 2: $\vdash (f; g) \longrightarrow g$ **using** *assms* **by** *blast*

have 3: $\vdash (f \wedge \text{more}); g \longrightarrow f; g$ **by** (*rule AndChopA*)

have 4: $\vdash (f \wedge \text{more}); g \longrightarrow g$ **using** 2 3 *lift-imp-trans* **by** *blast*

from 1 4 **show** *?thesis* **using** *ChopPlusElim* **by** *blast*

qed

lemma *ChopPlusEqvChopPlus*:

assumes $\vdash f = g$

shows $\vdash f; f^* = g; g^*$

proof –

have 1: $\vdash f = g$ **using** *assms* **by** *auto*

hence 2: $\vdash f \longrightarrow g$ **by** *auto*

hence 3: $\vdash f; f^* \longrightarrow g; g^*$ **by** (*rule ChopPlusImpChopPlus*)

have 4: $\vdash g \longrightarrow f$ **using** 1 **by** *auto*

hence 5: $\vdash g; g^* \longrightarrow f; f^*$ **by** (*rule ChopPlusImpChopPlus*)

from 3 5 **show** *?thesis* **by** *fastforce*

qed

lemma *CSEqvCS*:

assumes $\vdash f = g$

shows $\vdash f^* = g^*$

proof –

have 1: $\vdash f = g$ **using** *assms* **by** *auto*

hence 2: $\vdash f; f^* = g; g^*$ **by** (*rule ChopPlusEqvChopPlus*)

hence 3: $\vdash (\text{empty} \vee f; f^*) = (\text{empty} \vee g; g^*)$ **by** *auto*

from 3 **show** *?thesis* **using** *CSEqvOrChopCS* **by** (*metis int-eq*)

qed

lemma *AndCSA*:

$\vdash (f \wedge g)^* \longrightarrow f^*$

proof –

have 1: $\vdash f \wedge g \longrightarrow f$ **by** *auto*

from 1 **show** *?thesis* **using** *CSImpCS* **by** *blast*

qed

lemma *AndCSB*:

$\vdash (f \wedge g)^* \longrightarrow g^*$

proof –

have 1: $\vdash f \wedge g \longrightarrow g$ **by** *auto*

from 1 **show** *?thesis* **using** *CSImpCS* **by** *blast*

qed

lemma *CSIntro*:

assumes $\vdash f \wedge \text{more} \longrightarrow (g \wedge \text{more}); f$

shows $\vdash f \longrightarrow g^*$
proof –
have 1: $\vdash f \wedge \text{more} \longrightarrow (g \wedge \text{more}); f$
using *assms* **by** *auto*
have 2: $\vdash \text{more} = (\neg \text{empty})$
by (*auto simp: empty-d-def*)
have 3: $\vdash f \wedge \neg \text{empty} \longrightarrow (g \wedge \text{more}); f$
using 1 2 **by** *fastforce*
have 4: $\vdash g^* = (\text{empty} \vee (g \wedge \text{more}); g^*)$
by (*rule ChopstarEqv*)
hence 41: $\vdash (\neg(\text{empty} \vee (g \wedge \text{more}); g^*)) = (\neg \text{empty} \wedge \neg((g \wedge \text{more}); g^*))$
by *fastforce*
have 411: $\vdash (\neg \text{empty} \wedge \neg((g \wedge \text{more}); g^*)) = (\text{more} \wedge \neg((g \wedge \text{more}); g^*))$
using *NotEmptyEqvMore* **by** *fastforce*
have 42: $\vdash \neg(g^*) = (\text{more} \wedge \neg((g \wedge \text{more}); g^*))$
using 4 41 411 **by** *fastforce*
have 43: $\vdash f \wedge \neg(g^*) \longrightarrow f \wedge \text{more} \wedge \neg((g \wedge \text{more}); g^*)$
using 42 **by** *fastforce*
have 44: $\vdash f \wedge \text{more} \wedge \neg((g \wedge \text{more}); g^*) \longrightarrow (g \wedge \text{more}); f \wedge \neg((g \wedge \text{more}); g^*)$
using 3 43 1 **by** *auto*
have 5: $\vdash f \wedge \neg(g^*) \longrightarrow$
 $(g \wedge \text{more}); f \wedge \neg((g \wedge \text{more}); g^*)$
using 43 44 *lift-imp-trans* **by** *fastforce*
have 6: $\vdash g \wedge \text{more} \longrightarrow \text{more}$
by *auto*
from 5 6 **show** *?thesis* **using** *ChopContra* **by** *blast*
qed

lemma *CSElimWithoutMore*:

assumes $\vdash \text{empty} \longrightarrow g$
 $\vdash f; g \longrightarrow g$
shows $\vdash f^* \longrightarrow g$
proof –
have 1: $\vdash \text{empty} \longrightarrow g$ **using** *assms* **by** *blast*
have 2: $\vdash f; g \longrightarrow g$ **using** *assms* **by** *blast*
have 3: $\vdash (f \wedge \text{more}); g \longrightarrow f; g$ **by** (*rule AndChopA*)
have 4: $\vdash (f \wedge \text{more}); g \longrightarrow g$ **using** 2 3 *lift-imp-trans* **by** *blast*
from 1 4 **show** *?thesis* **using** *CSElim* **by** *blast*
qed

lemma *ChopAssocB*:

$\vdash (f;g);h = f;(g;h)$
using *ChopAssoc* **by** *fastforce*

lemma *CSChopEqvChopOrRule*:

assumes $\vdash f = (g^*; h)$
shows $\vdash f = ((g; f) \vee h)$
proof –
have 1: $\vdash f = (g^*; h)$ **using** *assms* **by** *auto*
have 2: $\vdash g^* = (\text{empty} \vee (g; g^*))$ **by** (*rule CSEqvOrChopCS*)

hence 3: $\vdash g^*; h = (h \vee ((g; g^*); h))$ **by** (*rule EmptyOrChopEqvRule*)
have 4: $\vdash (g; g^*); h = g; (g^*; h)$ **by** (*rule ChopAssocB*)
hence 41: $\vdash g^*; h = (h \vee g; (g^*; h))$ **using** 3 **by** *fastforce*
have 5: $\vdash g; f = g; (g^*; h)$ **using** 1 **by** (*rule RightChopEqvChop*)
hence 6: $\vdash (g^*; h) = (h \vee g; f)$ **using** 41 **by** *fastforce*
hence 61: $\vdash (g^*; h) = ((g; f) \vee h)$ **by** *auto*
from 1 61 **show** *?thesis* **by** *fastforce*
qed

lemma *CSChopIntroRule*:

assumes $\vdash f \wedge \neg h \longrightarrow g; f$
 $\vdash g \longrightarrow \text{more}$
shows $\vdash f \longrightarrow g^*; h$
proof –
have 1: $\vdash f \wedge \neg h \longrightarrow g; f$
using *assms* **by** *blast*
have 2: $\vdash g \longrightarrow \text{more}$
using *assms* **by** *blast*
hence 3: $\vdash g \longrightarrow g \wedge \text{more}$
by *auto*
hence 4: $\vdash g; f \longrightarrow (g \wedge \text{more}); f$
by (*rule LeftChopImpChop*)
have 5: $\vdash f \longrightarrow (g \wedge \text{more}); f \vee h$
using 1 4 **by** *fastforce*
have 6: $\vdash g^* = (\text{empty} \vee (g \wedge \text{more}); g^*)$
by (*rule ChopstarEqv*)
hence 7: $\vdash (g^*); h = (h \vee ((g \wedge \text{more}); g^*); h)$
by (*rule EmptyOrChopEqvRule*)
have 8: $\vdash ((g \wedge \text{more}); g^*); h = (g \wedge \text{more}); (g^*; h)$
by (*rule ChopAssocB*)
have 9: $\vdash (g^*); h = (h \vee (g \wedge \text{more}); (g^*; h))$
using 7 8 **by** *fastforce*
have 10: $\vdash f \wedge \neg (g^*; h) \longrightarrow (g \wedge \text{more}); f \wedge \neg ((g \wedge \text{more}); (g^*; h))$
using 5 9 **by** *fastforce*
have 11: $\vdash g \wedge \text{more} \longrightarrow \text{more}$
by *fastforce*
from 10 11 **show** *?thesis* **using** *ChopContra* **by** *blast*
qed

lemma *DiamondAndEmptyEqvAndEmpty*:

$\vdash (\Diamond f \wedge \text{empty}) = (f \wedge \text{empty})$
by (*auto simp: itl-defs*)

lemma *InitAndEmptyEqvAndEmpty*:

$\vdash ((\text{init } w) \wedge \text{empty}) = (w \wedge \text{empty})$

proof –

have 1: $\vdash ((\text{init } w) \wedge \text{empty}) = ((w \wedge \text{empty}); \# \text{True} \wedge \text{empty})$
by (*metis init-d-def int-eq lift-and-com*)

have 2: $\vdash ((w \wedge \text{empty}); \# \text{True} \wedge \text{empty}) = (w \wedge \text{empty}); (\# \text{True} \wedge \text{empty})$
by (*meson AndChopA ChopAndA ChopAndEmptyEqvEmptyChopEmpty lift-imp-trans Prop11 Prop12*)
have 3: $\vdash (w \wedge \text{empty}); (\# \text{True} \wedge \text{empty}) = (w \wedge \text{empty}); \text{empty}$
using *RightChopEqvChop* **by** *fastforce*
have 4: $\vdash (w \wedge \text{empty}); \text{empty} = (w \wedge \text{empty})$
using *ChopEmpty* **by** *blast*
from 1 2 3 4 **show** ?thesis **by** *fastforce*
qed

lemma *InitAndNotBoxInitImpNotEmpty*:

$\vdash \text{init } w \wedge \neg(\Box(\text{init } w)) \longrightarrow \neg \text{empty}$

proof –

have 1: $\vdash ((\text{init } w) \wedge \text{empty}) = (w \wedge \text{empty})$
by (*rule InitAndEmptyEqvAndEmpty*)
have 2: $\vdash (\neg(\Box(\text{init } w)) \wedge \text{empty}) = (\Diamond(\neg(\text{init } w)) \wedge \text{empty})$
by (*auto simp: always-d-def*)
have 3: $\vdash (\Diamond(\neg(\text{init } w)) \wedge \text{empty}) = (\neg(\text{init } w) \wedge \text{empty})$
by (*simp add: DiamondAndEmptyEqvAndEmpty*)
have 4: $\vdash (\neg(\text{init } w)) = (\text{init }(\neg w))$ **using** *Initprop(2)* **by** *blast*
have 5: $\vdash (\neg(\text{init } w) \wedge \text{empty}) = (\neg w \wedge \text{empty})$
using 4 *InitAndEmptyEqvAndEmpty* **by** (*metis inteq-reflection*)
have 6: $\vdash (\neg(\Box(\text{init } w)) \wedge \text{empty}) = (\neg w \wedge \text{empty})$
using 2 3 5 **by** *fastforce*
have 7: $\vdash \neg(\text{init } w \wedge \neg(\Box(\text{init } w)) \wedge \text{empty})$
using 1 6 **by** *fastforce*
from 7 **show** ?thesis **by** *auto*
qed

lemma *BoxImpTrueChopAndEmpty*:

$\vdash \Box f \longrightarrow \# \text{True}; (f \wedge \text{empty})$

using *BoxAndChopImport Finprop(3)* **by** *fastforce*

lemma *BoxInitAndMoreImpBoxInitAndMoreAndFinInit*:

$\vdash \Box(\text{init } w) \wedge \text{more} \longrightarrow (\Box(\text{init } w) \wedge \text{more}) \wedge \text{fin }(\text{init } w)$

proof –

have 1: $\vdash \text{fin }(\text{init } w) = \# \text{True}; (\text{init } w \wedge \text{empty})$ **using** *FinEqvTrueChopAndEmpty* **by** *blast*
have 2: $\vdash \Box(\text{init } w) \longrightarrow \# \text{True}; (\text{init } w \wedge \text{empty})$ **by** (*rule BoxImpTrueChopAndEmpty*)
from 1 2 **show** ?thesis **by** *fastforce*
qed

lemma *CSImpBox*:

assumes $\vdash f \longrightarrow \text{empty} \vee (\Box(\text{init } w) \wedge \text{more}); f$

shows $\vdash \text{init } w \wedge f \longrightarrow \Box(\text{init } w)$

proof –

have 1: $\vdash f \longrightarrow \text{empty} \vee (\Box(\text{init } w) \wedge \text{more}); f$
using *assms* **by** *auto*
have 2: $\vdash \text{init } w \wedge \neg(\Box(\text{init } w)) \longrightarrow \neg \text{empty}$
by (*rule InitAndNotBoxInitImpNotEmpty*)
have 3: $\vdash \text{init } w \wedge f \wedge \neg(\Box(\text{init } w)) \longrightarrow (\Box(\text{init } w) \wedge \text{more}); f$
using 1 2 **by** *fastforce*

have 4: $\vdash \Box (init\ w) \wedge more \longrightarrow (\Box (init\ w) \wedge more) \wedge fin\ (init\ w)$
by (rule *BoxInitAndMoreImpBoxInitAndMoreAndFinInit*)
hence 5: $\vdash (\Box (init\ w) \wedge more); f \longrightarrow ((\Box (init\ w) \wedge more) \wedge fin\ (init\ w)); f$
by (rule *LeftChopImpChop*)
have 6: $\vdash ((\Box (init\ w) \wedge more) \wedge fin\ (init\ w)); f =$
 $(\Box (init\ w) \wedge more); (init\ w \wedge f)$
by (rule *AndFinChopEqvStateAndChop*)
have 7: $\vdash \neg(\Box (init\ w)) \longrightarrow (\Box (init\ w))\ yields\ (\neg(\Box (init\ w)))$
by (rule *NotBoxStateImpBoxYieldsNotBox*)
have 8: $\vdash (\Box (init\ w))\ yields\ (\neg(\Box (init\ w))) \longrightarrow$
 $(\Box (init\ w) \wedge more)\ yields\ (\neg(\Box (init\ w)))$
by (rule *AndYieldsA*)
have 9: $\vdash (\Box (init\ w) \wedge more); (init\ w \wedge f) \wedge (\Box (init\ w) \wedge more)\ yields\ (\neg(\Box (init\ w)))$
 \longrightarrow
 $(\Box (init\ w) \wedge more); ((init\ w \wedge f) \wedge \neg(\Box (init\ w)))$
by (rule *ChopAndYieldsImp*)
have 10: $\vdash (init\ w \wedge f) \wedge \neg(\Box (init\ w)) \longrightarrow$
 $(\Box (init\ w) \wedge more); ((init\ w \wedge f) \wedge \neg(\Box (init\ w)))$
using 3 5 6 7 8 9 **by** *fastforce*
have 11: $\vdash (\Box (init\ w) \wedge more); ((init\ w \wedge f) \wedge \neg(\Box (init\ w))) \longrightarrow$
 $more; ((init\ w \wedge f) \wedge \neg(\Box (init\ w)))$
by (rule *AndChopB*)
have 12: $\vdash (init\ w \wedge f) \wedge \neg(\Box (init\ w)) \longrightarrow$
 $more; ((init\ w \wedge f) \wedge \neg(\Box (init\ w)))$
using 10 11 **by** *fastforce*
from 12 **show** ?thesis **using** *MoreChopContra* **by** *blast*
qed

lemma *BoxCSEqvBox*:

$\vdash (init\ w \wedge (\Box (init\ w))^*) = \Box (init\ w)$

proof –

have 1: $\vdash (\Box (init\ w))^* = (empty \vee (\Box (init\ w) \wedge more); (\Box (init\ w))^*)$
by (rule *ChopstarEqv*)
hence 2: $\vdash (\Box (init\ w))^* \longrightarrow empty \vee (\Box (init\ w) \wedge more); (\Box (init\ w))^*$
by *fastforce*
hence 3: $\vdash init\ w \wedge (\Box (init\ w))^* \longrightarrow \Box (init\ w)$
by (rule *CSImpBox*)
have 11: $\vdash \Box (init\ w) \longrightarrow (init\ w)$
using *BoxElim* **by** *blast*
have 12: $\vdash \Box (init\ w) \longrightarrow (\Box (init\ w))^*$
by (rule *ImpCS*)
have 13: $\vdash \Box (init\ w) \longrightarrow init\ w \wedge (\Box (init\ w))^*$
using 11 12 **by** *fastforce*
from 3 13 **show** ?thesis **by** *fastforce*
qed

lemma *BoxStateAndCSEqvCS*:

$\vdash (\Box (init\ w) \wedge f^*) = (init\ w \wedge (\Box (init\ w) \wedge f))^*$

proof –

have 1: $\vdash \Box (init\ w) \longrightarrow init\ w$

using *BoxElim* **by** *blast*
have 2: $\vdash (f^* \wedge \text{more}) = (f \wedge \text{more}); f^*$
by (*rule CSAndMoreEqvAndMoreChop*)
have 3: $\vdash (\Box(\text{init } w) \wedge ((f \wedge \text{more}); f^*)) =$
 $((\Box(\text{init } w) \wedge f \wedge \text{more}); (\Box(\text{init } w) \wedge f^*))$
by (*rule BoxStateAndChopEqvChop*)
have 4: $\vdash \Box(\text{init } w) \wedge f \wedge \text{more} \longrightarrow (\Box(\text{init } w) \wedge f) \wedge \text{more}$
by *auto*
hence 5: $\vdash (\Box(\text{init } w) \wedge f \wedge \text{more}); (\Box(\text{init } w) \wedge f^*) \longrightarrow$
 $((\Box(\text{init } w) \wedge f) \wedge \text{more}); (\Box(\text{init } w) \wedge f^*)$
by (*rule LeftChopImpChop*)
have 6: $\vdash (\Box(\text{init } w) \wedge f^*) \wedge \text{more} \longrightarrow$
 $((\Box(\text{init } w) \wedge f) \wedge \text{more}); (\Box(\text{init } w) \wedge f^*)$
using 2 3 5 **by** *fastforce*
hence 7: $\vdash \Box(\text{init } w) \wedge f^* \longrightarrow (\Box(\text{init } w) \wedge f)^*$
by (*rule CSIntro*)
have 71: $\vdash \text{init } w \wedge \Box(\text{init } w) \wedge f^* \longrightarrow \text{init } w \wedge (\Box(\text{init } w) \wedge f)^*$
using 7 **by** *fastforce*
have 8: $\vdash \Box(\text{init } w) \wedge f^* \longrightarrow \text{init } w \wedge (\Box(\text{init } w) \wedge f)^*$
using 1 71 **by** *fastforce*
have 11: $\vdash (\Box(\text{init } w) \wedge f)^* \longrightarrow (\Box(\text{init } w))^*$
by (*rule AndCSA*)
have 12: $\vdash (\text{init } w \wedge (\Box(\text{init } w))^*) = \Box(\text{init } w)$
by (*rule BoxCSEqvBox*)
have 13: $\vdash (\Box(\text{init } w) \wedge f)^* \longrightarrow f^*$
by (*rule AndCSB*)
have 14: $\vdash \text{init } w \wedge (\Box(\text{init } w) \wedge f)^* \longrightarrow \text{init } w \wedge (\Box(\text{init } w))^* \wedge f^*$
using 11 13 **by** *fastforce*
have 15: $\vdash \text{init } w \wedge (\Box(\text{init } w))^* \wedge f^* \longrightarrow \Box(\text{init } w) \wedge f^*$
using 12 **by** *auto*
have 16: $\vdash \text{init } w \wedge (\Box(\text{init } w) \wedge f)^* \longrightarrow \Box(\text{init } w) \wedge f^*$
using 14 15 *lift-imp-trans* **by** *blast*
from 8 16 **show** *?thesis* **by** *fastforce*
qed

lemma *BaCSImpCS*:

$\vdash \text{ba } (f \longrightarrow g) \longrightarrow f^* \longrightarrow g^*$

proof –

have 1: $\vdash f^* = (\text{empty} \vee (f \wedge \text{more}); f^*)$
by (*rule ChopstarEqv*)
have 2: $\vdash g^* = (\text{empty} \vee (g \wedge \text{more}); g^*)$
by (*rule ChopstarEqv*)
have 21: $\vdash \neg(g^*) = (\neg \text{empty} \wedge \neg((g \wedge \text{more}); g^*))$
using 2 **by** *fastforce*
hence 22: $\vdash \neg(g^*) = (\text{more} \wedge \neg((g \wedge \text{more}); g^*))$
using *NotEmptyEqvMore* **by** *fastforce*
have 3: $\vdash f^* \wedge \neg(g^*) \longrightarrow$
 $(\text{empty} \vee (f \wedge \text{more}); f^*) \wedge \text{more} \wedge \neg((g \wedge \text{more}); g^*)$
using 1 22 **by** *fastforce*
have 31: $\vdash ((\text{empty} \vee (f \wedge \text{more}); f^*) \wedge \text{more}) = ((f \wedge \text{more}); f^* \wedge \text{more})$

by (auto simp: empty-d-def)
 have 32: $\vdash f^* \wedge \neg (g^*) \longrightarrow (f \wedge \text{more}); f^* \wedge \neg ((g \wedge \text{more}); g^*)$
 using 3 31 by fastforce
 have 4: $\vdash (f \longrightarrow g) \longrightarrow (f \wedge \text{more} \longrightarrow g \wedge \text{more})$
 by auto
 hence 5: $\vdash \text{ba } (f \longrightarrow g) \longrightarrow \text{ba } (f \wedge \text{more} \longrightarrow g \wedge \text{more})$
 by (rule BaImpBa)
 have 6: $\vdash \text{ba } (f \wedge \text{more} \longrightarrow g \wedge \text{more}) \longrightarrow$
 $(f \wedge \text{more}); f^* \longrightarrow (g \wedge \text{more}); f^*$
 by (rule BaLeftChopImpChop)
 have 7: $\vdash \text{ba } (f \longrightarrow g) \wedge (f \wedge \text{more}); f^* \longrightarrow (g \wedge \text{more}); f^*$
 using 5 6 by fastforce
 have 8: $\vdash (g \wedge \text{more}); f^* \wedge \neg ((g \wedge \text{more}); g^*)$
 $\longrightarrow (g \wedge \text{more}); (f^* \wedge \neg (g^*))$
 by (rule ChopAndNotChopImp)
 have 9: $\vdash (g \wedge \text{more}); (f^* \wedge \neg (g^*)) \longrightarrow \text{more}; (f^* \wedge \neg (g^*))$
 by (rule AndChopB)
 have 10: $\vdash \text{ba } (f \longrightarrow g) \longrightarrow \text{more}; (f^* \wedge \neg (g^*)) \longrightarrow$
 $\text{more}; (\text{ba } (f \longrightarrow g) \wedge f^* \wedge \neg (g^*))$
 by (rule BaChopImpChopBa)
 have 11: $\vdash \text{ba } (f \longrightarrow g) \wedge f^* \wedge \neg (g^*) \longrightarrow$
 $\text{more}; (\text{ba } (f \longrightarrow g) \wedge f^* \wedge \neg (g^*))$
 using 32 7 8 9 10 by fastforce
 hence 12: $\vdash \neg (\text{ba } (f \longrightarrow g)) \wedge (f^*) \wedge (\neg (g^*))$
 using MoreChopLoop by blast
 from 12 show ?thesis using MP by fastforce
 qed

lemma BaCSEqvCS:

$\vdash \text{ba } (f = g) \longrightarrow (f^* = g^*)$

proof –

have 1: $\vdash \text{ba } (f = g) = (\text{ba } (f \longrightarrow g) \wedge \text{ba } (g \longrightarrow f))$ by (auto simp: itl-defs)
 have 2: $\vdash \text{ba } (f \longrightarrow g) \longrightarrow (f^* \longrightarrow g^*)$ by (rule BaCSImpCS)
 have 3: $\vdash \text{ba } (g \longrightarrow f) \longrightarrow (g^* \longrightarrow f^*)$ by (rule BaCSImpCS)
 have 4: $\vdash \text{ba } (f = g) \longrightarrow (f^* \longrightarrow g^*) \wedge (g^* \longrightarrow f^*)$ using 1 2 3 by fastforce
 have 5: $\vdash ((f^* \longrightarrow g^*) \wedge (g^* \longrightarrow f^*)) = (f^* = g^*)$ by auto
 from 4 5 show ?thesis by auto

qed

lemma BaAndCSImport:

$\vdash \text{ba } f \wedge g^* \longrightarrow (f \wedge g)^*$

proof –

have 1: $\vdash f \longrightarrow (g \longrightarrow f \wedge g)$ by auto
 hence 2: $\vdash \text{ba } f \longrightarrow \text{ba } (g \longrightarrow f \wedge g)$ by (rule BaImpBa)
 have 3: $\vdash \text{ba } (g \longrightarrow f \wedge g) \longrightarrow g^* \longrightarrow (f \wedge g)^*$ by (rule BaCSImpCS)
 from 2 3 show ?thesis by fastforce

qed

lemma CSSkip:

$\vdash \text{skip}^*$

by (*metis ChopPlusImpCS EmptyImpCS EmptyNextInducta next-d-def*)

5.8 Properties of While

lemma *WhileEqvIf*:

$\vdash \text{while } (\text{init } w) \text{ do } f = \text{if}_i (\text{init } w) \text{ then } (f; (\text{while } (\text{init } w) \text{ do } f)) \text{ else } \text{empty}$

proof –

have 1: $\vdash \text{while } (\text{init } w) \text{ do } f = (((\text{init } w) \wedge f)^* \wedge \text{fin } (\neg (\text{init } w)))$

by (*simp add: while-d-def*)

have 2: $\vdash (\text{init } w \wedge f)^* = (\text{empty} \vee ((\text{init } w \wedge f); (\text{init } w \wedge f)^*))$

by (*rule CSEqvOrChopCS*)

have 21: $\vdash (((\text{init } w) \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))) =$
 $((\text{empty} \vee ((\text{init } w \wedge f); (\text{init } w \wedge f)^*)) \wedge \text{fin } (\neg (\text{init } w)))$

using 2 **by** *fastforce*

have 22: $\vdash ((\text{empty} \vee ((\text{init } w \wedge f); (\text{init } w \wedge f)^*)) \wedge \text{fin } (\neg (\text{init } w))) =$
 $((\text{empty} \wedge \text{fin } (\neg (\text{init } w))) \vee ((\text{init } w \wedge f); (\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))))$

by *auto*

have 3: $\vdash (\text{empty} \wedge \text{fin } (\neg (\text{init } w))) = (\neg (\text{init } w) \wedge \text{empty})$

using *AndFinEqvChopAndEmpty EmptyChop* **by** (*metis int-eq*)

have 4: $\vdash (\text{init } w \wedge f); (\text{init } w \wedge f)^* = (\text{init } w \wedge (f; (\text{init } w \wedge f)^*))$

by (*rule StateAndChop*)

have 41: $\vdash (((\text{init } w \wedge f); (\text{init } w \wedge f)^*) \wedge \text{fin } (\neg (\text{init } w))) =$
 $(\text{init } w \wedge (f; (\text{init } w \wedge f)^*) \wedge \text{fin } (\neg (\text{init } w)))$

using 4 **by** *auto*

have 42: $\vdash (\text{init } w \wedge (f; (\text{init } w \wedge f)^*) \wedge \text{fin } (\neg (\text{init } w))) =$
 $(\text{init } w \wedge (f; (\text{init } w \wedge f)^*) \wedge \text{fin } (\neg (\text{init } w)))$

using *Initprop(2)* **by** (*metis StateAndEmptyChop int-eq*)

have 5: $\vdash ((f; ((\text{init } w \wedge f)^*)) \wedge (\text{fin } (\neg (\text{init } w)))) =$
 $(f; ((\text{init } w \wedge f)^* \wedge (\text{fin } (\neg (\text{init } w))))$

by (*rule ChopAndFin*)

have 51: $\vdash (f; ((\text{init } w \wedge f)^* \wedge (\text{fin } (\neg (\text{init } w)))) =$
 $(f; ((\text{init } w \wedge f)^* \wedge (\text{fin } (\neg (\text{init } w))))$

using *Initprop(2)* **by** (*metis FinAndChop int-eq*)

have 52: $\vdash (\text{init } w \wedge (f; (\text{init } w \wedge f)^*) \wedge \text{fin } (\neg (\text{init } w))) =$
 $(\text{init } w \wedge (f; ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))))$

using 42 5 51 **by** *fastforce*

have 6: $\vdash (f; ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w)))) = f; \text{while } (\text{init } w) \text{ do } f$

by (*simp add: while-d-def*)

have 61: $\vdash (\text{init } w \wedge (f; ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w)))) =$
 $(\text{init } w \wedge (f; \text{while } (\text{init } w) \text{ do } f))$ **using** 6

by *auto*

have 62: $\vdash (\text{empty} \wedge \text{fin } (\neg (\text{init } w))) \vee ((\text{init } w \wedge f); (\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w)))$
 $= (\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (f; \text{while } (\text{init } w) \text{ do } f))$

using 3 41 52 61 **by** *fastforce*

have 7: $\vdash \text{while } (\text{init } w) \text{ do } f$
 $= ((\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (f; \text{while } (\text{init } w) \text{ do } f)))$

using 1 21 22 62

by (*metis 3 41 42 5 51 inteq-reflection*)

have 71: $\vdash \text{if}_i (\text{init } w) \text{ then } (f; (\text{while } (\text{init } w) \text{ do } f)) \text{ else } \text{empty} =$
 $((\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (f; \text{while } (\text{init } w) \text{ do } f)))$

by (auto simp: ifthenelse-d-def)
 from 7 71 show ?thesis by fastforce
 qed

lemma *WhileChopEqvIf*:

$\vdash (\text{while } (\text{init } w) \text{ do } f); g = \text{if}_i(\text{init } w) \text{ then } (f; ((\text{while } (\text{init } w) \text{ do } f); g)) \text{ else } g$
proof –
 have 1: $\vdash \text{while } (\text{init } w) \text{ do } f =$
 $\text{if}_i (\text{init } w) \text{ then } (f; (\text{while } (\text{init } w) \text{ do } f)) \text{ else } \text{empty}$
 by (rule WhileEqvIf)
 hence 2: $\vdash (\text{while } (\text{init } w) \text{ do } f); g =$
 $\text{if}_i (\text{init } w) \text{ then } ((f; \text{while } (\text{init } w) \text{ do } f); g) \text{ else } (\text{empty}; g)$
 by (rule IfChopEqvRule)
 have 3: $\vdash \text{empty}; g = g$
 by (rule EmptyChop)
 have 4: $\vdash \text{if}_i (\text{init } w) \text{ then } ((f; \text{while } (\text{init } w) \text{ do } f); g) \text{ else } (\text{empty}; g) =$
 $\text{if}_i (\text{init } w) \text{ then } ((f; \text{while } (\text{init } w) \text{ do } f); g) \text{ else } g$
 using 3 using inteq-reflection by fastforce
 have 5: $\vdash ((f; \text{while } (\text{init } w) \text{ do } f); g) = (f; (\text{while } (\text{init } w) \text{ do } f; g))$
 by (rule ChopAssocB)
 have 6: $\vdash \text{if}_i (\text{init } w) \text{ then } ((f; \text{while } (\text{init } w) \text{ do } f); g) \text{ else } g =$
 $\text{if}_i (\text{init } w) \text{ then } (f; ((\text{while } (\text{init } w) \text{ do } f); g)) \text{ else } g$
 using 5 using inteq-reflection by fastforce
 from 1 2 4 6 show ?thesis by fastforce
 qed

lemma *WhileChopEqvIfRule*:

assumes $\vdash f = (\text{while } (\text{init } w) \text{ do } g); h$
shows $\vdash f = \text{if}_i (\text{init } w) \text{ then } (g; f) \text{ else } h$
proof –
 have 1: $\vdash f = (\text{while } (\text{init } w) \text{ do } g); h$
 using assms by auto
 have 2: $\vdash (\text{while } (\text{init } w) \text{ do } g); h =$
 $\text{if}_i (\text{init } w) \text{ then } (g; ((\text{while } (\text{init } w) \text{ do } g); h)) \text{ else } h$
 by (rule WhileChopEqvIf)
 have 3: $\vdash (g; f) = (g; ((\text{while } (\text{init } w) \text{ do } g); h))$
 using 1 by (rule RightChopEqvChop)
 have 4: $\vdash (g; ((\text{while } (\text{init } w) \text{ do } g); h)) = (g; f)$
 using 3 by auto
 have 5: $\vdash \text{if}_i (\text{init } w) \text{ then } (g; ((\text{while } (\text{init } w) \text{ do } g); h)) \text{ else } h =$
 $\text{if}_i (\text{init } w) \text{ then } (g; f) \text{ else } h$
 using 4 using inteq-reflection by fastforce
 from 1 2 5 show ?thesis by fastforce
 qed

lemma *WhileImpFin*:

$\vdash \text{while } (\text{init } w) \text{ do } f \longrightarrow \text{fin } (\neg (\text{init } w))$
proof –
 have 1: $\vdash (\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w)) \longrightarrow \text{fin } (\neg (\text{init } w))$ by auto
 from 1 show ?thesis by (simp add: while-d-def)

qed

lemma *WhileEqvEmptyOrChopWhile:*

$\vdash \text{while } (\text{init } w) \text{ do } f = ((\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (f \wedge \text{more}); \text{while } (\text{init } w) \text{ do } f))$

proof –

have 1: $\vdash (\text{init } w \wedge f)^* = (\text{empty} \vee ((\text{init } w \wedge f) \wedge \text{more}); (\text{init } w \wedge f)^*)$

by (rule *ChopstarEqv*)

have 2: $\vdash ((\text{init } w \wedge f) \wedge \text{more}) = (\text{init } w \wedge (f \wedge \text{more}))$

by *auto*

hence 3: $\vdash ((\text{init } w \wedge f) \wedge \text{more}); (\text{init } w \wedge f)^* = (\text{init } w \wedge f \wedge \text{more}); (\text{init } w \wedge f)^*$

by (rule *LeftChopEqvChop*)

have 4: $\vdash (\text{init } w \wedge f)^* = (\text{empty} \vee (\text{init } w \wedge f \wedge \text{more}); (\text{init } w \wedge f)^*)$

using 1 3 **by** *fastforce*

have 5: $\vdash ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))) =$

$((\text{empty} \wedge \text{fin } (\neg (\text{init } w))) \vee$
 $((\text{init } w \wedge f \wedge \text{more}); (\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))))$

using 1 4 **by** *fastforce*

have 6: $\vdash (\text{empty} \wedge \text{fin } (\neg (\text{init } w))) = (\neg (\text{init } w) \wedge \text{empty})$

using *AndFinEqvChopAndEmpty EmptyChop* **by** (*metis int-eq*)

have 7: $\vdash (\text{init } w \wedge f \wedge \text{more}); (\text{init } w \wedge f)^* = (\text{init } w \wedge (f \wedge \text{more}); (\text{init } w \wedge f)^*)$

by (rule *StateAndChop*)

have 71: $\vdash ((\text{init } w \wedge f \wedge \text{more}); (\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))) =$

$((\text{init } w \wedge (f \wedge \text{more}); (\text{init } w \wedge f)^*) \wedge \text{fin } (\neg (\text{init } w)))$

using 7 **by** *auto*

have 8: $\vdash ((f \wedge \text{more}); (\text{init } w \wedge f)^*) \wedge \text{fin } (\text{init } (\neg w)) =$

$((f \wedge \text{more}); ((\text{init } w \wedge f)^* \wedge \text{fin } (\text{init } (\neg w))))$

by (rule *ChopAndFin*)

have 81: $\vdash \text{fin } (\text{init } (\neg w)) = \text{fin } (\neg (\text{init } w))$

using *FinEqvFin Initprop(2)* **by** *fastforce*

have 83: $\vdash ((\text{init } w \wedge (f \wedge \text{more}); (\text{init } w \wedge f)^*) \wedge \text{fin } (\neg \text{init } w)) =$

$(\text{init } w \wedge (f \wedge \text{more}); ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))))$

by (*metis 81 ChopAndFin StateAndChop inteq-reflection*)

have 9: $\vdash ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))) =$

$((\neg (\text{init } w) \wedge \text{empty}) \vee$
 $(\text{init } w \wedge (f \wedge \text{more}); ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))))$

by (*metis 5 6 71 83 inteq-reflection*)

from 9 **show** *?thesis* **by** (*simp add: while-d-def*)

qed

lemma *WhileIntro:*

assumes $\vdash \neg (\text{init } w) \wedge f \longrightarrow \text{empty}$

$\vdash \text{init } w \wedge f \longrightarrow (g \wedge \text{more}); f$

shows $\vdash f \longrightarrow \text{while } (\text{init } w) \text{ do } g$

proof –

have 1: $\vdash \neg (\text{init } w) \wedge f \longrightarrow \text{empty}$

using *assms* **by** *blast*

have 2: $\vdash \text{init } w \wedge f \longrightarrow (g \wedge \text{more}); f$

using *assms* **by** *blast*

have 3: $\vdash \text{while } (\text{init } w) \text{ do } g =$

$((\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g))$

by (rule WhileEqvEmptyOrChopWhile)
 hence 31: $\vdash \neg (\text{while } (\text{init } w) \text{ do } g) =$
 $(\neg (\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g))$
 by fastforce
 hence 32: $\vdash (f \wedge \neg (\text{while } (\text{init } w) \text{ do } g)) =$
 $(f \wedge \neg (\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g))$
 by fastforce
 have 33: $\vdash (f \wedge \neg (\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)) =$
 $(f \wedge \neg (\neg (\text{init } w) \wedge \text{empty}) \wedge \neg (\text{init } w \wedge (g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g))$
 by auto
 have 34: $\vdash (f \wedge \neg (\neg (\text{init } w) \wedge \text{empty}) \wedge \neg ((\text{init } w) \wedge ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g))) =$
 $(f \wedge ((\text{init } w) \vee \text{more}) \wedge (\neg (\text{init } w) \vee \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)))$
 by (auto simp: empty-d-def)
 have 36: $\vdash (f \wedge \neg (\text{while } (\text{init } w) \text{ do } g)) =$
 $((f \wedge (\text{init } w) \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)) \vee$
 $(f \wedge (\text{init } w) \wedge \neg (\text{init } w)) \vee$
 $(f \wedge \text{more} \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)) \vee$
 $(f \wedge \text{more} \wedge \neg (\text{init } w)))$
 using 32 34 by fastforce
 have 37: $\vdash \neg (f \wedge \text{more} \wedge \neg (\text{init } w))$
 using 1 by (auto simp: empty-d-def)
 have 38: $\vdash (f \wedge \text{more} \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)) \longrightarrow$
 $((g \wedge \text{more}); f \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g))$
 using 1 2 by (auto simp: empty-d-def Valid-def)
 have 39: $\vdash (f \wedge (\text{init } w) \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)) \longrightarrow$
 $((g \wedge \text{more}); f \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g))$
 using 2 by auto
 have 40: $\vdash ((f \wedge (\text{init } w) \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)) \vee$
 $(f \wedge (\text{init } w) \wedge \neg (\text{init } w)) \vee$
 $(f \wedge \text{more} \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)) \vee$
 $(f \wedge \text{more} \wedge \neg (\text{init } w))) \longrightarrow$
 $(g \wedge \text{more}); f \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)$
 using 39 38 37 38 by fastforce
 have 4: $\vdash f \wedge \neg (\text{while } (\text{init } w) \text{ do } g) \longrightarrow$
 $(g \wedge \text{more}); f \wedge \neg ((g \wedge \text{more}); \text{while } (\text{init } w) \text{ do } g)$
 by (meson 36 40 Prop11 lift-imp-trans)
 have 5: $\vdash g \wedge \text{more} \longrightarrow \text{more}$
 by auto
 from 4 5 show ?thesis using ChopContra by blast
 qed

lemma WhileElim:

assumes $\vdash \neg (\text{init } w) \wedge \text{empty} \longrightarrow g$

$\vdash \text{init } w \wedge (f \wedge \text{more}); g \longrightarrow g$

shows $\vdash \text{while } (\text{init } w) \text{ do } f \longrightarrow g$

proof –

have 1: $\vdash \text{while } (\text{init } w) \text{ do } f =$

$((\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (f \wedge \text{more}); \text{while } (\text{init } w) \text{ do } f))$

by (rule WhileEqvEmptyOrChopWhile)

hence 11: $\vdash ((\text{while } (\text{init } w) \text{ do } f) \wedge \neg g) =$

$((\neg(\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (f \wedge \text{more}); \text{while } (\text{init } w) \text{ do } f)) \wedge \neg g$
by auto
have 2: $\vdash \neg (\text{init } w) \wedge \text{empty} \longrightarrow g$
using *assms* **by** *blast*
hence 21: $\vdash \neg g \longrightarrow \neg(\neg (\text{init } w) \wedge \text{empty})$
by auto
have 22: $\vdash ((\neg (\text{init } w) \wedge \text{empty}) \vee (\text{init } w \wedge (f \wedge \text{more}); \text{while } (\text{init } w) \text{ do } f)) \wedge \neg g \longrightarrow$
 $(\text{init } w \wedge (f \wedge \text{more}); \text{while } (\text{init } w) \text{ do } f)$
using 21 **by auto**
have 23: $\vdash (\text{while } (\text{init } w) \text{ do } f) \wedge \neg g \longrightarrow$
 $(\text{init } w \wedge (f \wedge \text{more}); \text{while } (\text{init } w) \text{ do } f) \wedge \neg g$
using 11 21 **by fastforce**
have 3: $\vdash (\text{init } w) \wedge ((f \wedge \text{more}); g) \longrightarrow g$
using *assms* **by** *blast*
hence 31: $\vdash \neg g \longrightarrow \neg((\text{init } w) \wedge ((f \wedge \text{more}); g))$
by fastforce
have 32: $\vdash (\text{init } w \wedge (f \wedge \text{more}); \text{while } (\text{init } w) \text{ do } f) \wedge \neg g \longrightarrow$
 $((f \wedge \text{more}); (\text{while } (\text{init } w) \text{ do } f)) \wedge \neg ((f \wedge \text{more}); g) \wedge \neg g$
using 31 **by auto**
have 4: $\vdash (\text{while } (\text{init } w) \text{ do } f) \wedge \neg g \longrightarrow$
 $((f \wedge \text{more}); (\text{while } (\text{init } w) \text{ do } f)) \wedge \neg ((f \wedge \text{more}); g)$
using 23 32 **by fastforce**
have 5: $\vdash f \wedge \text{more} \longrightarrow \text{more}$
by auto
from 4 5 **show** *?thesis* **using** *ChopContra* **by** *blast*
qed

lemma *BaWhileImpWhile*:

$\vdash \text{ba } (f \longrightarrow g) \longrightarrow (\text{while } (\text{init } w) \text{ do } f) \longrightarrow (\text{while } (\text{init } w) \text{ do } g)$
proof –
have 1: $\vdash (f \longrightarrow g) \longrightarrow ((\text{init } w \wedge f) \longrightarrow (\text{init } w \wedge g))$
by auto
hence 2: $\vdash \text{ba } (f \longrightarrow g) \longrightarrow \text{ba } ((\text{init } w \wedge f) \longrightarrow (\text{init } w \wedge g))$
by (*rule BaImpBa*)
have 3: $\vdash \text{ba } ((\text{init } w \wedge f) \longrightarrow (\text{init } w \wedge g)) \longrightarrow ((\text{init } w \wedge f)^* \longrightarrow (\text{init } w \wedge g)^*)$
by (*rule BaCSImpCS*)
have 4: $\vdash \text{ba } (f \longrightarrow g) \longrightarrow ((\text{init } w \wedge f)^* \wedge \text{fin } (\neg (\text{init } w))) \longrightarrow (\text{init } w \wedge g)^* \wedge \text{fin } (\neg (\text{init } w)))$
using 2 3 **by fastforce**
from 4 **show** *?thesis* **by** (*simp add: while-d-def*)
qed

lemma *WhileImpWhile*:

assumes $\vdash f \longrightarrow g$
shows $\vdash (\text{while } (\text{init } w) \text{ do } f) \longrightarrow (\text{while } (\text{init } w) \text{ do } g)$
proof –
have 1: $\vdash f \longrightarrow g$
using *assms* **by auto**
hence 2: $\vdash \text{ba } (f \longrightarrow g)$
by (*rule BaGen*)
have 3: $\vdash \text{ba } (f \longrightarrow g) \longrightarrow (\text{while } (\text{init } w) \text{ do } f) \longrightarrow (\text{while } (\text{init } w) \text{ do } g)$

by (rule BaWhileImpWhile)
 from 2 3 show ?thesis using MP by blast
 qed

5.9 Properties of Halt

lemma *WnextAndMoreEqvNext*:

$\vdash (wnext\ f \wedge more) = \bigcirc f$
 by (auto simp: itl-defs)

lemma *BoxStateAndEmptyEqvStateAndEmpty*:

$\vdash (\Box(empty = (init\ w)) \wedge empty) = ((init\ w) \wedge empty)$
 by (auto simp: itl-defs)

lemma *BoxEmptyEqvIStateqvEmptyAndStateOrNotStateNext*:

$\vdash \Box(empty = (init\ w)) = ((empty \wedge init\ w) \vee (\neg(init\ w) \wedge \bigcirc(\Box(empty = (init\ w)))))$

proof –

have 1: $\vdash \Box(empty = (init\ w)) =$
 $((\Box(empty = (init\ w)) \wedge empty) \vee (\Box(empty = (init\ w)) \wedge more))$
 by (auto simp: empty-d-def)
have 2: $\vdash (\Box(empty = (init\ w)) \wedge empty) = ((init\ w) \wedge empty)$
 using *BoxStateAndEmptyEqvStateAndEmpty* by blast
have 3: $\vdash \Box(empty = (init\ w)) = ((empty = (init\ w)) \wedge wnext(\Box(empty = (init\ w))))$
 using *BoxEqvAndWnextBox* by blast
hence 4: $\vdash (\Box(empty = (init\ w)) \wedge more) =$
 $((empty = (init\ w)) \wedge more) \wedge (wnext(\Box(empty = (init\ w))) \wedge more)$
 by auto
have 5: $\vdash ((empty = (init\ w)) \wedge more) = (\neg(init\ w) \wedge more)$
 by (auto simp: empty-d-def)
have 6: $\vdash (wnext(\Box(empty = (init\ w))) \wedge more) = \bigcirc(\Box(empty = (init\ w)))$
 using *WnextAndMoreEqvNext* by metis
have 7: $\vdash (\Box(empty = (init\ w)) \wedge more) =$
 $((\neg(init\ w) \wedge more) \wedge (wnext(\Box(empty = (init\ w))) \wedge more))$
 using 4 5 by fastforce
have 8: $\vdash ((\neg(init\ w) \wedge more) \wedge (wnext(\Box(empty = (init\ w))) \wedge more)) =$
 $((\neg(init\ w)) \wedge (wnext(\Box(empty = (init\ w))) \wedge more))$ by auto
have 9: $\vdash ((\neg(init\ w)) \wedge (wnext(\Box(empty = (init\ w))) \wedge more)) =$
 $((\neg(init\ w)) \wedge \bigcirc(\Box(empty = (init\ w))))$ using 8 6 by auto
have 10: $\vdash \Box(empty = (init\ w)) = (((init\ w) \wedge empty) \vee (\Box(empty = (init\ w)) \wedge more))$
 using 1 2 by fastforce
 show ?thesis using 10 7 9 by fastforce
 qed

lemma *HaltStateEqvIfStateThenEmptyElseNext*:

$\vdash halt\ (init\ w) = if_i\ (init\ w)\ then\ empty\ else\ (\bigcirc(halt\ (init\ w)))$

proof –

have 1: $\vdash halt\ (init\ w) = \Box(empty = (init\ w))$
 by (simp add: halt-d-def)
have 2: $\vdash \Box(empty = (init\ w)) =$
 $((empty \wedge init\ w) \vee (\neg(init\ w) \wedge \bigcirc(\Box(empty = (init\ w)))))$

by (rule *BoxEmptyEqvIStateqvEmptyAndStateOrNotStateNext*)
 have 21: $\vdash ((\text{empty} \wedge \text{init } w) \vee (\neg(\text{init } w) \wedge \bigcirc(\Box(\text{empty} = (\text{init } w)))) =$
 $((\text{init } w \wedge \text{empty}) \vee (\neg(\text{init } w) \wedge \bigcirc(\Box(\text{empty} = (\text{init } w))))$
 by auto
 have 3: $\vdash \text{if}_i (\text{init } w) \text{ then } \text{empty} \text{ else } (\bigcirc(\text{halt } (\text{init } w))) =$
 $((\text{init } w \wedge \text{empty}) \vee (\neg(\text{init } w) \wedge \bigcirc(\text{halt } (\text{init } w))))$
 by (simp add: *ifthenelse-d-def*)
 show ?thesis by (metis 1 2 21 3 *int-eq*)
 qed

lemma *HaltChopEqv*:

$\vdash ((\text{halt } (\text{init } w)) ; f) = (\text{if}_i (\text{init } w) \text{ then } (f) \text{ else } (\bigcirc(\text{halt } (\text{init } w)); f))$
proof –
 have 1: $\vdash \text{halt}(\text{init } w) =$
 $(\text{if}_i (\text{init } w) \text{ then } \text{empty} \text{ else } (\bigcirc(\text{halt } (\text{init } w))))$
 by (rule *HaltStateEqvIfStateThenEmptyElseNext*)
 hence 2: $\vdash ((\text{halt}(\text{init } w)); f) =$
 $(\text{if}_i (\text{init } w) \text{ then } (\text{empty}; f) \text{ else } (\bigcirc(\text{halt } (\text{init } w)); f))$
 by (rule *IfChopEqvRule*)
 have 3: $\vdash \text{empty} ; f = f$
 by (rule *EmptyChop*)
 have 4: $\vdash (\bigcirc(\text{halt } (\text{init } w))); f = \bigcirc(\text{halt } (\text{init } w); f)$
 by (rule *NextChop*)
 from 2 3 4 show ?thesis by (metis *inteq-reflection*)
 qed

lemma *AndHaltChopImp*:

$\vdash \text{init } w \wedge (\text{halt } (\text{init } w); f) \longrightarrow f$
proof –
 have 1: $\vdash \text{halt } (\text{init } w); f = \text{if}_i (\text{init } w) \text{ then } f \text{ else } (\bigcirc(\text{halt } (\text{init } w); f))$
 by (rule *HaltChopEqv*)
 have 2: $\vdash \text{init } w \wedge \text{if}_i (\text{init } w) \text{ then } f \text{ else } (\bigcirc(\text{halt } (\text{init } w); f)) \longrightarrow f$
 by (auto simp: *ifthenelse-d-def*)
 from 1 2 show ?thesis by fastforce
 qed

lemma *NotAndHaltChopImpNext*:

$\vdash \neg (\text{init } w) \wedge (\text{halt } (\text{init } w); f) \longrightarrow \bigcirc(\text{halt } (\text{init } w); f)$
proof –
 have 1: $\vdash \text{halt } (\text{init } w); f = \text{if}_i (\text{init } w) \text{ then } f \text{ else } (\bigcirc(\text{halt } (\text{init } w); f))$
 by (rule *HaltChopEqv*)
 have 2: $\vdash \neg (\text{init } w) \wedge \text{if}_i (\text{init } w) \text{ then } f \text{ else } (\bigcirc(\text{halt } (\text{init } w); f)) \longrightarrow$
 $\bigcirc(\text{halt } (\text{init } w); f)$
 by (auto simp: *ifthenelse-d-def*)
 from 1 2 show ?thesis by fastforce
 qed

lemma *NotAndHaltChopImpSkipYields*:

$\vdash \neg (\text{init } w) \wedge (\text{halt } (\text{init } w); f) \longrightarrow \text{skip yields } (\text{halt } (\text{init } w); f)$
proof –

have 1: $\vdash \neg (init\ w) \wedge (halt\ (init\ w); f) \longrightarrow \bigcirc (halt\ (init\ w); f)$
by (rule NotAndHaltChopImpNext)
have 2: $\vdash \bigcirc (halt\ (init\ w); f) \longrightarrow skip\ yields\ (halt\ (init\ w); f)$
by (rule NextImpSkipYields)
from 1 2 **show** ?thesis **by** fastforce
qed

lemma TrueChopAndEmptyEqvChopAndEmpty:
 $\vdash ((\#True; (f \wedge empty)) \wedge g) = (g; (f \wedge empty))$
using AndFinEqvChopAndEmpty FinEqvTrueChopAndEmpty **by** (metis int-eq lift-and-com)

lemma WprevEqvEmptyOrPrev:
 $\vdash wprev\ f = (empty \vee prev\ f)$
by (auto simp: itl-defs)

lemma NotChopSkipEqvMoreAndNotChopSkip:
 $\vdash (\neg f); skip = (more \wedge \neg(f; skip))$
proof –
have 1: $\vdash wprev\ f = (empty \vee prev\ f)$ **using** WprevEqvEmptyOrPrev **by** auto
hence 2: $\vdash (\neg(wprev\ f)) = (\neg(empty \vee prev\ f))$ **by** auto
have 3: $\vdash \neg(wprev\ f) = ((\neg f); skip)$ **by** (simp add: wprev-d-def prev-d-def)
have 31: $\vdash (empty \vee prev\ f) = (empty \vee (f; skip))$ **by** (simp add: prev-d-def)
have 32: $\vdash (empty \vee (f; skip)) = (\neg more \vee \neg \neg(f; skip))$ **by** (simp add: empty-d-def)
have 33: $\vdash (\neg more \vee \neg \neg(f; skip)) = (\neg(more \wedge \neg(f; skip)))$ **by** fastforce
have 34: $\vdash (empty \vee prev\ f) = (\neg(more \wedge \neg(f; skip)))$ **using** 31 32 33 **by** (metis int-eq)
have 4: $\vdash \neg(empty \vee prev\ f) = (more \wedge \neg(f; skip))$ **using** 34 **by** fastforce
from 2 3 4 **show** ?thesis **by** fastforce
qed

lemma HaltChopImpNotHaltChopNot:
 $\vdash halt\ (init\ w); f \longrightarrow \neg (halt\ (init\ w); (\neg f))$
proof –
have 1: $\vdash halt\ (init\ w); f = if_i\ (init\ w)\ then\ f\ else\ (\bigcirc (halt\ (init\ w); f))$
by (rule HaltChopEqv)
have 2: $\vdash if_i\ (init\ w)\ then\ f\ else\ (\bigcirc (halt\ (init\ w); f)) \longrightarrow$
 $(((init\ w) \longrightarrow f) \wedge (\neg(init\ w) \longrightarrow (\bigcirc (halt\ (init\ w); f))))$
by (rule IfThenElseImp)
have 3: $\vdash halt\ (init\ w); (\neg f) =$
 $if_i\ (init\ w)\ then\ (\neg f)\ else\ (\bigcirc (halt\ (init\ w); (\neg f)))$
by (rule HaltChopEqv)
have 4: $\vdash if_i\ (init\ w)\ then\ (\neg f)\ else\ (\bigcirc (halt\ (init\ w); (\neg f))) \longrightarrow$
 $(((init\ w) \longrightarrow \neg f) \wedge (\neg(init\ w) \longrightarrow (\bigcirc (halt\ (init\ w); (\neg f))))$
by (rule IfThenElseImp)
have 5: $\vdash halt\ (init\ w); f \wedge halt\ (init\ w); (\neg f) \longrightarrow$
 $(((init\ w) \longrightarrow f) \wedge (\neg(init\ w) \longrightarrow (\bigcirc (halt\ (init\ w); f)))) \wedge$
 $(((init\ w) \longrightarrow \neg f) \wedge (\neg(init\ w) \longrightarrow (\bigcirc (halt\ (init\ w); (\neg f))))$
using 1 2 3 4 **by** fastforce
have 6: $\vdash ((init\ w) \longrightarrow f) \wedge (\neg(init\ w) \longrightarrow (\bigcirc (halt\ (init\ w); f))) \wedge$

$$((init\ w) \longrightarrow \neg f) \wedge (\neg(init\ w) \longrightarrow (\bigcirc(halt\ (init\ w); (\neg f)))) \longrightarrow$$

$$(\bigcirc(halt\ (init\ w); f)) \wedge (\bigcirc(halt\ (init\ w); (\neg f)))$$
by *auto*
have 7: $\vdash halt\ (init\ w); f \wedge halt\ (init\ w); (\neg f) \longrightarrow$

$$(\bigcirc(halt\ (init\ w); f)) \wedge (\bigcirc(halt\ (init\ w); (\neg f)))$$
using 5 6 *lift-imp-trans* **by** *blast*
have 8: $\vdash ((\bigcirc(halt\ (init\ w); f)) \wedge (\bigcirc(halt\ (init\ w); (\neg f)))) =$

$$\bigcirc(halt\ (init\ w); f \wedge halt\ (init\ w); (\neg f))$$
using *NextAndEqvNextAndNext* **by** *fastforce*
have 9: $\vdash halt\ (init\ w); f \wedge halt\ (init\ w); (\neg f) \longrightarrow$

$$\bigcirc(halt\ (init\ w); f \wedge halt\ (init\ w); (\neg f))$$
using 7 8 **by** *fastforce*
hence 10: $\vdash \neg(halt\ (init\ w); f \wedge halt\ (init\ w); (\neg f))$
using *NextLoop* **by** *blast*
from 10 **show** *?thesis* **by** *auto*
qed

lemma *HaltChopImpHaltYields*:

$\vdash halt\ (init\ w); f \longrightarrow (halt\ (init\ w))\ yields\ f$
proof –
have 1: $\vdash halt\ (init\ w); f \longrightarrow \neg(halt\ (init\ w); (\neg f))$ **by** (*rule HaltChopImpNotHaltChopNot*)
from 1 **show** *?thesis* **by** (*simp add: yields-d-def*)
qed

lemma *HaltChopAnd*:

$\vdash (halt\ (init\ w)); f \wedge (halt\ (init\ w)); g \longrightarrow (halt\ (init\ w)); (f \wedge g)$
proof –
have 1: $\vdash (halt\ (init\ w)); g \longrightarrow (halt\ (init\ w))\ yields\ g$ **by** (*rule HaltChopImpHaltYields*)
hence 2: $\vdash (halt\ (init\ w)); f \wedge (halt\ (init\ w)); g \longrightarrow$

$$(halt\ (init\ w)); f \wedge (halt\ (init\ w))\ yields\ g$$
 by *auto*
have 3: $\vdash (halt\ (init\ w)); f \wedge (halt\ (init\ w))\ yields\ g \longrightarrow$

$$(halt\ (init\ w)); (f \wedge g)$$
 by (*rule ChopAndYieldsImp*)
from 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *HaltAndChopAndHaltChopImpHaltAndChopAnd*:

$\vdash (halt\ (init\ w) \wedge f); f1 \wedge (halt\ (init\ w); g) \longrightarrow (halt\ (init\ w) \wedge f); (f1 \wedge g)$
proof –
have 1: $\vdash f1 \longrightarrow \neg g \vee (f1 \wedge g)$
by *auto*
hence 2: $\vdash (halt\ (init\ w) \wedge f); f1 \longrightarrow$

$$(halt\ (init\ w) \wedge f); (\neg g) \vee ((halt\ (init\ w) \wedge f); (f1 \wedge g))$$
by (*rule ChopOrImpRule*)
have 3: $\vdash (halt\ (init\ w) \wedge f); (\neg g) \longrightarrow halt\ (init\ w); (\neg g)$
by (*rule AndChopA*)
have 31: $\vdash (halt\ (init\ w) \wedge f); f1 \longrightarrow$

$$halt\ (init\ w); (\neg g) \vee ((halt\ (init\ w) \wedge f); (f1 \wedge g))$$
using 23 **by** *fastforce*
have 4: $\vdash halt\ (init\ w); g \longrightarrow \neg(halt\ (init\ w); (\neg g))$
by (*rule HaltChopImpNotHaltChopNot*)

hence 41: $\vdash (\text{halt } (\text{init } w); (\neg g)) \longrightarrow \neg(\text{halt } (\text{init } w); g)$
by *auto*
have 42: $\vdash (\text{halt } (\text{init } w) \wedge f); f1 \longrightarrow$
 $\neg(\text{halt } (\text{init } w); g) \vee ((\text{halt } (\text{init } w) \wedge f); (f1 \wedge g))$
using 31 41 **by** *fastforce*
from 42 **show** ?thesis **by** *auto*
qed

lemma *HaltImpBoxYields*:

$\vdash (\text{halt } (\text{init } w)); f \longrightarrow (\Box(\neg (\text{init } w))) \text{ yields } ((\text{halt } (\text{init } w)); f)$
proof –
have 1: $\vdash (\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f)) \longrightarrow \text{di } (\Box(\neg (\text{init } w)))$
by (*rule ChopImpDi*)
have 2: $\vdash \Box(\neg (\text{init } w)) \longrightarrow \neg (\text{init } w)$
by (*rule BoxElim*)
hence 3: $\vdash \text{di } (\Box(\neg (\text{init } w))) \longrightarrow \text{di } (\neg (\text{init } w))$
by (*rule DiImpDi*)
have 4: $\vdash \text{di } (\text{init } (\neg w)) = (\text{init } (\neg w))$
by (*rule DiState*)
have 41: $\vdash (\text{init } (\neg w)) = (\neg (\text{init } w))$
using *Initprop(2)* **by** *fastforce*
have 42: $\vdash \text{di } (\neg (\text{init } w)) = (\neg (\text{init } w))$
using 4 41 **by** (*metis inteq-reflection*)
have 5: $\vdash ((\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f))) \longrightarrow \neg (\text{init } w)$
using 1 2 42 **using** 3 **by** *fastforce*
hence 51: $\vdash (\text{halt } (\text{init } w); f) \wedge ((\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f))) \longrightarrow$
 $(\text{halt } (\text{init } w); f) \wedge \neg (\text{init } w)$
by *fastforce*
have 6: $\vdash \text{halt } (\text{init } w); f = \text{if}_i (\text{init } w) \text{ then } f \text{ else } (\bigcirc(\text{halt } (\text{init } w); f))$
by (*rule HaltChopEqv*)
hence 61: $\vdash (\text{halt } (\text{init } w); f \wedge \neg (\text{init } w)) =$
 $((\text{if}_i (\text{init } w) \text{ then } f \text{ else } (\bigcirc(\text{halt } (\text{init } w); f))) \wedge \neg (\text{init } w))$
using 6 **by** *auto*
have 62: $\vdash (\text{if}_i (\text{init } w) \text{ then } f \text{ else } (\bigcirc(\text{halt } (\text{init } w); f))) \wedge$
 $\neg (\text{init } w) \longrightarrow (\bigcirc(\text{halt } (\text{init } w); f))$
by (*auto simp: ifthenelse-d-def*)
have 63: $\vdash \text{halt } (\text{init } w); f \wedge \neg (\text{init } w) \longrightarrow (\bigcirc(\text{halt } (\text{init } w); f))$
using 61 62 **by** *fastforce*
have 7: $\vdash (\text{halt } (\text{init } w); f) \wedge (\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f)) \longrightarrow$
 $\bigcirc((\text{halt } (\text{init } w)); f)$
using 51 63 **using** *lift-imp-trans* **by** *blast*
have 8: $\vdash \Box(\neg (\text{init } w)) \longrightarrow \text{empty} \vee \bigcirc(\Box(\neg (\text{init } w)))$
using *BoxBoxImpBox BoxEqvAndEmptyOrNextBox* **by** *fastforce*
hence 9: $\vdash ((\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f))) \longrightarrow$
 $\neg (\text{halt } (\text{init } w); f) \vee \bigcirc((\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f)))$
by (*rule EmptyOrNextChopImpRule*)
hence 10: $\vdash ((\text{halt } (\text{init } w)); f) \wedge (\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f)) \longrightarrow$
 $\bigcirc((\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f)))$
by *fastforce*
have 11: $\vdash (\text{halt } (\text{init } w)); f \wedge (\Box(\neg (\text{init } w))); (\neg (\text{halt } (\text{init } w); f)) \longrightarrow$

```

      ○(( halt (init w)); f) ∧ ○((□(¬ (init w))); (¬ ( halt (init w); f)))
using 7 10 by fastforce
have 12: ⊢ ○(( halt (init w)); f) ∧ ○((□(¬ (init w))); (¬ ( halt (init w); f)))
      → ○((( halt (init w)); f) ∧ ((□(¬ (init w))); (¬ ( halt (init w); f))))
using NextAndEqvNextAndNext by fastforce
have 13: ⊢ ( halt (init w)); f ∧ (□ (¬ (init w))); (¬ ( halt (init w); f)) →
      ○((( halt (init w)); f) ∧ ((□(¬ (init w))); (¬ ( halt (init w); f))))
using 11 12 by fastforce
hence 14: ⊢ ¬ (( halt (init w)); f ∧ (□ (¬ (init w))); (¬ ( halt (init w); f)))
using NextLoop by blast
hence 15: ⊢ ( halt (init w)); f → ¬ ((□ (¬ (init w))); (¬ ( halt (init w); f)))
by auto
from 15 show ?thesis by (simp add: yields-d-def)
qed

```

5.10 Properties of Groups of chops

```

lemma NestedChopImpChop:
  assumes ⊢ init w ∧ f → g; (init w1 ∧ f1)
    ⊢ init w1 ∧ f1 → g1; (init w2 ∧ f2)
  shows ⊢ init w ∧ f → g; (g1; (init w2 ∧ f2))
proof –
  have 1: ⊢ init w ∧ f → g; (init w1 ∧ f1) using assms(1) by auto
  have 2: ⊢ init w1 ∧ f1 → g1; (init w2 ∧ f2) using assms(2) by auto
  hence 3: ⊢ g; (init w1 ∧ f1) → g; (g1; (init w2 ∧ f2)) by (rule RightChopImpChop)
  from 1 3 show ?thesis by fastforce
qed

```

end

6 First Order Finite ITL theorems

theory FOTheorems

imports

Theorems

begin

We give the proofs of a list of first order Finite ITL theorems.

lemma EExI-unl:

```

w ⊨ f x ⇒ w ⊨ (∃ ∃ x. f x)
using EExVal by auto

```

lemma EExNoDep:

```

⊢ (∃ ∃ x. g) = g

```

proof –

have 1: $\vdash g \longrightarrow (\exists \exists x. g)$ **by** (*meson EExI*)
have 2: $\bigwedge x. \vdash g \longrightarrow g$ **by** *simp*
have 3: $\vdash (\exists \exists x. g) \longrightarrow g$ **using** 2 **by** (*meson EExE*)
from 1 3 **show** *?thesis* **using** *int-iffI* **by** *blast*
qed

lemma *AAxNoDep*:
 $\vdash (\forall \forall x. g) = g$
using *EExNoDep*[*of LIFT*($\neg g$)] *AAxDef* *EExE* *EExI*
by (*simp add: exist-state-d-def forall-state-d-def intI*)

lemma *EExEqvRule*:
assumes $\bigwedge x. \vdash f x = g x$
shows $\vdash (\exists \exists x. f x) = (\exists \exists x. g x)$
by (*metis EExE EExI assms int-iffD1 int-iffD2 int-iffI lift-imp-trans*)

lemma *AAxImpEEx*:
 $\vdash (\forall \forall x. f x) \longrightarrow (\exists \exists x. f x)$
by (*simp add: exist-state-d-def forall-state-d-def intI*)

lemma *EExImpRule*:
assumes $\vdash f x \longrightarrow g x$
shows $\vdash (\exists \exists x. f x \longrightarrow g x)$
using *assms* **by** (*meson MP EExI*)

lemma *EExImpRuleDist*:
assumes $\vdash f x \longrightarrow g x$
shows $\vdash (\forall \forall x. f x) \longrightarrow (\exists \exists x. g x)$
proof –
have 1: $\vdash (f x) \longrightarrow (\exists \exists x. g x)$ **using** *EExI* *assms* *lift-imp-trans* **by** *blast*
have 2: $\vdash \neg(f x) \vee (\exists \exists x. g x)$ **using** 1 **by** *auto*
have 3: $\vdash \neg(f x) \longrightarrow (\exists \exists x. \neg(f x))$ **by** (*meson EExI*)
have 4: $\vdash (\exists \exists x. \neg(f x)) = (\neg(\forall \forall x. f x))$ **using** *AAxDef* **by** *fastforce*
from 2 3 4 **show** *?thesis* **by** *fastforce*
qed

lemma *EExImpNoDepDist*:
assumes $\vdash f \longrightarrow g x$
shows $\vdash f \longrightarrow (\exists \exists x. g x)$
using *assms* **by** (*metis EExI lift-imp-trans*)

lemma *EExOrDist-1*:
 $\vdash (\exists \exists x. h x) \longrightarrow (\exists \exists x. (f x) \vee (h x))$
proof –
have 1: $\bigwedge x. \vdash h x \longrightarrow f x \vee h x$ **by** (*simp add: Valid-def*)
have 2: $\bigwedge x. \vdash f x \vee h x \longrightarrow (\exists \exists x. (f x) \vee (h x))$ **by** (*meson EExI*)
have 3: $\bigwedge x. \vdash h x \longrightarrow (\exists \exists x. (f x) \vee (h x))$ **using** 1 2 **by** (*meson lift-imp-trans*)
from 3 **show** *?thesis* **using** *EExE* **by** *blast*
qed

lemma *EEExOrDist-2*:

$\vdash (\exists \exists x. f x) \longrightarrow (\exists \exists x. (f x) \vee (h x))$

proof –

have 1: $\bigwedge x. \vdash f x \longrightarrow f x \vee h x$ **by** (*simp add: Valid-def*)

have 2: $\bigwedge x. \vdash f x \vee h x \longrightarrow (\exists \exists x. (f x) \vee (h x))$ **by** (*meson EEExI*)

have 3: $\bigwedge x. \vdash f x \longrightarrow (\exists \exists x. (f x) \vee (h x))$ **using** 1 2 **by** (*meson lift-imp-trans*)

from 3 **show** *?thesis* **using** *EEExE* **by** *blast*

qed

lemma *EEExOrDist-3*:

$\vdash (\exists \exists x. f x) \vee (\exists \exists x. h x) \longrightarrow (\exists \exists x. (f x) \vee (h x))$

using *EEExOrDist-2 EEExOrDist-1* **by** *fastforce*

lemma *EEExOrDist-4*:

$\vdash (\exists \exists x. (f x) \vee (h x)) \longrightarrow (\exists \exists x. f x) \vee (\exists \exists x. h x)$

proof –

have 1: $\bigwedge x. \vdash (f x) \vee (h x) \longrightarrow (\exists \exists x. f x) \vee (\exists \exists x. h x)$

by (*simp add: EEExI-unl intI*)

from 1 **show** *?thesis* **by** (*simp add: EEExE*)

qed

lemma *EEExOrDist*:

$\vdash ((\exists \exists x. f x) \vee (\exists \exists x. h x)) = (\exists \exists x. (f x) \vee (h x))$

using *EEExOrDist-3 EEExOrDist-4* **by** *fastforce*

lemma *EEExOrImport-1*:

$\vdash g \longrightarrow (\exists \exists x. g \vee (f x))$

by (*simp add: EEExI-unl Valid-def*)

lemma *EEExOrImport-2*:

$\vdash (\exists \exists x. f x) \longrightarrow (\exists \exists x. g \vee (f x))$

by (*simp add: EEExOrDist-1*)

lemma *EEExOrImport-3*:

$\vdash (g \vee (\exists \exists x. f x)) \longrightarrow (\exists \exists x. g \vee (f x))$

using *EEExOrImport-1 EEExOrImport-2* **by** *fastforce*

lemma *EEExOrImport-4*:

$\vdash (\exists \exists x. g \vee f x) \longrightarrow (g \vee (\exists \exists x. f x))$

proof –

have 1: $\bigwedge x. \vdash g \vee f x \longrightarrow g \vee (\exists \exists x. f x)$ **by** (*meson EEExI int-iffD2 int-simps(27) Prop04 Prop08*)

from 1 **show** *?thesis* **by** (*simp add: EEExE*)

qed

lemma *EEExOrImport*:

$\vdash (g \vee (\exists \exists x. f x)) = (\exists \exists x. g \vee f x)$

by (*metis EEExOrImport-3 EEExOrImport-4 int-iffI*)

lemma *EEExAndImport-1*:

$\vdash g \wedge (\exists \exists x. f x) \longrightarrow (\exists \exists x. g \wedge f x)$

proof –
have 1: $\vdash (g \wedge (\exists \exists x. f x) \longrightarrow (\exists \exists x. g \wedge f x)) =$
 $((\exists \exists x. f x) \longrightarrow (g \longrightarrow (\exists \exists x. g \wedge f x)))$ **by** *fastforce*
have 2: $\bigwedge x. \vdash f x \longrightarrow (g \longrightarrow (\exists \exists x. g \wedge f x))$ **by** (*metis EExI int-eq lift-and-com Prop09*)
hence 3: $\vdash (\exists \exists x. f x) \longrightarrow (g \longrightarrow (\exists \exists x. g \wedge f x))$ **by** (*simp add: EExE*)
from 1 3 **show** *?thesis* **by** *auto*
qed

lemma *EExAndImport-2*:
 $\vdash (\exists \exists x. g \wedge f x) \longrightarrow g \wedge (\exists \exists x. f x)$
proof –
have 1: $\bigwedge x. \vdash g \wedge f x \longrightarrow g \wedge (\exists \exists x. f x)$
by (*metis EExI int-iffD2 lift-and-com lift-imp-trans Prop12*)
from 1 **show** *?thesis* **by** (*simp add: EExE*)
qed

lemma *EExAndImport*:
 $\vdash (g \wedge (\exists \exists x. f x)) = (\exists \exists x. g \wedge f x)$
by (*simp add: EExAndImport-1 EExAndImport-2 int-iffI*)

lemma *EExAndDist*:
assumes $\vdash f x \wedge g x$
shows $\vdash (\exists \exists x. f x) \wedge (\exists \exists x. g x)$
proof –
have 1: $\vdash f x$ **using** *assms* **by** *fastforce*
have 2: $\vdash g x$ **using** *assms* **by** *fastforce*
have 3: $\vdash (\exists \exists x. f x)$ **using** 1 **by** (*meson EExI MP*)
have 4: $\vdash (\exists \exists x. g x)$ **using** 2 **by** (*meson EExI MP*)
from 3 4 **show** *?thesis* **by** *fastforce*
qed

lemma *EExAndNoDepDist*:
assumes $\vdash f \wedge g x$
shows $\vdash f \wedge (\exists \exists x. g x)$
proof –
have 1: $\vdash f$ **using** *assms* **by** *fastforce*
have 2: $\vdash g x$ **using** *assms* **by** *fastforce*
have 3: $\vdash (\exists \exists x. g x)$ **using** 2 **by** (*meson EExI MP*)
from 1 3 **show** *?thesis* **by** *fastforce*
qed

lemma *Spec*:
 $\vdash (\forall \forall x. f x) \longrightarrow f x$
proof –
have 1: $\vdash \neg(f x) \longrightarrow (\exists \exists x. \neg(f x))$ **by** (*meson EExI*)
have 2: $\vdash \neg(\exists \exists x. \neg(f x)) \longrightarrow f x$ **using** 1 **by** *auto*
from 2 **show** *?thesis* **using** *AAxDef* **by** *fastforce*
qed

lemma *AAxE*:
assumes $\vdash (\forall \forall x. f x)$
 $\vdash f x \longrightarrow g$
shows $\vdash g$
using *MP Spec assms(1) assms(2) by blast*

lemma *AAxI*:
assumes $\bigwedge x. \vdash f x$
shows $\vdash (\forall \forall x. f x)$
using *assms by (simp add: Valid-def exist-state-d-def forall-state-d-def)*

lemma *AAxEqvRule*:
assumes $\bigwedge x. \vdash f x = g x$
shows $\vdash (\forall \forall x. f x) = (\forall \forall x. g x)$
by (*metis (mono-tags, lifting) AAxDef EExEqvRule assms int-iffD1 int-iffI*
inteq-reflection lift-imp-neg)

lemma *AAxAndDist*:
 $\vdash (\forall \forall x. (f x) \wedge (g x)) = ((\forall \forall x. f x) \wedge (\forall \forall x. g x))$
proof –
have 1: $\vdash ((\exists \exists x. \neg(f x)) \vee (\exists \exists x. \neg(g x))) = (\exists \exists x. \neg(f x) \vee \neg(g x))$
by (*simp add: EExOrDist*)
have 2: $\vdash ((\exists \exists x. \neg(f x))) = (\neg(\forall \forall x. f x))$
using *AAxDef by fastforce*
have 3: $\vdash ((\exists \exists x. \neg(g x))) = (\neg(\forall \forall x. g x))$
using *AAxDef by fastforce*
have 4: $\vdash ((\exists \exists x. \neg(f x)) \vee (\exists \exists x. \neg(g x))) = (\neg(\forall \forall x. f x) \vee \neg(\forall \forall x. g x))$
using 2 3 **by** *fastforce*
have 5: $\bigwedge x. \vdash (\neg(f x) \vee \neg(g x)) = (\neg((f x) \wedge (g x)))$
by *auto*
have 6: $\vdash (\exists \exists x. \neg(f x) \vee \neg(g x)) = (\exists \exists x. \neg((f x) \wedge (g x)))$
using 5 **by** (*simp add: EExEqvRule*)
have 7: $\vdash (\exists \exists x. \neg((f x) \wedge (g x))) = (\neg(\forall \forall x. (f x) \wedge (g x)))$
using *AAxDef by fastforce*
have 8: $\vdash (\neg(\forall \forall x. f x) \vee \neg(\forall \forall x. g x)) = (\neg((\forall \forall x. f x) \wedge (\forall \forall x. g x)))$
by *fastforce*
have 9: $\vdash (\neg((\forall \forall x. f x) \wedge (\forall \forall x. g x))) = (\neg(\forall \forall x. (f x) \wedge (g x)))$
using 1 4 6 7 8 **by** *fastforce*
from 9 **show** *?thesis by fastforce*
qed

lemma *AAxAndImport*:
 $\vdash (g \wedge (\forall \forall x. f x)) = (\forall \forall x. g \wedge f x)$
proof –
have 1: $\vdash (\neg g \vee (\exists \exists x. \neg(f x))) = (\exists \exists x. \neg g \vee \neg(f x))$ **by** (*simp add: EExOrImport*)
have 2: $\vdash ((\exists \exists x. \neg(f x))) = (\neg(\forall \forall x. f x))$ **using** *AAxDef by fastforce*
have 3: $\vdash (\neg g \vee (\exists \exists x. \neg(f x))) = (\neg(g \wedge (\forall \forall x. f x)))$ **using** 2 **by** *fastforce*
have 4: $\bigwedge x. \vdash (\neg g \vee \neg(f x)) = (\neg(g \wedge f x))$ **by** *auto*
have 5: $\vdash (\exists \exists x. \neg g \vee \neg(f x)) = (\exists \exists x. \neg(g \wedge f x))$ **using** 4 **by** (*simp add: EExEqvRule*)

have 6: $\vdash (\exists \exists x. \neg(g \wedge f x)) = (\neg(\forall \forall x. g \wedge f x))$ **using** *AAxDef* **by** *fastforce*
have 7: $\vdash (\neg(g \wedge (\forall \forall x. f x))) = (\neg(\forall \forall x. g \wedge f x))$ **using** 1 3 5 6 **by** *fastforce*
from 7 **show** *?thesis* **by** *fastforce*
qed

lemma *AAxOrImport*:

$\vdash (g \vee (\forall \forall x. f x)) = (\forall \forall x. g \vee f x)$

proof –

have 1: $\vdash (\neg g \wedge (\exists \exists x. \neg(f x))) = (\exists \exists x. \neg g \wedge \neg(f x))$ **by** (*simp add: EExAndImport*)
have 2: $\vdash (\exists \exists x. \neg(f x)) = (\neg((\forall \forall x. f x)))$ **using** *AAxDef* **by** *fastforce*
have 3: $\vdash (\neg g \wedge (\exists \exists x. \neg(f x))) = (\neg(g \vee (\forall \forall x. f x)))$ **using** 2 **by** *fastforce*
have 4: $\bigwedge x. \vdash (\neg g \wedge \neg(f x)) = (\neg(g \vee f x))$ **by** *auto*
have 5: $\vdash (\exists \exists x. \neg g \wedge \neg(f x)) = (\exists \exists x. \neg(g \vee f x))$ **using** 4 **by** (*simp add: EExEqvRule*)
have 6: $\vdash (\exists \exists x. \neg(g \vee f x)) = (\neg(\forall \forall x. g \vee f x))$ **using** *AAxDef* **by** *fastforce*
have 7: $\vdash (\neg(g \vee (\forall \forall x. f x))) = (\neg(\forall \forall x. g \vee f x))$ **using** 1 3 5 6 **by** *fastforce*
from 7 **show** *?thesis* **by** *auto*

qed

lemma *EExImpChopRule*:

assumes $\vdash f x \longrightarrow g x$

shows $\vdash (\exists \exists x. h; (f x) \longrightarrow h; (g x))$

using *RightChopImpChop*[*of f x g x h*]

EExImpRule[*of* $\lambda x. \text{LIFT}(h; (f x))$ $x \lambda x. \text{LIFT}(h; (g x))$] *assms* **by** *auto*

lemma *EExChopRight*:

$\vdash (\exists \exists x. (f x); g) \longrightarrow (\exists \exists x. f x); g$

proof –

have 1: $\bigwedge x. \vdash (f x); g \longrightarrow (\exists \exists x. f x); g$ **by** (*simp add: EExI LeftChopImpChop*)

from 1 **show** *?thesis* **by** (*simp add: EExE*)

qed

lemma *EExChopRightNoDep*:

$\vdash (\exists \exists x. (f x); g) = (\exists \exists x. f x); g$

by (*auto simp add: exist-state-d-def Valid-def itl-defs*)

lemma *EExChopLeft* :

$\vdash (\exists \exists x. g; (f x)) \longrightarrow g; (\exists \exists x. f x)$

proof –

have 1: $\bigwedge x. \vdash g; (f x) \longrightarrow g; (\exists \exists x. f x)$ **by** (*simp add: EExI RightChopImpChop*)

from 1 **show** *?thesis* **by** (*simp add: EExE*)

qed

lemma *EExChopLeftNoDep*:

$\vdash (\exists \exists x. g; (f x)) = g; (\exists \exists x. f x)$

by (*auto simp add: exist-state-d-def Valid-def itl-defs*)

lemma *EExEExChopEqvEExEExChop*:

$\vdash (\exists \exists v. (\exists \exists y. (f v); (g y))) = (\exists \exists y. (\exists \exists v. (f v); (g y)))$

by (*simp add: exist-state-d-def Valid-def itl-defs*) *blast*

lemma *EEExEEExChopEqvEEExChopEEExA*:
 $\vdash (\exists \exists v. (\exists \exists y. (f\ v);(g\ y))) = (\exists \exists v. (f\ v);(\exists \exists y. (g\ y)))$
by (*simp add: exist-state-d-def Valid-def itl-defs*) *blast*

lemma *EEExEEExChopEqvEEExChopEEExB*:
 $\vdash (\exists \exists y. (\exists \exists v. (f\ v);(g\ y))) = (\exists \exists y. (\exists \exists v. (f\ v)); (g\ y))$
by (*simp add: exist-state-d-def Valid-def itl-defs*) *blast*

lemma *EEExEEExChopEqvEEExChopEEExC*:
 $\vdash (\exists \exists v. (\exists \exists y. (f\ v);(g\ y))) = (\exists \exists v. (f\ v));(\exists \exists y. (g\ y))$
using *EEExChopRightNoDep*[*of f LIFT(($\exists \exists y. g\ y$))*]
EEExEEExChopEqvEEExChopEEExA[*of f g*]
by *fastforce*

lemma *ExLen*:
 $\vdash \exists n. \text{len}(n)$
by (*simp add: Valid-def itl-defs*)

lemma *CSPowerChop*:
 $\vdash (f^*) = (\exists n. \text{power}\ (f \wedge \text{more})\ n)$
by (*simp add: chopstar-d-def powerstar-d-def Valid-def*)

lemma *ExChopRightNoDep*:
 $\vdash (\exists x. (f\ x);g) = (\exists x. (f\ x));g$
by (*auto simp add: Valid-def itl-defs*)

lemma *ExChopLeftNoDep*:
 $\vdash (\exists x. g;(f\ x)) = g;(\exists x. f\ x)$
by (*auto simp add: Valid-def itl-defs*)

lemma *ExExEqvExEx*:
 $\vdash (\exists x. (\exists y. (f\ x);(g\ y))) = (\exists y. (\exists x. (f\ x);(g\ y)))$
by (*auto simp add: Valid-def itl-defs*)

lemma *TassignEqvExFin*:
 $\vdash v \leftarrow e = (\exists c. \#c=e \wedge \text{fin}(\$v = \#c))$
by (*simp add: Valid-def itl-defs*)

lemma *MoreImpNextassignEqvExNext*:
 $\vdash \text{more} \longrightarrow (v := e) = (\exists c. \#c=e \wedge \bigcirc(\$v = \#c))$
by (*simp add: Valid-def next-assign-d-def next-val-d-def itl-defs*)

lemma *MoreImpPrevassignEqvExPrevFin*:
 $\vdash \text{more} \longrightarrow (v =: e) = (\exists c. \#c=e \wedge \text{prev}(\text{fin}(\$v = \#c)))$
by (*auto simp add: Valid-def prev-assign-d-def itl-defs penult-val-d-def*)

end

7 Time Reversal

```

theory TimeReversal
imports
  Theorems FOTheorems
begin

```

Time reversal operator is defined in [6].

7.1 Definition

```

definition reverse-d :: ('a::world, 'b) formfun  $\Rightarrow$  ('a, 'b) formfun
where reverse-d F  $\equiv \lambda s. irev s \models F$ 

```

```

syntax
  -reverse-d      :: lift  $\Rightarrow$  lift      ((r-) [85] 85)

```

```

syntax (ASCII)
  -reverse-d      :: lift  $\Rightarrow$  lift      ((reverse -) [85] 85)

```

```

translations
  -reverse-d       $\equiv$  CONST reverse-d

```

7.2 Time reversal Rules

```

lemma EExRev :
   $\vdash (\exists \exists x. F x)^r = (\exists \exists x. (F x)^r)$ 
by (simp add: Valid-def exist-state-d-def reverse-d-def)

```

```

lemma rev-const :
   $\vdash (\#c)^r = \#c$ 
by (auto simp: reverse-d-def)

```

```

lemma rev-fun1 :
   $\vdash (f<x>)^r = f<x^r>$ 
by (auto simp: reverse-d-def)

```

```

lemma rev-fun2:
   $\vdash (f<x,y>)^r = f<x^r,y^r>$ 
by (auto simp: reverse-d-def)

```

```

lemma rev-fun3:
   $\vdash (f<x,y,z>)^r = f<x^r,y^r,z^r>$ 
by (auto simp: reverse-d-def)

```

```

lemma rev-forall:
   $\vdash (\forall x. P x)^r = (\forall x. (P x)^r)$ 
by (auto simp: reverse-d-def)

```

```

lemma rev-exists:
   $\vdash (\exists x. P x)^r = (\exists x. (P x)^r)$ 

```

by (*auto simp: reverse-d-def*)

lemma *rev-exists1*:

$\vdash (\exists! x. P x)^r = (\exists! x. (P x)^r)$

by (*auto simp: reverse-d-def*)

lemma *rev-current*:

$\vdash (\$v)^r = (!v)$

by (*auto simp: irev-inth itl-def reverse-d-def*)

lemma *rev-next*:

$\vdash (v\$)^r = (v!)$

by (*auto simp: irev-inth itl-def reverse-d-def*)

lemma *rev-penult*:

$\vdash (v!)^r = (v\$)$

by (*auto simp: irev-inth itl-def reverse-d-def*)

lemma *rev-fin*:

$\vdash (!v)^r = (\$v)$

by (*auto simp: irev-inth itl-def reverse-d-def*)

lemma *EqvReverseReverse*:

$\vdash (f^r)^r = f$

by (*simp add: Valid-def reverse-d-def*)

lemma *ReverseEqv*:

$(\vdash f) \longleftrightarrow (\vdash f^r)$

by (*metis Valid-def irev-swap reverse-d-def*)

lemma *RevSkip*:

$\vdash \text{skip}^r = \text{skip}$

by (*simp add: Valid-def reverse-d-def skip-defs*)

lemma *RevChop*:

$\vdash (f;g)^r = (g^r;f^r)$

proof (*auto simp add: Valid-def itl-def reverse-d-def*)

show $\bigwedge w n. n \leq \text{ilen } w \implies$

$f (\text{prefix } n (\text{irev } w)) \implies$

$g (\text{suffix } n (\text{irev } w)) \implies$

$\exists n \leq \text{ilen } w. g (\text{irev } (\text{prefix } n w)) \wedge f (\text{irev } (\text{suffix } n w))$

by (*metis diff-diff-cancel irev-prefix irev-suffix suffix-ilen-bound suffix-ilen*)

show $\bigwedge w n. n \leq \text{ilen } w \implies$

$g (\text{irev } (\text{prefix } n w)) \implies$

$f (\text{irev } (\text{suffix } n w)) \implies$

$\exists n \leq \text{ilen } w. f (\text{prefix } n (\text{irev } w)) \wedge g (\text{suffix } n (\text{irev } w))$

by (*metis irev-prefix irev-suffix suffix-ilen-bound suffix-ilen*)

qed

lemma *RMoreEqvMore*:

$\vdash \text{more}^r = \text{more}$
by (*simp add: Valid-def itl-def reverse-d-def*)

lemma *REmptyEqvEmpty*:
 $\vdash \text{empty}^r = \text{empty}$
by (*metis RMoreEqvMore empty-d-def int-eq rev-fun1*)

lemma *PowerCommute*:
 $\vdash ((f \wedge \text{more}); (\text{power } (f \wedge \text{more}) \ n)) = (\text{power } (f \wedge \text{more}) \ n); (f \wedge \text{more})$
proof
 (*induct n*)
case 0
then show ?*case* **by** (*metis ChopEmpty EmptyChop inteq-reflection pow-0*)
next
case (*Suc n*)
then show ?*case* **by** (*metis ChopAssoc inteq-reflection pow-Suc*)
qed

lemma *REqvRule*:
assumes $\vdash f = g$
shows $\vdash (f^r) = (g^r)$
using *assms*
using *inteq-reflection* **by** *force*

lemma *RevPowerChop*:
 $\vdash (\text{power } (f \wedge \text{more}) \ n)^r = (\text{power } ((f \wedge \text{more})^r) \ n)$
proof
 (*induct n*)
case 0
then show ?*case* **using** *REmptyEqvEmpty* **by** *auto*
next
case (*Suc n*)
then show ?*case*
by (*metis PowerCommute RevChop inteq-reflection pow-Suc*)
qed

lemma *RevChopstar*:
 $\vdash (f^*)^r = (f^r)^*$
proof –
have 1: $\vdash (f^*) = (\exists n. \text{power } (f \wedge \text{more}) \ n)$
by (*simp add: chopstar-d-def powerstar-d-def Valid-def*)
have 2: $\vdash (f^*)^r = (\exists n. \text{power } (f \wedge \text{more}) \ n)^r$
using *REqvRule 1* **by** *blast*
have 3: $\vdash (\exists n. \text{power } (f \wedge \text{more}) \ n)^r = (\exists n. (\text{power } (f \wedge \text{more}) \ n)^r)$
by (*simp add: rev-exists*)
have 4: $\vdash (\exists n. (\text{power } (f \wedge \text{more}) \ n)^r) = (\exists n. (\text{power } ((f \wedge \text{more})^r) \ n))$
by (*simp add: RevPowerChop ExEqvRule*)
have 5: $\vdash (f \wedge \text{more})^r = (f^r \wedge \text{more})$
by (*metis RMoreEqvMore inteq-reflection rev-fun2*)
hence 6: $\vdash (\exists n. (\text{power } ((f \wedge \text{more})^r) \ n)) = (\exists n. (\text{power } ((f^r \wedge \text{more}) \ n)))$

by (metis 4 inteq-reflection)
 have 7: $\vdash (\exists n. (\text{power } ((f^r \wedge \text{more})) n)) = (f^r)^*$
 by (simp add: chopstar-d-def powerstar-d-def Valid-def)
 from 2 3 4 6 7 show ?thesis by fastforce
 qed

lemmas all-rev = rev-const rev-fun1 rev-fun2 rev-fun3 rev-forall rev-exists
 rev-exists1 rev-current rev-next rev-penult rev-fin RevSkip RevChop RevChopstar

lemmas all-rev-unl = all-rev[THEN intD]
 lemmas all-rev-eq = all-rev[THEN inteq-reflection]

7.3 Properties of Time Reversal

lemma RNot:
 $\vdash (\neg f)^r = (\neg f^r)$
 by (simp add: rev-fun1)

lemma RNot:
 $\vdash (\neg(f^r))^r = (\neg f)$
 by (metis EquReverseReverse int-eq rev-fun1)

lemma RTrue:
 $\vdash (\# \text{True})^r = \# \text{True}$
 using rev-const by fastforce

lemma ROr:
 $\vdash (f \vee g)^r = (f^r \vee g^r)$
 by (simp add: rev-fun2)

lemma RROr:
 $\vdash (f^r \vee g^r)^r = (f \vee g)$
 proof –
 have 1: $\vdash (f^r \vee g^r)^r = ((f^r)^r \vee (g^r)^r)$ using ROr by blast
 have 2: $\vdash ((f^r)^r \vee (g^r)^r) = (f \vee g)$ using EquReverseReverse by (metis inteq-reflection)
 from 1 2 show ?thesis by fastforce
 qed

lemma RAnd:
 $\vdash (f \wedge g)^r = (f^r \wedge g^r)$
 by (simp add: rev-fun2)

lemma RRAnd:
 $\vdash (f^r \wedge g^r)^r = (f \wedge g)$
 proof –
 have 1: $\vdash (f^r \wedge g^r)^r = ((f^r)^r \wedge (g^r)^r)$ using RAnd by blast
 have 2: $\vdash ((f^r)^r \wedge (g^r)^r) = (f \wedge g)$ using EquReverseReverse by (metis inteq-reflection)
 from 1 2 show ?thesis by fastforce
 qed

lemma *RImpRule*:
assumes $\vdash f \longrightarrow g$
shows $\vdash f^r \longrightarrow g^r$
using *assms* **by** (*simp add: Valid-def reverse-d-def*)

lemma *RAndEmptyEqvAndEmpty*:
 $\vdash (f \wedge \text{empty})^r = (f \wedge \text{empty})$
by (*simp add: Valid-def itl-defs reverse-d-def, metis INil-ilen irev.simps(1)*)

lemma *RNextEqvPrev*:
 $\vdash (\bigcirc f)^r = \text{prev } (f^r)$
by (*metis RevChop RevSkip inteq-reflection next-d-def prev-d-def*)

lemma *RRNextEqvPrev*:
 $\vdash (\bigcirc (f^r))^r = \text{prev } (f)$
proof –
have 1: $\vdash (\bigcirc (f^r))^r = \text{prev } ((f^r)^r)$ **using** *RNextEqvPrev* **by** *blast*
have 2: $\vdash \text{prev } ((f^r)^r) = \text{prev } f$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *RWNextEqvWPrev*:
 $\vdash (\text{wnext } f)^r = \text{wprev } (f^r)$
by (*simp add: all-rev-eq(12) all-rev-eq(13) all-rev-eq(2) next-d-def prev-d-def wnext-d-def wprev-d-def*)

lemma *RRWNextEqvWPrev*:
 $\vdash (\text{wnext } (f^r))^r = \text{wprev } (f)$
proof –
have 1: $\vdash (\text{wnext } (f^r))^r = \text{wprev } ((f^r)^r)$ **using** *RWNextEqvWPrev* **by** *blast*
have 2: $\vdash \text{wprev } ((f^r)^r) = \text{wprev } f$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *RPrevEqvNext*:
 $\vdash (\text{prev } f)^r = \bigcirc (f^r)$
by (*metis RevChop RevSkip inteq-reflection next-d-def prev-d-def*)

lemma *RRPrevEqvNext*:
 $\vdash (\text{prev } (f^r))^r = \bigcirc (f)$
proof –
have 1: $\vdash (\text{prev } (f^r))^r = \bigcirc ((f^r)^r)$ **using** *RPrevEqvNext* **by** *blast*
have 2: $\vdash \bigcirc ((f^r)^r) = \bigcirc f$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *RWPrevEqvWNext*:
 $\vdash (\text{wprev } f)^r = \text{wnext } (f^r)$

by (*metis* *EqvReverseReverse* *RRWNextEqvWPrev* *int-eq*)

lemma *RRWPrevEqvWNext*:

$\vdash (wprev (f^r))^r = wnext(f)$

proof –

have 1: $\vdash (wprev (f^r))^r = wnext ((f^r)^r)$ **using** *RRWPrevEqvWNext* **by** *blast*

have 2: $\vdash wnext ((f^r)^r) = wnext f$ **using** *EqvReverseReverse* **by** (*metis* *inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RDiamondEqvDi*:

$\vdash (\Diamond f)^r = di (f^r)$

by (*simp* *add*: *di-d-def* *sometimes-d-def*, *metis* *RevChop* *RTrue* *inteq-reflection*)

lemma *RRDiamondEqvDi*:

$\vdash (\Diamond (f^r))^r = di (f)$

proof –

have 1: $\vdash (\Diamond (f^r))^r = di ((f^r)^r)$ **using** *RDiamondEqvDi* **by** *blast*

have 2: $\vdash di ((f^r)^r) = di f$ **using** *EqvReverseReverse* **by** (*metis* *inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RBoxEqvBi*:

$\vdash (\Box f)^r = bi (f^r)$

by (*simp* *add*: *always-d-def* *bi-d-def*, *metis* *RDiamondEqvDi* *int-eq* *rev-fun1*)

lemma *RRBoxEqvBi*:

$\vdash (\Box (f^r))^r = bi (f)$

proof –

have 1: $\vdash (\Box (f^r))^r = bi ((f^r)^r)$ **using** *RBoxEqvBi* **by** *blast*

have 2: $\vdash bi ((f^r)^r) = bi f$ **using** *EqvReverseReverse* **by** (*metis* *inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RDIEqvDiamond*:

$\vdash (di f)^r = \Diamond (f^r)$

by (*simp* *add*: *di-d-def* *sometimes-d-def*, *metis* *RevChop* *RTrue* *inteq-reflection*)

lemma *RRDiEqvDiamond*:

$\vdash (di (f^r))^r = \Diamond (f)$

proof –

have 1: $\vdash (di (f^r))^r = \Diamond ((f^r)^r)$ **using** *RDIEqvDiamond* **by** *blast*

have 2: $\vdash \Diamond ((f^r)^r) = \Diamond f$ **using** *EqvReverseReverse* **by** (*metis* *inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RBIEqvBox*:

$\vdash (bi f)^r = \Box (f^r)$

by (*simp* *add*: *always-d-def* *bi-d-def*, *metis* *RDIEqvDiamond* *rev-fun1* *int-eq*)

lemma *RRBiEqvBox*:

$\vdash (bi\ (f^r))^r = \square\ (f)$

proof –

have 1: $\vdash (bi\ (f^r))^r = \square\ ((f^r)^r)$ **using** *RBiEqvBox* **by** *blast*

have 2: $\vdash \square\ ((f^r)^r) = \square\ f$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RDaEqvDa*:

$\vdash (da\ f)^r = da(f^r)$

proof –

have 1: $\vdash (\#True;(f;\#True))^r = (f;\#True)^r; \#True^r$ **using** *RevChop* **by** *blast*

have 2: $\vdash (f;\#True)^r; \#True^r = (f;\#True)^r; \#True$ **using** *RTrue RightChopEqvChop* **by** *blast*

have 3: $\vdash (f;\#True)^r; \#True = (\#True^r;f^r);\#True$ **by** (*simp add: RevChop LeftChopEqvChop*)

have 4: $\vdash (\#True^r;f^r);\#True = (\#True;f^r);\#True$ **by** (*metis 3 RTrue int-eq*)

have 5: $\vdash (\#True;f^r);\#True = \#True;(f^r;\#True)$ **using** *ChopAssocB* **by** *blast*

have 6: $\vdash (\#True;(f;\#True))^r = \#True;(f^r;\#True)$ **using** 1 2 3 4 5 **by** *fastforce*

from 6 **show** *?thesis* **by** (*simp add: da-d-def*)

qed

lemma *RRDaEqvDa*:

$\vdash (da\ (f^r))^r = da(f)$

proof –

have 1: $\vdash (da\ (f^r))^r = da\ ((f^r)^r)$ **using** *RDaEqvDa* **by** *blast*

have 2: $\vdash da\ ((f^r)^r) = da\ f$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RBaEqvBa*:

$\vdash (ba\ f)^r = ba(f^r)$

by (*simp add: ba-d-def, metis RDaEqvDa int-eq rev-fun1*)

lemma *RRBaEqvBa*:

$\vdash (ba\ (f^r))^r = ba(f)$

proof –

have 1: $\vdash (ba\ (f^r))^r = ba\ ((f^r)^r)$ **using** *RBaEqvBa* **by** *blast*

have 2: $\vdash ba\ ((f^r)^r) = ba\ f$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *ChopCsImpCSChop*:

$\vdash f;f^* \longrightarrow f^*;f$

by (*meson CSChopEqvChopOrRule CSChopEqvOrChopPlusChop ChopAssocB ChopPlusElimWithoutMore EmptyYields Prop03 Prop04 Prop06*)

lemma *CSChopImpChopCS*:

$\vdash f^*;f \longrightarrow f;f^*$

proof –

have 1: $\vdash (f^r);(f^r)^* \longrightarrow (f^r)^*; (f^r)$
using *ChopCsImpCSChop* **by** *blast*
hence 2: $\vdash ((f^r);(f^r)^* \longrightarrow (f^r)^*; (f^r))^r$
using *ReverseEqv* **by** *blast*
have 3: $\vdash (((f^r);(f^r)^* \longrightarrow (f^r)^*; (f^r))^r) = ((f^r);(f^r)^*)^r \longrightarrow ((f^r)^*; (f^r))^r$
by (*simp add: rev-fun2*)
have 4: $\vdash ((f^r);(f^r)^*)^r = ((f^r)^*)^r; (f^r)^r$
by (*simp add: RevChop*)
have 5: $\vdash ((f^r)^*)^r; (f^r)^r = ((f^r)^r)^*; (f^r)^r$
by (*simp add: LeftChopEqvChop RevChopstar*)
have 6: $\vdash (f^r)^r = f$
using *EqvReverseReverse* **by** *blast*
have 7: $\vdash ((f^r)^r)^*; (f^r)^r = f^*; f$
using 6 *CSEqvCS ChopEqvChop* **by** *blast*
have 8: $\vdash ((f^r);(f^r)^*)^r = f^*; f$
using 7 5 **using** 4 **by** *fastforce*
have 9: $\vdash ((f^r)^*; (f^r))^r = (f^r)^r; ((f^r)^*)^r$
by (*simp add: RevChop*)
have 10: $\vdash (f^r)^r; ((f^r)^*)^r = (f^r)^r; ((f^r)^r)^*$
by (*simp add: RevChopstar RightChopEqvChop*)
have 11: $\vdash (f^r)^r; ((f^r)^r)^* = f; f^*$
using 6 *ChopPlusEqvChopPlus* **by** *blast*
have 12: $\vdash ((f^r);(f^r)^*)^r = f; f^*$
using 9 10 11 **by** (*metis* 4 5 *ChopCsImpCSChop RImpRule int-eq int-iffI*)
from 2 3 8 12 **show** *?thesis* **by** *fastforce*
qed

lemma *CSChopEqvChopCS*:

$\vdash f; f^* = f^*; f$

using *ChopCsImpCSChop CSChopImpChopCS* **by** *fastforce*

lemma *TrueChopSkipEqvSkipChopTrue*:

$\vdash \#True; skip = skip; \#True$

proof –

have 1: $\vdash skip; skip^* = skip^*; skip$ **using** *CSChopEqvChopCS* **by** *blast*

have 2: $\vdash skip^* = \#True$ **using** *CSSkip* **by** *simp*

have 3: $\vdash skip; skip^* = skip; \#True$ **using** 2 **using** *RightChopEqvChop* **by** *blast*

have 4: $\vdash skip^*; skip = \#True; skip$ **using** 2 **using** *LeftChopEqvChop* **by** *blast*

from 1 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *RInitEqvFin*:

$\vdash (init\ f)^r = fin(f)$

proof –

have 1: $\vdash (init\ f)^r = ((f \wedge empty); \#True)^r$

by (*metis AndChopCommute REqvRule init-d-def*)

have 2: $\vdash ((f \wedge empty); \#True)^r = (\#True; (f \wedge empty))^r$

using *RTrue* **by** (*metis RevChop int-eq*)

have 3: $\vdash \#True; (f \wedge empty)^r = \#True; (f^r \wedge empty)$

by (*metis RAnd REmptyEqvEmpty RightChopEqvChop int-eq*)

have 4: $\vdash \#True; (f^r \wedge \text{empty}) = \#True; (f \wedge \text{empty})$
using *RAndEmptyEqvAndEmpty*
by (*metis REmptyEqvEmpty RightChopEqvChop all-rev-eq(3) int-eq*)
have 5: $\vdash \#True; (f \wedge \text{empty}) = \text{fin}(f)$
using *FinEqvTrueChopAndEmpty* **by** *fastforce*
from 1 2 3 4 5 **show** *?thesis* **by** *fastforce*
qed

lemma *RFinEqvInit*:

$\vdash (\text{fin } f)^r = \text{init } (f)$
proof –
have 1: $\vdash \text{fin } f = \#True; (f \wedge \text{empty})$
using *FinEqvTrueChopAndEmpty* **by** *auto*
have 2: $\vdash (\text{fin } f)^r = (\#True; (f \wedge \text{empty}))^r$
using 1 *REqvRule* **by** *blast*
have 3: $\vdash (\#True; (f \wedge \text{empty}))^r = (f \wedge \text{empty})^r; \#True$
using *RTrue* **by** (*metis RevChop int-eq*)
have 4: $\vdash (f \wedge \text{empty})^r; \#True = (f^r \wedge \text{empty}); \#True$
using *LeftChopEqvChop RAnd REmptyEqvEmpty* **by** (*metis int-eq*)
have 5: $\vdash (f \wedge \text{empty})^r; \#True = (f \wedge \text{empty}); \#True$
by (*simp add: RAndEmptyEqvAndEmpty LeftChopEqvChop*)
have 6: $\vdash (f \wedge \text{empty}); \#True = \text{init}(f)$
by (*simp add: AndChopCommute init-d-def*)
from 1 2 3 4 5 6 **show** *?thesis* **by** *fastforce*
qed

lemma *RHaltEqvInitonly*:

$\vdash (\text{halt } f)^r = \text{initonly } (f^r)$
proof –
have 1: $\vdash (\text{halt } f)^r = (\Box (\text{empty} = f))^r$ **by** (*simp add: halt-d-def*)
have 2: $\vdash (\Box (\text{empty} = f))^r = \text{bi } (\text{empty} = f)^r$ **by** (*simp add: RBoxEqvBi*)
have 3: $\vdash (\text{empty} = f)^r = (\text{empty} = f^r)$ **by** (*metis REmptyEqvEmpty inteq-reflection rev-fun2*)
hence 4: $\vdash \text{bi } (\text{empty} = f)^r = \text{bi}(\text{empty} = f^r)$ **by** (*simp add: BiEqvBi*)
have 5: $\vdash \text{bi}(\text{empty} = f^r) = \text{initonly}(f^r)$ **by** (*simp add: initonly-d-def*)
from 1 2 4 5 **show** *?thesis* **by** *fastforce*
qed

lemma *RInitonlyEqvHalt*:

$\vdash (\text{initonly } f)^r = \text{halt}(f^r)$
proof –
have 1: $\vdash (\text{initonly } f)^r = (\text{bi } (\text{empty} = f))^r$ **by** (*simp add: initonly-d-def*)
have 2: $\vdash (\text{bi } (\text{empty} = f))^r = \Box((\text{empty} = f)^r)$ **by** (*simp add: RBiEqvBox*)
have 3: $\vdash (\text{empty} = f)^r = (\text{empty} = f^r)$ **by** (*metis REmptyEqvEmpty inteq-reflection rev-fun2*)
hence 4: $\vdash \Box((\text{empty} = f)^r) = \Box(\text{empty} = f^r)$ **by** (*simp add: BoxEqvBox*)
have 5: $\vdash \Box(\text{empty} = f^r) = \text{halt}(f^r)$ **by** (*simp add: halt-d-def*)
from 1 2 4 5 **show** *?thesis* **by** *fastforce*

qed

lemma *RRHaltEqvInitonly*:

$\vdash (\text{halt } (f^r))^r = \text{initonly } (f)$

proof –

have 1: $\vdash (\text{halt } (f^r))^r = \text{initonly } ((f^r)^r)$ **using** *RRHaltEqvInitonly* **by** *blast*

have 2: $\vdash \text{initonly } ((f^r)^r) = \text{initonly } (f)$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RRInitonlyEqvHalt* :

$\vdash (\text{initonly } (f^r))^r = \text{halt } (f)$

proof –

have 1: $\vdash (\text{initonly } (f^r))^r = \text{halt } ((f^r)^r)$ **using** *RRInitonlyEqvHalt* **by** *blast*

have 2: $\vdash \text{halt } ((f^r)^r) = \text{halt } (f)$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RKeepEqvKeep* :

$\vdash (\text{keep } f)^r = \text{keep } (f^r)$

proof –

have 1: $\vdash (\text{keep } f)^r = (\text{ba } (\text{skip} \longrightarrow f))^r$ **by** (*simp add: keep-d-def*)

have 2: $\vdash (\text{ba } (\text{skip} \longrightarrow f))^r = \text{ba } ((\text{skip} \longrightarrow f)^r)$ **by** (*simp add: RBaEqvBa*)

have 3: $\vdash (\text{skip} \longrightarrow f)^r = (\text{skip} \longrightarrow f^r)$ **by** (*metis all-rev-eq(12) rev-fun2*)

hence 4: $\vdash \text{ba } ((\text{skip} \longrightarrow f)^r) = \text{ba } (\text{skip} \longrightarrow f^r)$ **by** (*simp add: BaEqvBa*)

have 5: $\vdash \text{ba } (\text{skip} \longrightarrow f^r) = \text{keep } (f^r)$ **by** (*simp add: keep-d-def*)

from 1 2 4 5 **show** *?thesis* **by** *fastforce*

qed

lemma *RRKeepEqvKeep* :

$\vdash (\text{keep } (f^r))^r = \text{keep } (f)$

proof –

have 1: $\vdash (\text{keep } (f^r))^r = \text{keep } ((f^r)^r)$ **using** *RKeepEqvKeep* **by** *blast*

have 2: $\vdash \text{keep } ((f^r)^r) = \text{keep } (f)$ **using** *EqvReverseReverse* **by** (*metis inteq-reflection*)

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *NextDiamondEqvDiamondNext*:

$\vdash \bigcirc(\Diamond f) = \Diamond(\bigcirc f)$

proof –

have 1: $\vdash \# \text{True}; \text{skip} = \text{skip}; \# \text{True}$ **by** (*rule TrueChopSkipEqvSkipChopTrue*)

hence 2: $\vdash (\# \text{True}; \text{skip}); f = (\text{skip}; \# \text{True}); f$ **using** *LeftChopEqvChop* **by** *blast*

have 3: $\vdash (\# \text{True}; \text{skip}); f = \# \text{True}; (\text{skip}; f)$ **by** (*simp add: ChopAssocB*)

have 4: $\vdash (\text{skip}; \# \text{True}); f = \text{skip}; (\# \text{True}; f)$ **by** (*simp add: ChopAssocB*)

from 2 3 4 **show** *?thesis* **by** (*metis int-eq next-d-def sometimes-d-def*)

qed

lemma *WeakNextBoxInduct*:

assumes $\vdash \text{wnext } (\Box f) \longrightarrow f$

shows $\vdash f$

proof –

have 1: $\vdash \text{wnext } (\Box f) \longrightarrow f$ **using** *assms* **by** *blast*
hence 2: $\vdash \neg f \longrightarrow \neg (\text{wnext } (\Box f))$ **by** *fastforce*
hence 3: $\vdash \neg f \longrightarrow \bigcirc (\neg (\Box f))$ **by** (*simp add: wnext-d-def*)
have 4: $\vdash (\neg (\Box f)) = (\Diamond (\neg f))$ **by** (*auto simp: always-d-def*)
hence 5: $\vdash \bigcirc (\neg (\Box f)) = \bigcirc (\Diamond (\neg f))$ **using** *NextEqvNext* **by** *blast*
have 6: $\vdash \neg f \longrightarrow \bigcirc (\Diamond (\neg f))$ **using** 3 5 **by** *fastforce*
have 7: $\vdash \bigcirc (\Diamond (\neg f)) = \Diamond (\bigcirc (\neg f))$ **using** *NextDiamondEqvDiamondNext* **by** *blast*
have 8: $\vdash \neg f \longrightarrow \Diamond (\bigcirc (\neg f))$ **using** 6 7 **by** *fastforce*
have 9: $\vdash \Diamond (\neg f) \longrightarrow \Diamond (\Diamond (\bigcirc (\neg f)))$ **using** 8 *DiamondImpDiamond* **by** *blast*
have 10: $\vdash \Diamond (\Diamond (\bigcirc (\neg f))) = \Diamond (\bigcirc (\neg f))$ **using** *DiamondDiamondEqvDiamond* **by** *blast*
have 11: $\vdash \Diamond (\neg f) \longrightarrow \Diamond (\bigcirc (\neg f))$ **using** 9 10 **by** *fastforce*
have 12: $\vdash \Diamond (\neg f) \longrightarrow \bigcirc (\Diamond (\neg f))$ **using** 7 11 **by** *fastforce*
hence 13: $\vdash \neg (\Diamond (\neg f))$ **using** *NextLoop* **by** *blast*
hence 14: $\vdash \Box f$ **by** (*simp add: always-d-def*)
have 15: $\vdash \Box f \longrightarrow f$ **using** *BoxElim* **by** *blast*
from 14 15 **show** *?thesis* **using** *MP* **by** *blast*

qed

lemma *RassignEqvTAssign*:

$\vdash (\$v = e)^r = (v \leftarrow e^r)$

proof –

have 1: $\vdash (\$v = e)^r = ((\$v)^r = e^r)$ **by** (*simp add: rev-fun2*)
have 2: $\vdash ((\$v)^r = e^r) = (!v = e^r)$ **by** (*simp add: all-rev-eq(8)*)
have 3: $\vdash (!v = e^r) = (v \leftarrow e^r)$ **by** (*simp add: intI temporal-assign-d-def*)
from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *RTAssignEqvAssign*:

$\vdash (v \leftarrow e)^r = (\$v = e^r)$

proof –

have 1: $\vdash (v \leftarrow e)^r = (!v = e)^r$ **by** (*simp add: REqvRule intI temporal-assign-d-def*)
have 2: $\vdash (!v = e)^r = (\$v = e^r)$ **by** (*metis all-rev-eq(11) rev-fun2*)
from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RNextAssignEqvPrevAssign*:

$\vdash (v := e)^r = (v =: e^r)$

proof –

have 1: $\vdash (v := e)^r = (v\$ = e)^r$ **by** (*simp add: REqvRule intI next-assign-d-def*)
have 2: $\vdash (v\$ = e)^r = (v! = e^r)$ **by** (*metis all-rev-eq(9) rev-fun2*)
have 3: $\vdash (v! = e^r) = (v =: e^r)$ **by** (*simp add: prev-assign-d-def*)
from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *RPrevAssignEqvNextAssign*:

$\vdash (v =: e)^r = (v := e^r)$

proof –

have 1: $\vdash (v =: e)^r = (v! = e)^r$ **by** (*simp add: REqvRule intI prev-assign-d-def*)
have 2: $\vdash (v! = e)^r = (v\$ = e^r)$ **by** (*metis all-rev-eq(10) rev-fun2*)

have 3: $\vdash (v\$ = e^r) = (v := e^r)$ **by** (*simp add: next-assign-d-def*)
from 1 2 3 **show** ?thesis **by** fastforce
qed

lemma *RGetsEqvBaSkipImp*:

$\vdash (v \text{ gets } e)^r = \text{ba}(\text{skip} \longrightarrow (\$v = e^r))$
proof –
have 1: $\vdash (v \text{ gets } e)^r = (\text{ba}(\text{skip} \longrightarrow (!v = e)))^r$
using gets-d-def temporal-assign-d-def keep-d-def REqvRule
by (metis Prop04 ba-d-def int-simps(15))
have 2: $\vdash (\text{ba}(\text{skip} \longrightarrow (!v = e)))^r = \text{ba} ((\text{skip} \longrightarrow (!v = e))^r)$
by (simp add: RBaEqvBa)
have 3: $\vdash (\text{skip} \longrightarrow (!v = e))^r = (\text{skip} \longrightarrow (\$v = e^r))$
by (simp add: all-rev-eq(11) all-rev-eq(12) all-rev-eq(3))
hence 4: $\vdash \text{ba} ((\text{skip} \longrightarrow (!v = e))^r) = \text{ba} (\text{skip} \longrightarrow (\$v = e^r))$
by (simp add: BaEqvBa)
from 1 2 4 **show** ?thesis **by** fastforce
qed

lemma *RIfThenElse*:

$\vdash (\text{if}_i f0 \text{ then } f1 \text{ else } f2)^r = \text{if}_i (f0^r) \text{ then } (f1^r) \text{ else } (f2^r)$
by (simp add: all-rev-eq(2) all-rev-eq(3) ifthenelse-d-def)

lemma *RWhile*:

$\vdash (\text{init } f \wedge \text{while } f0 \text{ do } f1)^r = (\text{fin}(f) \wedge ((f0^r) \wedge (f1^r))^* \wedge \text{init } (\neg(f0)))$
proof –
have 1: $\vdash (\text{init } f \wedge \text{while } f0 \text{ do } f1)^r = (\text{init } f \wedge (f0 \wedge f1)^* \wedge \text{fin } (\neg f0))^r$
by (simp add: while-d-def)
have 2: $\vdash (\text{init } f \wedge (f0 \wedge f1)^* \wedge \text{fin } (\neg f0))^r = ((\text{init } f)^r \wedge ((f0 \wedge f1)^*)^r \wedge (\text{fin } (\neg f0))^r)$
by (simp add: all-rev-eq(3))
have 3: $\vdash (\text{init } f)^r = \text{fin}(f)$
by (simp add: RInitEqvFin)
have 4: $\vdash ((f0 \wedge f1)^*)^r = ((f0^r) \wedge (f1^r))^*$
by (metis RevChopstar all-rev-eq(3))
have 5: $\vdash (\text{fin } (\neg f0))^r = \text{init } (\neg(f0))$
by (metis RFinEqvInit)
have 6: $\vdash ((\text{init } f)^r \wedge ((f0 \wedge f1)^*)^r \wedge (\text{fin } (\neg f0))^r) =$
 $(\text{fin}(f) \wedge ((f0^r) \wedge (f1^r))^* \wedge \text{init } (\neg(f0)))$ **using** 3 4 5 **by** fastforce
from 1 2 6 **show** ?thesis **by** fastforce
qed

lemma *AAxRev*:

$\vdash (\forall \forall x. F x)^r = (\forall \forall x. (F x)^r)$
proof –
have 1: $\vdash (\forall \forall x. F x) = (\neg(\exists \exists x. \neg(F x)))$ **using** AAxDef **by** blast
have 2: $\vdash (\forall \forall x. F x)^r = (\neg(\exists \exists x. \neg(F x)))^r$ **using** REqvRule 1 **by** blast
have 3: $\vdash (\neg(\exists \exists x. \neg(F x)))^r = (\neg((\exists \exists x. (\neg(F x))^r)))$ **by** (simp add: rev-fun1)
have 4: $\vdash ((\exists \exists x. (\neg(F x))^r) = ((\exists \exists x. (\neg(F x))^r))$ **by** (simp add: EExRev)
hence 5: $\vdash (\neg((\exists \exists x. (\neg(F x))^r))) = (\neg(\exists \exists x. (\neg(F x))^r))$ **by** auto
have 51: $\bigwedge x. \vdash (\neg(F x))^r = (\neg(F x)^r)$ **by** (simp add: rev-fun1)

```

hence 52:  $\vdash (\exists \exists x. (\neg (F x))^r) = (\exists \exists x. \neg (F x)^r)$ 
  using EEqRule[of  $(\lambda x. \text{LIFT}(\neg (F x))^r) (\lambda x. \text{LIFT}(\neg (F x)^r))$ ]
  by fastforce
hence 6:  $\vdash (\neg(\exists \exists x. (\neg (F x))^r)) = (\neg(\exists \exists x. \neg (F x)^r))$  by fastforce
have 7:  $\vdash (\neg(\exists \exists x. \neg (F x)^r)) = (\forall \forall x. (F x)^r)$  using AAxDef by fastforce
from 1 2 3 5 6 7 show ?thesis by fastforce
qed

end

```

8 Projection operator

```

theory Projection
imports Fuse TimeReversal
begin

```

This theory introduces the projection operator [4]. The projection operator is defined and we prove the soundness of the rules and axiom system.

8.1 Definitions

```

primrec filt :: 'a interval  $\Rightarrow$  index  $\Rightarrow$  'a interval
  where filt  $\sigma \langle l \rangle = \langle \text{inth } \sigma l \rangle$ 
  | filt  $\sigma (x \odot ls) = (\text{inth } \sigma x) \odot \text{filt } \sigma ls$ 

primrec lsum :: 'a interval interval  $\Rightarrow$  nat  $\Rightarrow$  index
  where lsum  $\langle xs \rangle a = \langle a + (\text{ilen } xs) \rangle$ 
  | lsum  $(xs \odot xxs) a = (a + (\text{ilen } xs)) \odot (\text{lsum } xxs (a + (\text{ilen } xs)))$ 

definition addzero :: index  $\Rightarrow$  index
  where addzero  $ls = (\text{if } \text{ilen } ls = 0 \text{ then}$ 
     $(\text{if } \text{ifirst } ls = 0 \text{ then } ls \text{ else } 0 \odot ls) \text{ else } 0 \odot ls)$ 

definition powerinterval :: ('a::world) formula  $\Rightarrow$  'a interval  $\Rightarrow$  index  $\Rightarrow$  bool
  where powerinterval  $F \sigma l = (\forall i. i < \text{ilen } l \longrightarrow ((\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma) \models F))$ 

definition cpl :: ('a::world) formula  $\Rightarrow$  'a formula  $\Rightarrow$  'a interval  $\Rightarrow$  index
  where cpl  $f g \sigma = (\epsilon l. \text{index-sequence } 0 l \wedge \text{powerinterval } f \sigma l \wedge$ 
     $(\text{inth } l (\text{ilen } l)) = \text{ilen } \sigma \wedge ((\text{filt } \sigma l) \models g))$ 

primrec lcpl :: ('a::world) formula  $\Rightarrow$  'a formula  $\Rightarrow$  'a interval  $\Rightarrow$  index  $\Rightarrow$  nat interval interval
  where lcpl  $f g \sigma \langle x \rangle = \langle (\text{imap } (\text{shift } x) (\text{cpl } f g (\text{sub } x x \sigma))) \rangle$ 
  | lcpl  $f g \sigma (x \odot xs) =$ 
     $(\text{case } xs \text{ of } \langle y \rangle \Rightarrow \langle (\text{imap } (\text{shift } x) (\text{cpl } f g (\text{sub } x y \sigma))) \rangle$ 
     $| y \odot ys \Rightarrow (\text{imap } (\text{shift } x) (\text{cpl } f g (\text{sub } x y \sigma))) \odot (\text{lcpl } f g \sigma xs))$ 

definition projection-d :: ('a::world) formula  $\Rightarrow$  'a formula  $\Rightarrow$  'a formula
  where projection-d  $F G \equiv \lambda s. (\exists l. \text{index-sequence } 0 l \wedge (\text{inth } l (\text{ilen } l)) = (\text{ilen } s) \wedge$ 
     $\text{powerinterval } F s l \wedge ((\text{filt } s l) \models G)$ 

```

)

syntax

-projection-d :: $[lift, lift] \Rightarrow lift$ ((- Δ -) [84,84] 83)

syntax (ASCII)

-projection-d :: $[lift, lift] \Rightarrow lift$ ((- *proj* -) [84,84] 83)

translations

-projection-d \Rightarrow *CONST projection-d*

definition *uprojection-d* :: ('a:: world) formula \Rightarrow 'a formula \Rightarrow 'a formula

where *uprojection-d* $F\ G \equiv LIFT(\neg(F \Delta (\neg G)))$

syntax

-uprojection-d :: $[lift, lift] \Rightarrow lift$ ((- ∇ -) [84,84] 83)

syntax (ASCII)

-uprojection-d :: $[lift, lift] \Rightarrow lift$ ((- *uproj* -) [84,84] 83)

translations

-uprojection-d \Rightarrow *CONST uprojection-d*

definition *dp-d* :: ('a:: world) formula \Rightarrow 'a formula

where *dp-d* $F \equiv LIFT(\#True \Delta F)$

definition *bp-d* :: ('a:: world) formula \Rightarrow 'a formula

where *bp-d* $F \equiv LIFT(\#True \nabla F)$

syntax

-dp-d :: $lift \Rightarrow lift$ ((*dp* -) [88] 87)

-bp-d :: $lift \Rightarrow lift$ ((*bp* -) [88] 87)

syntax (ASCII)

-dp-d :: $lift \Rightarrow lift$ ((*dp* -) [88] 87)

-bp-d :: $lift \Rightarrow lift$ ((*bp* -) [88] 87)

translations

-dp-d \Rightarrow *CONST dp-d*

-bp-d \Rightarrow *CONST bp-d*

8.2 Lemmas

8.2.1 filt Lemmas

lemma *filt-ilen*:

ilen(filt s0 l) = ilen l

by (*induct l, simp, simp*)

lemma *filt-inth*:

```

assumes  $i \leq \text{ilen } (\text{filt } s0 \ l)$ 
shows  $(\text{inth } (\text{filt } s0 \ l) \ i) = (\text{inth } s0 \ (\text{inth } l \ i))$ 
using assms
proof (induct l arbitrary: i)
case (INil x)
then show ?case by simp
next
case (ICons x1a l)
then show ?case by simp (metis Nitpick.case-nat-unfold Suc-eq-plus1 le-diff-conv)
qed

```

lemma *filt-expand:*

```

 $(s1 = (\text{filt } s0 \ l)) =$ 
 $(\text{ilen } s1 = \text{ilen } l \wedge$ 
 $(\forall \ i \leq \text{ilen } s1. (\text{inth } s1 \ i) = (\text{inth } s0 \ (\text{inth } l \ i))))$ 
by (metis filt-ilen filt-inth interval-eq-inth-eq)

```

lemma *filt-fuse:*

```

 $\text{filt } xs \ (\text{fuse } l1 \ l2) = (\text{fuse } (\text{filt } xs \ l1) \ (\text{filt } xs \ l2))$ 
by (induct l1 arbitrary: l2 xs) simp-all

```

lemma *fuse-filt-ilen:*

```

assumes index-sequence 0 l
 $(\text{inth } l \ (\text{ilen } l)) = \text{ilen } xs$ 
shows  $\text{ilen } (\text{filt } (\text{fuse } (\text{prefix } (\text{inth } l \ n) \ xs) \ (\text{suffix } (\text{inth } l \ n) \ xs)) \ l) =$ 
 $\text{ilen } (\text{fuse } (\text{filt } (\text{prefix } (\text{inth } l \ n) \ xs) \ (\text{prefix } n \ l))$ 
 $(\text{filt } (\text{suffix } (\text{inth } l \ n) \ xs) \ (\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l) \ )))$ 
proof –
have 1:  $\text{ilen } (\text{filt } (\text{fuse } (\text{prefix } (\text{inth } l \ n) \ xs) \ (\text{suffix } (\text{inth } l \ n) \ xs)) \ l) = \text{ilen } l$ 
by (simp add: filt-ilen)
have 2:  $\text{ilen } (\text{fuse } (\text{filt } (\text{prefix } (\text{inth } l \ n) \ xs) \ (\text{prefix } n \ l))$ 
 $(\text{filt } (\text{suffix } (\text{inth } l \ n) \ xs) \ (\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l) \ ))) =$ 
 $(\text{ilen}((\text{filt } (\text{prefix } (\text{inth } l \ n) \ xs) \ (\text{prefix } n \ l))) +$ 
 $\text{ilen}(\text{filt } (\text{suffix } (\text{inth } l \ n) \ xs) \ (\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l) \ )))$ 
using fuse-ilen-a by blast
have 3:  $\text{ilen}((\text{filt } (\text{prefix } (\text{inth } l \ n) \ xs) \ (\text{prefix } n \ l))) = \text{ilen}(\text{prefix } n \ l)$ 
using filt-ilen by blast
have 4:  $\text{ilen}(\text{filt } (\text{suffix } (\text{inth } l \ n) \ xs) \ (\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l) \ ))) =$ 
 $\text{ilen}(\text{suffix } n \ l)$ 
by (simp add: filt-ilen)
have 5:  $\text{ilen}(\text{prefix } n \ l) + \text{ilen}(\text{suffix } n \ l) = \text{ilen } l$ 
by (simp)
from 1 2 3 4 5 show ?thesis by auto
qed

```

lemma *fuse-filt-inth-a:*

assumes *index-sequence* 0 l
 (inth l (ilen l)) = ilen xs
 i ≤ ilen l
 n ≤ ilen l
shows inth (filt (fuse (prefix (inth l n) xs) (suffix (inth l n) xs)) l) i =
 inth xs (inth l i)
proof –
have 1: filt (fuse (prefix (inth l n) xs) (suffix (inth l n) xs)) l =
 filt xs l
using *assms* **by** (metis fuse-prefix-suffix idx-less-last-1 le-neq-implies-less
 less-imp-le-nat not-less)
have 2: inth (filt xs l) i = inth xs (inth l i)
using *assms* **by** (metis filt-inth filt-ilen)
from 1 2 **show** ?thesis **by** auto
qed

lemma *fuse-filt-inth-b*:

assumes *index-sequence* 0 l
 (inth l (ilen l)) = ilen xs
 i ≤ ilen l
 n ≤ ilen l
shows inth (fuse (filt (prefix (inth l n) xs) (prefix n l))
 (filt (suffix (inth l n) xs) (imap (shiftn (inth l n)) (suffix n l)))) i =
 inth xs (inth l i)
proof –
have 1: i ≤ ilen(fuse (filt (prefix (inth l n) xs) (prefix n l))
 (filt (suffix (inth l n) xs) (imap (shiftn (inth l n)) (suffix n l))))
using *assms*
by (metis filt-ilen fuse-ilen-a fuse-prefix-suffix-ilen ilen-imap)
have 2: ilast(filt (prefix (inth l n) xs) (prefix n l)) = inth xs (inth l n)
using *assms* filt-inth[of - (prefix (inth l n) xs) (prefix n l)]
by (metis filt-ilen idx-less-equal ilast-prefix prefix-ilen-good order-refl)
have 3: ifirst(filt (suffix (inth l n) xs) (imap (shiftn (inth l n)) (suffix n l))) =
 inth (filt (suffix (inth l n) xs) (imap (shiftn (inth l n)) (suffix n l))) 0
by *simp*
have 4: inth (filt (suffix (inth l n) xs) (imap (shiftn (inth l n)) (suffix n l))) 0 =
 inth (suffix (inth l n) xs) (inth (imap (shiftn (inth l n)) (suffix n l)) 0)
using *filt-inth* **by** *blast*
have 5: inth (suffix (inth l n) xs) (inth (imap (shiftn (inth l n)) (suffix n l)) 0) =
 inth (suffix (inth l n) xs) (inth l (n+0) – (inth l n))
by (*simp* *add: assms inth-imap shiftn-def*)
have 6: inth (suffix (inth l n) xs) (inth l (n+0) – (inth l n)) =
 inth (suffix (inth l n) xs) 0
by *simp*
have 7: inth (suffix (inth l n) xs) 0 = inth xs (inth l n)
using *assms* **by** (metis Nat.add-0-right inth-suffix le0)
have 8: ilast(filt (prefix (inth l n) xs) (prefix n l)) =
 ifirst(filt (suffix (inth l n) xs) (imap (shiftn (inth l n)) (suffix n l)))
using 2 4 5 7 **by** *auto*

have 10: $\text{inth} (\text{fuse} (\text{filt} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{prefix } n \ l))$
 $(\text{filt} (\text{suffix} (\text{inth } l \ n) \ xs) (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \))) \ i =$
 $(\text{if } i \leq \text{ilen} (\text{filt} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{prefix } n \ l))$
 $\text{then } \text{inth} (\text{filt} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{prefix } n \ l)) \ i$
 $\text{else } \text{inth} (\text{filt} (\text{suffix} (\text{inth } l \ n) \ xs) (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \)))$
 $(i - \text{ilen} (\text{filt} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{prefix } n \ l))))$
using 1 8 fuse-inth by auto
have 11: $\text{ilen} (\text{filt} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{prefix } n \ l)) = n$
using assms by (metis filt-ilen prefix-ilen-good)
have 12: $i \leq n \longrightarrow \text{inth} (\text{filt} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{prefix } n \ l)) \ i =$
 $\text{inth} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{inth} (\text{prefix } n \ l) \ i)$
by (simp add: 11 filt-inth)
have 13: $i \leq n \longrightarrow \text{inth} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{inth} (\text{prefix } n \ l) \ i) =$
 $\text{inth} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{inth } l \ i)$
by (simp add: assms)
have 15: $i \leq n \longrightarrow \text{inth} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{inth } l \ i) = \text{inth } xs (\text{inth } l \ i)$
using assms(1) assms(2) assms(4) idx-less-equal inth-prefix by blast
have 16: $\text{inth} (\text{filt} (\text{suffix} (\text{inth } l \ n) \ xs) (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \)))$
 $(i - \text{ilen} (\text{filt} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{prefix } n \ l)))) =$
 $\text{inth} (\text{filt} (\text{suffix} (\text{inth } l \ n) \ xs) (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \)))$
 $(i - n)$
by (simp add: 11)
have 17: $\text{inth} (\text{filt} (\text{suffix} (\text{inth } l \ n) \ xs) (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \))) (i - n) =$
 $\text{inth} (\text{suffix} (\text{inth } l \ n) \ xs) (\text{inth} (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \)) (i - n))$
using assms filt-inth[of $i - n$ ($\text{suffix} (\text{inth } l \ n) \ xs$) ($\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \)$]
by (simp add: filt-ilen)
have 18: $i > n \longrightarrow \text{inth} (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \) (i - n) =$
 $\text{inth } l \ (n + (i - n)) - (\text{inth } l \ n)$
using assms
by (simp add: idx-shiftn-suffix-inth)
have 19: $i > n \longrightarrow \text{inth} (\text{suffix} (\text{inth } l \ n) \ xs) (\text{inth} (\text{imap} (\text{shiftn} (\text{inth } l \ n)) (\text{suffix } n \ l) \) (i - n))$
 $= \text{inth} (\text{suffix} (\text{inth } l \ n) \ xs) ((\text{inth } l \ i) - (\text{inth } l \ n))$
by (simp add: 18)
have 20 : $i > n \longrightarrow \text{inth} (\text{suffix} (\text{inth } l \ n) \ xs) ((\text{inth } l \ i) - (\text{inth } l \ n)) =$
 $\text{inth } xs ((\text{inth } l \ n) + ((\text{inth } l \ i) - (\text{inth } l \ n)))$
using assms by (metis diff-le-mono idx-less-last-1 inth-suffix
 $\text{le-neq-implies-less less-imp-le-nat nat-le-linear})$
have 22: $i > n \longrightarrow (\text{inth } l \ n) + ((\text{inth } l \ i) - (\text{inth } l \ n)) = (\text{inth } l \ i)$
using assms using idx-less-equal by fastforce
have 23: $i > n \longrightarrow \text{inth } xs ((\text{inth } l \ n) + ((\text{inth } l \ i) - (\text{inth } l \ n))) = \text{inth } xs (\text{inth } l \ i)$
by (simp add: 22)
from 10 show ?thesis by (simp add: 11 12 13 15 17 19 20 23)
qed

lemma fuse-filt-inth:

assumes $\text{index-sequence } 0 \ l$
 $(\text{inth } l \ (\text{ilen } l)) = \text{ilen } xs$
 $i \leq \text{ilen } l$
 $n \leq \text{ilen } l$

shows $\text{inth} (\text{filt} (\text{fuse} (\text{prefix} (\text{inth } l \ n) \ xs) (\text{suffix} (\text{inth } l \ n) \ xs)) \ l) \ i =$

$$\text{inth} (\text{fuse} (\text{filt} (\text{prefix} (\text{inth} \ l \ n) \ xs) (\text{prefix} \ n \ l))$$

$$(\text{filt} (\text{suffix} (\text{inth} \ l \ n) \ xs) (\text{imap} (\text{shiftn} (\text{inth} \ l \ n)) (\text{suffix} \ n \ l)))) \ i$$
using *assms fuse-filt-inth-a*[of *l xs i n*]

$$\text{fuse-filt-inth-b}$$
[of *l xs i n*] **by** *simp*

lemma *fuse-filt*:

assumes *index-sequence 0 l*

$$(\text{inth} \ l \ (\text{ilen} \ l)) = \text{ilen} \ xs$$

$$n \leq \text{ilen} \ l$$
shows
$$\text{filt} (\text{fuse} (\text{prefix} (\text{inth} \ l \ n) \ xs) (\text{suffix} (\text{inth} \ l \ n) \ xs)) \ l =$$

$$\text{fuse} (\text{filt} (\text{prefix} (\text{inth} \ l \ n) \ xs) (\text{prefix} \ n \ l))$$

$$(\text{filt} (\text{suffix} (\text{inth} \ l \ n) \ xs) (\text{imap} (\text{shiftn} (\text{inth} \ l \ n)) (\text{suffix} \ n \ l)))$$
using *assms fuse-filt-ilen fuse-filt-inth interval-eq-inth-eq* **by** (*metis filt-ilen*)

lemma *fuse-filt-a*:

assumes *index-sequence 0 l1*

$$\text{index-sequence} (\text{ilast} \ l1) \ l2$$

$$\text{ilast} \ l2 = \text{ilen} \ xs$$
shows
$$\text{filt} (\text{fuse} (\text{prefix} (\text{ilast} \ l1) \ xs) (\text{suffix} (\text{ifirst} \ l2) \ xs)) (\text{fuse} \ l1 \ l2) =$$

$$\text{fuse} (\text{filt} (\text{prefix} (\text{ilast} \ l1) \ xs) \ l1)$$

$$(\text{filt} (\text{suffix} (\text{ifirst} \ l2) \ xs) (\text{imap} (\text{shiftn} (\text{ifirst} \ l2)) \ l2))$$
proof –
have 1:
$$\text{ilast} \ l1 = \text{ifirst} \ l2$$

using *assms* **by** (*metis index-sequence-def*)
have 2: *index-sequence 0 (fuse l1 l2)*
using *assms* **by** (*metis index-sequence-def idx-fuse*)
have 3:
$$\text{ilast} (\text{fuse} \ l1 \ l2) = \text{ilen} \ xs$$

using *assms* **by** (*metis 1 add.left-neutral fuse-inth-a fuse-ilen-a le-add2*)
have 4:
$$\text{ilen} \ l1 \leq \text{ilen} (\text{fuse} \ l1 \ l2)$$

by (*simp add: fuse-ilen-a*)
have 5:
$$\text{filt} (\text{fuse} (\text{prefix} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1)) \ xs)$$

$$(\text{suffix} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1)) \ xs))$$

$$(\text{fuse} \ l1 \ l2)$$

$$=$$

$$\text{fuse} (\text{filt} (\text{prefix} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1)) \ xs) (\text{prefix} (\text{ilen} \ l1) (\text{fuse} \ l1 \ l2)))$$

$$(\text{filt} (\text{suffix} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1)) \ xs)$$

$$(\text{imap} (\text{shiftn} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1))) (\text{suffix} (\text{ilen} \ l1) (\text{fuse} \ l1 \ l2))))$$
using 2 3 4 *fuse-filt* **by** *auto*
have 6:
$$\text{filt} (\text{fuse} (\text{prefix} (\text{ilast} \ l1) \ xs) (\text{suffix} (\text{ifirst} \ l2) \ xs)) (\text{fuse} \ l1 \ l2) =$$

$$\text{filt} (\text{fuse} (\text{prefix} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1)) \ xs)$$

$$(\text{suffix} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1)) \ xs))$$

$$(\text{fuse} \ l1 \ l2)$$
using 1 4 *ilast-prefix*[of *ilast l1 fuse l1 l2*]
prefix-fuse **by** (*simp add: fuse-inth*)
have 7:
$$(\text{filt} (\text{prefix} (\text{ilast} \ l1) \ xs) \ l1) =$$

$$(\text{filt} (\text{prefix} (\text{inth} (\text{fuse} \ l1 \ l2) (\text{ilen} \ l1)) \ xs) (\text{prefix} (\text{ilen} \ l1) (\text{fuse} \ l1 \ l2)))$$
using 1 4 *ilast-prefix*[of (*ilast l1*) *fuse l1 l2*]
prefix-fuse
by (*simp add: prefix-fuse fuse-inth*)

have 8: (filt (suffix (ifirst l2) xs) (imap (shiftn (ifirst l2)) l2)) =
 (filt (suffix (inth (fuse l1 l2) (ilen l1)) xs)
 (imap (shiftn (inth (fuse l1 l2) (ilen l1))) (suffix (ilen l1) (fuse l1 l2))))
using 1 4 ilast-prefix prefix-fuse suffix-fuse **by** metis
show ?thesis **using** 5 6 7 8 **by** auto
qed

lemma filt-prefix:
assumes $n \leq \text{ilen } l$
shows $\text{prefix } n (\text{filt } \sigma \ l) = \text{filt } \sigma (\text{prefix } n \ l)$
proof –
have 1: $\text{ilen } (\text{prefix } n (\text{filt } \sigma \ l)) = \text{ilen } (\text{filt } \sigma (\text{prefix } n \ l))$
by (simp add: assms filt-ilen)
have 2: $\forall i \leq \text{ilen}(\text{prefix } n (\text{filt } \sigma \ l)).$
 $(\text{inth } (\text{prefix } n (\text{filt } \sigma \ l)) \ i) = (\text{inth } (\text{filt } \sigma (\text{prefix } n \ l)) \ i)$
by (simp add: assms filt-inth filt-ilen le-trans)
show ?thesis **using** 1 2 interval-eq-inth-eq **by** blast
qed

lemma filt-prefix-idx:
assumes $n \leq \text{ilen } \sigma$
 $\text{index-sequence } 0 \ l$
 $\text{inth } l (\text{ilen } l) = n$
shows $\text{filt } \sigma \ l = \text{filt } (\text{prefix } n \ \sigma) \ l$
proof –
have 1: $\text{ilen}(\text{filt } \sigma \ l) = \text{ilen } (\text{filt } (\text{prefix } n \ \sigma) \ l)$
by (simp add: filt-ilen)
have 2: $\forall i \leq \text{ilen}(\text{filt } \sigma \ l). (\text{inth } (\text{filt } \sigma \ l) \ i) = (\text{inth } (\text{filt } (\text{prefix } n \ \sigma) \ l) \ i)$
by (metis assms filt-inth filt-ilen idx-less-equal inth-prefix prefix-ilen-good order-refl)
show ?thesis
by (simp add: 1 2 interval-eq-inth-eq)
qed

lemma filt-suffix:
assumes $n \leq \text{ilen } l$
shows $\text{suffix } n (\text{filt } \sigma \ l) = \text{filt } \sigma (\text{suffix } n \ l)$
proof –
have 1: $\text{ilen } (\text{suffix } n (\text{filt } \sigma \ l)) = \text{ilen } (\text{filt } \sigma (\text{suffix } n \ l))$
by (simp add: assms filt-ilen)
have 2: $\forall i \leq \text{ilen } (\text{suffix } n (\text{filt } \sigma \ l)).$
 $(\text{inth } (\text{suffix } n (\text{filt } \sigma \ l)) \ i) = (\text{inth } (\text{filt } \sigma (\text{suffix } n \ l)) \ i)$
by (simp add: assms filt-inth filt-ilen
 ordered-cancel-comm-monoid-diff-class.le-diff-conv2)
show ?thesis **using** 1 2 interval-eq-inth-eq **by** blast
qed

lemma filt-suffix-idx-ilen:

assumes $n \leq \text{ilen } \sigma$
 $\text{index-sequence } 0 \ l$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma - n$
shows $\text{filt } \sigma \ (\text{imap } (\text{shift } n) \ l) = \text{ilen } (\text{filt } (\text{suffix } n \ \sigma) \ l)$
using *assms* **by** (*simp add: filt-ilen*)

lemma *filt-suffix-idx-inth*:
assumes $n \leq \text{ilen } \sigma$
 $\text{index-sequence } 0 \ l$
 $i \leq \text{ilen } l$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma - n$
shows $\text{inth } (\text{filt } \sigma \ (\text{imap } (\text{shift } n) \ l)) \ i = \text{inth } (\text{filt } (\text{suffix } n \ \sigma) \ l) \ i$
proof –
have 1: $\text{inth } (\text{filt } \sigma \ (\text{imap } (\text{shift } n) \ l)) \ i = \text{inth } \sigma \ (\text{inth } (\text{imap } (\text{shift } n) \ l) \ i)$
by (*simp add: assms filt-inth filt-ilen*)
have 2: $\text{inth } \sigma \ (\text{inth } (\text{imap } (\text{shift } n) \ l) \ i) = (\text{inth } \sigma \ ((\text{inth } l \ i) + n))$
by (*simp add: shift-def inth-imap*)
have 3: $\text{inth } (\text{filt } (\text{suffix } n \ \sigma) \ l) \ i = \text{inth } (\text{suffix } n \ \sigma) \ (\text{inth } l \ i)$
by (*simp add: assms filt-inth filt-ilen*)
have 4: $\text{inth } (\text{suffix } n \ \sigma) \ (\text{inth } l \ i) = (\text{inth } \sigma \ ((\text{inth } l \ i) + n))$
by (*metis add.commute assms eq-iff idx-less-equal inth-suffix suffix-ilen-good*)
show ?thesis **by** (*simp add: 1 2 3 4*)
qed

lemma *filt-suffix-idx*:
assumes $n \leq \text{ilen } \sigma$
 $\text{index-sequence } 0 \ l$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma - n$
shows $\text{filt } \sigma \ (\text{imap } (\text{shift } n) \ l) = \text{filt } (\text{suffix } n \ \sigma) \ l$
using *assms* *filt-suffix-idx-ilen* *filt-suffix-idx-inth* *interval-eq-inth-eq*
by (*simp add: filt-suffix-idx-inth interval-eq-inth-eq filt-ilen*)

lemma *filt-filt*:
 $(\text{inth } (\text{filt } (\text{filt } \text{xs } l1) \ l2) \ k) =$
 $(\text{inth } \text{xs } (\text{inth } l1 \ (\text{inth } l2 \ k)))$
by (*metis filt-expand ilen-imap inth-imap*)

lemma *filt-filt-imap*:
 $(\text{filt } (\text{filt } \text{xs } l1) \ l2) = (\text{filt } \text{xs } (\text{imap } (\lambda x. (\text{inth } l1 \ x)) \ l2))$
by (*metis filt-expand ilen-imap inth-imap*)

lemma *filt-imap*:
 $\text{filt } \text{xs } l = \text{imap } (\lambda x. \text{inth } \text{xs } x) \ l$
by (*metis filt-expand ilen-imap inth-imap*)

lemma *filt-imap-filt*:
 $(\text{filt } (\text{filt } \text{xs } l1) \ l2) = \text{filt } \text{xs } (\text{filt } l1 \ l2)$
by (*metis filt-filt-imap filt-imap*)

lemma *filt-sub*:
assumes $k \leq n$
 $n \leq \text{ilen } l$
shows $(\text{sub } k \ n \ (\text{filt } \sigma \ l)) = (\text{filt } \sigma \ (\text{sub } k \ n \ l))$
using *assms*
by (*simp add: sub-def filt-prefix filt-suffix*)

lemma *filt-lfuse-imap*:
 $(\text{filt } \sigma \ (\text{lfuse } (x\text{xs}))) =$
 $(\text{lfuse } (\text{imap } (\lambda \text{xs} . (\text{filt } \sigma \ \text{xs})) \ x\text{xs}))$
proof (*induct xxs*)
case (*INil x*)
then show ?*case* **by** *simp*
next
case (*ICons x1a xxs*)
then show ?*case* **by** (*simp add: filt-fuse*)
qed

8.2.2 powerinterval lemmas

lemma *powerinterval-split0*:
assumes *index-sequence 0 l*
 $n \leq \text{ilen } l$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma$
 $i < \text{ilen}(\text{prefix } n \ l)$
shows $(\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)) =$
 $(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)$

proof –
have 01: $(\text{inth } l \ n) \leq \text{ilen } \sigma$
by (*metis assms(1) assms(2) assms(3) idx-less-last-1 le-less*)
have 02: $(\text{inth } (\text{prefix } n \ l) \ i) \leq (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i))$
using *assms(1) assms(3) assms(4) idx-expand* **by** *fastforce*
have 03: $(\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \leq (\text{inth } l \ n)$
using *assms(1) assms(2) assms(3) assms(4) idx-less-equal* **by** *fastforce*
have 1: $\text{ilen } (\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)) =$
 $\text{ilen}(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ (\sigma))$
using *ilen-sub*
using 01 02 03 *assms(4)* **by** *auto*
have 2: $(\forall j \leq \text{ilen } (\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)).$
 $(\text{inth } (\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)) \ j) =$
 $(\text{inth } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ (\sigma)) \ j))$

proof
fix *j*
show $j \leq \text{ilen } (\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)) \longrightarrow$
 $(\text{inth } (\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)) \ j) =$
 $(\text{inth } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ (\sigma)) \ j)$

proof –

have 21: $\text{ilen}(\text{sub}(\text{inth}(\text{prefix } n \ l) \ i) (\text{inth}(\text{prefix } n \ l) (\text{Suc } i)) (\text{prefix}(\text{inth } l \ n) \ \sigma)) =$
 $(\text{inth } l (\text{Suc } i)) - (\text{inth } l \ i)$
using 1 *assms*(1) *assms*(3) *assms*(4) *idx-expand* **by** *fastforce*

have 22: $(\text{inth}(\text{prefix } n \ l) \ i) = (\text{inth } l \ i)$
using *assms* **by** *auto*

have 23: $(\text{inth}(\text{prefix } n \ l) (\text{Suc } i)) = (\text{inth } l (\text{Suc } i))$
using *assms* **by** *auto*

have 24: $j \leq (\text{inth } l (\text{Suc } i)) - (\text{inth } l \ i) \longrightarrow$
 $\text{inth}(\text{sub}(\text{inth } l \ i) (\text{inth } l (\text{Suc } i)) \ \sigma) \ j = (\text{inth } \sigma ((\text{inth } l \ i) + j))$
using *assms* *idx-expand* *inth-sub* *less-le-trans* **by** *fastforce*

have 25: $j \leq (\text{inth } l (\text{Suc } i)) - (\text{inth } l \ i) \longrightarrow (\text{inth } l (\text{Suc } i)) \leq \text{ilen}(\text{prefix}(\text{inth } l \ n) \ \sigma)$
using *assms*(1) *assms*(2) *assms*(3) *assms*(4) *idx-expand* *idx-less-equal*
by *fastforce*

have 26: $j \leq (\text{inth } l (\text{Suc } i)) - (\text{inth } l \ i) \longrightarrow$
 $(\text{inth}(\text{sub}(\text{inth}(\text{prefix } n \ l) \ i) (\text{inth}(\text{prefix } n \ l) (\text{Suc } i)) (\text{prefix}(\text{inth } l \ n) \ \sigma)) \ j =$
 $(\text{inth}(\text{prefix}(\text{inth } l \ n) \ \sigma) ((\text{inth } l \ i) + j))$
using 02 22 23 25 **by** *auto*

have 27: $(\text{inth } l (\text{Suc } i)) \leq (\text{inth } l \ n)$
using *assms*(1) *assms*(2) *assms*(3) *assms*(4) *idx-less-equal* **by** *fastforce*

have 28: $j \leq (\text{inth } l (\text{Suc } i)) - (\text{inth } l \ i) \longrightarrow (\text{inth } l \ i) + j \leq (\text{inth } l \ n)$
using 27 *assms*(1) *assms*(3) *assms*(4) *idx-expand* **by** *fastforce*

have 29: $j \leq (\text{inth } l (\text{Suc } i)) - (\text{inth } l \ i) \longrightarrow$
 $(\text{inth}(\text{prefix}(\text{inth } l \ n) \ \sigma) ((\text{inth } l \ i) + j)) = (\text{inth } \sigma ((\text{inth } l \ i) + j))$
using *assms* *inth-prefix* **using** 28 **by** *blast*

show ?thesis

using 21 22 23 24 26 29 **by** *auto*

qed

qed

show ?thesis

using 1 2 *interval-eq-inth-eq* **by** *blast*

qed

lemma *powerinterval-splita*:

assumes *index-sequence* 0 *l*

$n \leq \text{ilen } l$

$\text{inth } l (\text{ilen } l) = \text{ilen } \sigma$

powerinterval *f* σ *l*

shows *powerinterval* *f* $(\text{prefix}(\text{inth } l \ n) \ \sigma) (\text{prefix } n \ l)$

proof –

have 0: $(\forall \ i. \ i < \text{ilen } l \longrightarrow ((\text{sub}(\text{inth } l \ i) (\text{inth } l (\text{Suc } i)) \ \sigma) \models f))$

using *assms* **by** (*simp* *add*: *powerinterval-def*)

have 1: *powerinterval* *f* $(\text{prefix}(\text{inth } l \ n) \ \sigma) (\text{prefix } n \ l) =$

$(\forall \ i. \ i < \text{ilen}(\text{prefix } n \ l) \longrightarrow$

$((\text{sub}(\text{inth}(\text{prefix } n \ l) \ i) (\text{inth}(\text{prefix } n \ l) (\text{Suc } i)) (\text{prefix}(\text{inth } l \ n) \ \sigma)) \models f))$

using *powerinterval-def* **by** *blast*

have 2: $(\forall \ i. \ i < \text{ilen}(\text{prefix } n \ l) \longrightarrow$

$((\text{sub}(\text{inth}(\text{prefix } n \ l) \ i) (\text{inth}(\text{prefix } n \ l) (\text{Suc } i)) (\text{prefix}(\text{inth } l \ n) \ \sigma)) \models f))$

proof

fix *i*

```

show  $i < \text{ilen}(\text{prefix } n \ l) \longrightarrow$ 
   $((\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)) \models f)$ 
proof –
  have 21:  $i < n \longrightarrow ((\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f)$ 
    using 0 assms less-le-trans by blast
  have 22:  $i < \text{ilen}(\text{prefix } n \ l) \longrightarrow$ 
     $((\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ \sigma) \models f)$ 
    by (simp add: 21 assms)
  have 23:  $i < \text{ilen}(\text{prefix } n \ l) \longrightarrow$ 
     $((\text{sub } (\text{inth } (\text{prefix } n \ l) \ i) \ (\text{inth } (\text{prefix } n \ l) \ (\text{Suc } i)) \ (\text{prefix } (\text{inth } l \ n) \ \sigma)) \models f)$ 
    using 22 assms powerinterval-split0 by fastforce
  show ?thesis using 23 by blast
qed
qed
show ?thesis using 1 2 by blast
qed

lemma powerinterval-splitb0:
assumes index-sequence 0  $l$ 
   $n \leq \text{ilen } l$ 
   $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma$ 
   $i < (\text{ilen } l - n)$ 
shows  $(\text{sub } (\text{inth } (((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) \ i)$ 
   $(\text{inth } (((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) \ (\text{Suc } i))$ 
   $(\text{suffix } (\text{inth } l \ n) \ \sigma) =$ 
   $(\text{sub } (\text{inth } l \ (i+n)) \ (\text{inth } l \ ((\text{Suc } i)+n)) \ \sigma)$ 
proof –
  have 1:  $\text{ilen}(((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) = (\text{ilen } l) - n$ 
    by (simp add: assms)
  have 2:  $i < (\text{ilen } l - n) \longrightarrow$ 
     $(\text{inth } (((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) \ i) =$ 
     $(\text{inth } l \ (n + i)) - (\text{inth } l \ n)$ 
    using assms idx-shiftn-suffix-inth by force
  have 3:  $i < (\text{ilen } l - n) \longrightarrow$ 
     $(\text{inth } (((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) \ (\text{Suc } i)) =$ 
     $(\text{inth } l \ (n + (\text{Suc } i))) - (\text{inth } l \ n)$ 
    using assms idx-shiftn-suffix-inth by fastforce
  have 4:  $i < (\text{ilen } l - n) \longrightarrow$ 
     $(\text{sub } (\text{inth } (((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) \ i)$ 
     $(\text{inth } (((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) \ (\text{Suc } i))$ 
     $(\text{suffix } (\text{inth } l \ n) \ \sigma) =$ 
     $(\text{sub } (\text{inth } l \ (n + i)) - (\text{inth } l \ n))$ 
     $(\text{inth } l \ (n + (\text{Suc } i))) - (\text{inth } l \ n)$ 
     $(\text{suffix } (\text{inth } l \ n) \ \sigma)$ 
    by (simp add: 2 3)
  have 5:  $i < (\text{ilen } l - n) \longrightarrow (\text{inth } l \ (n + i)) < (\text{inth } l \ (n + (\text{Suc } i)))$ 
    using assms index-sequence-def by auto
  have 6:  $i < (\text{ilen } l - n) \longrightarrow (\text{inth } l \ n) \leq (\text{inth } l \ (n + i))$ 
    using assms by (metis add.commute idx-less-equal le-add1 less-diff-conv less-imp-le-nat)
  have 7:  $i < (\text{ilen } l - n) \longrightarrow (\text{inth } l \ n) \leq (\text{inth } l \ (n + (\text{Suc } i)))$ 

```

```

    using 5 6 by linarith
have 8:  $i < (\text{ilen } l - n) \longrightarrow (\text{inth } l (n + i)) - (\text{inth } l n) < (\text{inth } l (n + (\text{Suc } i))) - (\text{inth } l n)$ 
    using 5 6 diff-less-mono by blast
have 9:  $i < (\text{ilen } l - n) \longrightarrow (\text{inth } l (n + (\text{Suc } i))) - (\text{inth } l n) \leq \text{ilen } \sigma - (\text{inth } l n)$ 
    by (metis Suc-leI add.commute assms diff-le-mono idx-less-last-1 le-eq-less-or-eq
        less-diff-conv ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 10:  $i < (\text{ilen } l - n) \longrightarrow$ 
    (sub ( (inth l (n + i)) - (inth l n) )
      ( (inth l (n + (Suc i))) - (inth l n) )
      (suffix (inth l n)  $\sigma$ )) =
    (sub (inth l (i+n)) (inth l ((Suc i)+n))  $\sigma$ )
    using sub-suffix
    by (metis 6 7 8 9 add.commute le-add-diff-inverse2)
show ?thesis
using 2 3 10 assms by auto
qed

```

lemma *powerinterval-splitb*:

```

assumes index-sequence 0 l
     $n \leq \text{ilen } l$ 
     $\text{inth } l (\text{ilen } l) = \text{ilen } \sigma$ 
    powerinterval f  $\sigma$  l

```

shows $\text{powerinterval } f (\text{suffix } (\text{inth } l n) \sigma) ((\text{imap } (\text{shiftn } (\text{inth } l n)) (\text{suffix } n l)))$

proof –

```

have 0:  $(\forall i. i < \text{ilen } l \longrightarrow ((\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma) \models f))$ 
    using assms by (simp add: powerinterval-def)
have 1:  $\text{powerinterval } f (\text{suffix } (\text{inth } l n) \sigma) ((\text{imap } (\text{shiftn } (\text{inth } l n)) (\text{suffix } n l))) =$ 
     $(\forall i. i < \text{ilen } (((\text{imap } (\text{shiftn } (\text{inth } l n)) (\text{suffix } n l)))) \longrightarrow$ 
    ( (sub (inth (((imap (shiftn (inth l n)) (suffix n l)))) i)
      (inth (((imap (shiftn (inth l n)) (suffix n l)))) (Suc i))
      (suffix (inth l n)  $\sigma$ ))  $\models f$ ))
    by (simp add: powerinterval-def)
have 2:  $(\forall i. i < \text{ilen } (((\text{imap } (\text{shiftn } (\text{inth } l n)) (\text{suffix } n l)))) \longrightarrow$ 
    ( (sub (inth (((imap (shiftn (inth l n)) (suffix n l)))) i)
      (inth (((imap (shiftn (inth l n)) (suffix n l)))) (Suc i))
      (suffix (inth l n)  $\sigma$ ))  $\models f$ ))

```

proof

fix i

```

show  $i < \text{ilen } (((\text{imap } (\text{shiftn } (\text{inth } l n)) (\text{suffix } n l)))) \longrightarrow$ 
    ( (sub (inth (((imap (shiftn (inth l n)) (suffix n l)))) i)
      (inth (((imap (shiftn (inth l n)) (suffix n l)))) (Suc i))
      (suffix (inth l n)  $\sigma$ ))  $\models f$  )

```

proof –

```

have 21:  $i < (\text{ilen } l) - n \longrightarrow$ 
     $((\text{sub } (\text{inth } l (i+n)) (\text{inth } l ((\text{Suc } i)+n)) \sigma) \models f)$ 
    by (simp add: 0)

```

show ?thesis

using 21 assms powerinterval-splitb0 **by** fastforce

qed

qed

show *?thesis* **using** 1 2 **by** *blast*
qed

lemma *powerinterval-split*:

assumes *index-sequence* 0 *l*

$n \leq \text{ilen } l$

$\text{inth } l (\text{ilen } l) = \text{ilen } \sigma$

shows $\text{powerinterval } f \ \sigma \ l =$

$(\text{powerinterval } f (\text{prefix } (\text{inth } l \ n) \ \sigma) (\text{prefix } n \ l) \wedge$
 $\text{powerinterval } f (\text{suffix } (\text{inth } l \ n) \ \sigma) ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) (\text{suffix } n \ l))))$

proof –

have 1: $\text{powerinterval } f \ \sigma \ l \implies$

$\text{powerinterval } f (\text{prefix } (\text{inth } l \ n) \ \sigma) (\text{prefix } n \ l)$

by (*simp add: assms powerinterval-splita*)

have 2: $\text{powerinterval } f \ \sigma \ l \implies$

$\text{powerinterval } f (\text{suffix } (\text{inth } l \ n) \ \sigma) ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) (\text{suffix } n \ l)))$

by (*simp add: assms powerinterval-splitb*)

have 3: $\text{powerinterval } f (\text{prefix } (\text{inth } l \ n) \ \sigma) (\text{prefix } n \ l) =$

$(\forall i. i < n \longrightarrow (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f)$

by (*metis assms prefix-ilen-good powerinterval-def powerinterval-splita0*)

have 4: $\text{powerinterval } f (\text{suffix } (\text{inth } l \ n) \ \sigma) ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) (\text{suffix } n \ l))) =$

$(\forall i. i < (\text{ilen } l) - n \longrightarrow ((\text{sub } (\text{inth } l \ (i+n)) (\text{inth } l \ ((\text{Suc } i)+n)) \ \sigma) \models f))$

by (*simp add: assms powerinterval-def powerinterval-splitb0*)

have 5: $(\forall i. i < (\text{ilen } l) - n \longrightarrow ((\text{sub } (\text{inth } l \ (i+n)) (\text{inth } l \ ((\text{Suc } i)+n)) \ \sigma) \models f)) =$

$(\forall i. n \leq i \wedge i < (\text{ilen } l) \longrightarrow ((\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f))$

by (*metis le-add2 le-add-diff-inverse2 less-diff-conv plus-nat.simps(2)*)

have 5: $(\text{powerinterval } f (\text{prefix } (\text{inth } l \ n) \ \sigma) (\text{prefix } n \ l) \wedge$

$\text{powerinterval } f (\text{suffix } (\text{inth } l \ n) \ \sigma) ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) (\text{suffix } n \ l)))) \implies$

$\text{powerinterval } f \ \sigma \ l$

using 3 4 5 *assms powerinterval-def*

by (*metis not-less-eq not-less-less-Suc-eq order.order-iff-strict*)

show *?thesis*

using 1 2 5 **by** *blast*

qed

lemma *powerinterval-fuse*:

assumes *index-sequence* 0 *l1*

index-sequence 0 *l2*

ilast *l1* = *cp*

$cp \leq \text{ilen } \sigma$

$\text{ilast } l2 = \text{ilen } \sigma - cp$

shows $\text{powerinterval } f \ \sigma \ (\text{fuse } l1 \ (\text{imap } (\text{shift } cp) \ l2)) =$

$(\text{powerinterval } f (\text{prefix } cp \ \sigma) \ l1 \wedge$

$\text{powerinterval } f (\text{suffix } cp \ \sigma) \ l2)$

proof –

have 1: *index-sequence* 0 (*fuse* *l1* (*imap* (*shift* *cp*) *l2*))

using *assms idx-fuse*[of *l1* (*imap* (*shift* *cp*) *l2*)]

by (*metis index-sequence-def idx-link*)

have 2: $cp = (\text{inth } l1 (\text{ilen } l1))$

using *assms inth-ilen-ilast* **by** *blast*

```

have 3:  $ilen\ l1 \leq ilen\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))$ 
  by (simp add: fuse-ilen-a)
have 4:  $inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ (ilen\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))) = ilen\ \sigma$ 
  by (metis assms idx-fuse-ilen ilen-gr-zero)
have 5:  $cp = inth\ (fuse\ l1\ (imap\ (shift\ cp)\ l2))\ (ilen\ l1)$ 
  by (metis 3 assms fuse-inth idx-fuse-ifirst-ilast order-refl)
have 6:  $(imap\ (shiftn\ cp)\ (imap\ (shift\ cp)\ l2)) = l2$ 
  using assms by (metis idx-link lsk-ls)
have 7:  $ilast\ l1 = ifirst\ (imap\ (shift\ cp)\ l2)$ 
  by (metis assms idx-fuse-ifirst-ilast)
show ?thesis
using 1 3 4 5 6 7 prefix-fuse suffix-fuse powerinterval-split
by fastforce
qed

```

lemma *powerinterval-idx:*

```

(powerinterval (LIFT( $f \wedge more$ ))  $\sigma\ l \wedge (inth\ l\ 0) = 0 \wedge (inth\ l\ (ilen\ l)) = ilen\ \sigma =$ 
 $(index-sequence\ 0\ l \wedge (inth\ l\ (ilen\ l)) = ilen\ \sigma \wedge powerinterval\ f\ \sigma\ l)$ )
proof (auto simp add: powerinterval-def index-sequence-def more-defs)
show  $\bigwedge n. \forall i < ilen\ l.$ 
   $f\ (sub\ (inth\ l\ i)\ (inth\ l\ (Suc\ i))\ \sigma) \wedge$ 
   $0 < ilen\ (sub\ (inth\ l\ i)\ (inth\ l\ (Suc\ i))\ \sigma) \implies$ 
   $inth\ l\ 0 = 0 \implies$ 
   $inth\ l\ (ilen\ l) = ilen\ \sigma \implies n < ilen\ l \implies$ 
   $inth\ l\ n < inth\ l\ (Suc\ n)$ 
by (simp add: sub-def)
show  $\bigwedge i. inth\ l\ 0 = 0 \implies$ 
   $\forall n < ilen\ l. inth\ l\ n < inth\ l\ (Suc\ n) \implies$ 
   $inth\ l\ (ilen\ l) = ilen\ \sigma \implies$ 
   $\forall i < ilen\ l. f\ (sub\ (inth\ l\ i)\ (inth\ l\ (Suc\ i))\ \sigma) \implies$ 
   $i < ilen\ l \implies 0 < ilen\ (sub\ (inth\ l\ i)\ (inth\ l\ (Suc\ i))\ \sigma)$ 
by (simp add: sub-def)
  (metis index-sequence-def idx-less-last-1)
qed

```

8.2.3 cpl lemmas

lemma *cpl-expand:*

```

assumes  $(\exists l. index-sequence\ 0\ l \wedge powerinterval\ f\ \sigma\ l \wedge (inth\ l\ (ilen\ l)) = ilen\ \sigma \wedge$ 
 $((filt\ \sigma\ l) \models g))$ 
shows  $index-sequence\ 0\ (cpl\ f\ g\ \sigma) \wedge powerinterval\ f\ \sigma\ (cpl\ f\ g\ \sigma) \wedge$ 
 $(inth\ (cpl\ f\ g\ \sigma)\ (ilen\ (cpl\ f\ g\ \sigma))) = ilen\ \sigma \wedge ((filt\ \sigma\ (cpl\ f\ g\ \sigma)) \models g)$ 
proof –
have 0:  $cpl\ f\ g\ \sigma = (\epsilon\ l. index-sequence\ 0\ l \wedge powerinterval\ f\ \sigma\ l \wedge$ 
 $(inth\ l\ (ilen\ l)) = ilen\ \sigma \wedge ((filt\ \sigma\ l) \models g))$ 
  using cpl-def by force
have 1:  $(\exists l. index-sequence\ 0\ l \wedge powerinterval\ f\ \sigma\ l \wedge (inth\ l\ (ilen\ l)) = ilen\ \sigma \wedge$ 
 $((filt\ \sigma\ l) \models g))$ 
  using assms by auto

```

have 2: *index-sequence* 0 (*cpl f g* σ) \wedge *powerinterval* *f* σ (*cpl f g* σ) \wedge
 (*inth* (*cpl f g* σ) (*ilen* (*cpl f g* σ))) = *ilen* $\sigma \wedge ((\text{filt } \sigma \text{ (cpl f g } \sigma)) \models g)$
using 0 1
someI-ex[of $\lambda l. \text{index-sequence } 0 \ l \wedge \text{powerinterval } f \ \sigma \ l \wedge (\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma \wedge$
 $((\text{filt } \sigma \ l) \models g)$] **by** *simp*
show ?thesis
using 2 **by** *blast*
qed

lemma *cpl-projection*:

$(\sigma \models f \triangle g) =$
 (*index-sequence* 0 (*cpl f g* σ) \wedge *powerinterval* *f* σ (*cpl f g* σ) \wedge
 (*inth* (*cpl f g* σ) (*ilen* (*cpl f g* σ))) = *ilen* $\sigma \wedge g \ (\text{filt } \sigma \text{ (cpl f g } \sigma))$)
using *cpl-expand* **by** (*simp add: projection-d-def, blast*)

lemma *cpl-empty*:

assumes *ilen* $\sigma = 0 \wedge (\sigma \models f \triangle g)$
shows (*cpl f g* σ) = $\langle 0 \rangle$
using *assms cpl-projection idx-less-last-1 INil-ilen* **by** *fastforce*

lemma *cpl-empty-a*:

assumes *ilen* $\sigma = 0$
 (*cpl f g* σ) = $\langle 0 \rangle$
 $g(\text{filt } \sigma \ \langle 0 \rangle)$
shows ($\sigma \models f \triangle g$)

proof –

have 1: *index-sequence* 0 $\langle 0 \rangle$
by (*simp add: index-sequence-def*)
have 2: *powerinterval* *f* σ $\langle 0 \rangle$
by (*simp add: powerinterval-def*)
have 3: (*inth* $\langle 0 \rangle$ (*ilen* $\langle 0 \rangle$)) = *ilen* σ
by (*simp add: assms*)
have 4: $g(\text{filt } \sigma \ \langle 0 \rangle)$
using *assms* **by** *blast*
from 1 2 3 4 **show** ?thesis
by (*simp add: assms cpl-projection*)
qed

lemma *cpl-more*:

assumes *ilen* $\sigma > 0$
 $(\sigma \models f \triangle g)$
shows *ilen*(*cpl f g* σ) > 0
by (*metis assms cpl-projection gr0I index-sequence-def*)

lemma *cpl-more-than-first*:

assumes *ilen* $\sigma > 0$
 $(\sigma \models f \triangle g)$
shows (*inth* (*cpl f g* σ) 0) = 0
using *assms cpl-projection index-sequence-def* **by** *auto*

lemma *cpl-more-than-last*:

assumes $ilen\ \sigma > 0$

$(\sigma \models f \triangle g)$

shows $(inth\ (cpl\ f\ g\ \sigma)\ (ilen\ (cpl\ f\ g\ \sigma))) = ilen\ \sigma$

using *assms cpl-projection* **by** *blast*

lemma *cpl-sub-more*:

assumes $n < k$

$k \leq ilen\ \sigma$

$((sub\ n\ k\ \sigma) \models f \triangle g)$

shows $ilen(cpl\ f\ g\ (sub\ n\ k\ \sigma)) > 0$

using *assms*

by (*simp add: cpl-more*)

lemma *cpl-bounds*:

assumes $n < k$

$k \leq ilen\ \sigma$

$((sub\ n\ k\ \sigma) \models f \triangle g)$

$i < ilen\ (cpl\ f\ g\ (sub\ n\ k\ \sigma))$

shows $0 \leq (inth\ (cpl\ f\ g\ (sub\ n\ k\ \sigma))\ i) \wedge (inth\ (cpl\ f\ g\ (sub\ n\ k\ \sigma))\ i) \leq k - n$

using *assms*

by (*metis cpl-projection idx-less-last-1 ilen-sub le0 less-imp-le-nat*)

lemma *cpl-imap-bounds*:

assumes $n < k$

$k \leq ilen\ \sigma$

$((sub\ n\ k\ \sigma) \models f \triangle g)$

$i < ilen\ (imap\ (shift\ n)\ (cpl\ f\ g\ (sub\ n\ k\ \sigma)))$

shows $n \leq (inth\ (imap\ (shift\ n)\ (cpl\ f\ g\ (sub\ n\ k\ \sigma)))\ i) \wedge$

$(inth\ (imap\ (shift\ n)\ (cpl\ f\ g\ (sub\ n\ k\ \sigma)))\ i) \leq k$

using *assms*

by (*metis shift-def Nat.le-diff-conv2 cpl-bounds ilen-imap*

inth-imap le-add2 less-imp-le-nat)

lemma *cpl-ifirst*:

assumes $(sub\ x1a\ (ifirst\ l)\ \sigma) \models f \triangle g$

shows $ifirst((imap\ (shift\ x1a)\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma)))) = x1a$

proof –

have 1: $(index\ sequence\ 0\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma)) \wedge$

$powerinterval\ f\ (sub\ x1a\ (ifirst\ l)\ \sigma)\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma)) \wedge$

$(inth\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma))\ (ilen\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma)))) =$

$ilen\ (sub\ x1a\ (ifirst\ l)\ \sigma) \wedge$

$g\ (filt\ (sub\ x1a\ (ifirst\ l)\ \sigma)\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma)))$)

using *cpl-projection assms* **by** *auto*

have 2: $index\ sequence\ 0\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma))$

using 1 **by** *auto*

have 3: $(inth\ (cpl\ f\ g\ (sub\ x1a\ (ifirst\ l)\ \sigma))\ 0) = 0$

using 2 *index-sequence-def* **by** *blast*

show *?thesis*

by (metis 3 shift-def add.left-neutral inth-imap)
qed

lemma cpl-ifirst-same:

assumes (sub x1a x1a σ) ⊨ f Δ g
shows ifirst((imap (shift x1a) (cpl f g (sub x1a x1a σ)))) = x1a
proof –
have 1: ifirst (⟨ x1a ⟩) = x1a
by auto
from 1 cpl-ifirst show ?thesis by (metis assms)
qed

lemma cpl-ilast:

assumes ((sub x (ifirst l) σ) ⊨ f Δ g) ∧ x < ifirst l ∧ ifirst l ≤ ilen σ
shows ilast((imap (shift x) (cpl f g (sub x (ifirst l) σ)))) = ifirst l
proof –
have 01: (index-sequence 0 (cpl f g (sub x (ifirst l) σ)) ∧
powerinterval f (sub x (ifirst l) σ) (cpl f g (sub x (ifirst l) σ)) ∧
(inth (cpl f g (sub x (ifirst l) σ)) (ilen (cpl f g (sub x (ifirst l) σ)))) =
ilen (sub x (ifirst l) σ) ∧
g (filt (sub x (ifirst l) σ) (cpl f g (sub x (ifirst l) σ))))
using cpl-projection assms by (simp add: cpl-projection)
have 02: (inth (cpl f g (sub x (ifirst l) σ)) (ilen (cpl f g (sub x (ifirst l) σ)))) =
ilen (sub x (ifirst l) σ)
using 01 by auto
have 03: ilast(cpl f g (sub x (ifirst l) σ)) =
(inth (cpl f g (sub x (ifirst l) σ)) (ilen (cpl f g (sub x (ifirst l) σ))))
by simp
have 04: x < ifirst l
using assms by blast
have 05: ilen (sub x (ifirst l) σ) = (ifirst l) - x
using assms ilen-sub less-imp-le-nat by blast
have 06: ilast((imap (shift x) (cpl f g (sub x (ifirst l) σ)))) =
((ifirst l) - x) + x
by (metis 02 05 shift-def ilen-imap inth-imap)
show ?thesis using 04 06 by auto
qed

lemma cpl-ilast-i:

assumes ((sub (inth l i) (inth l (Suc i)) σ) ⊨ f Δ g)
(inth l i) < (inth l (Suc i))
(inth l (Suc i)) ≤ ilen σ
shows ilast((imap (shift (inth l i)) (cpl f g (sub (inth l i) (inth l (Suc i)) σ)))) =
(inth l (Suc i))
proof –
have 01: (index-sequence 0 (cpl f g (sub (inth l i) (inth l (Suc i)) σ)) ∧
powerinterval f (sub (inth l i) (inth l (Suc i)) σ)
(cpl f g (sub (inth l i) (inth l (Suc i)) σ)) ∧
(inth (cpl f g (sub (inth l i) (inth l (Suc i)) σ))
(ilen (cpl f g (sub (inth l i) (inth l (Suc i)) σ)))) =

```

      ilen (sub (inth l i) (inth l (Suc i)) σ) ∧
      g (filt (sub (inth l i) (inth l (Suc i)) σ) (cpl f g (sub (inth l i) (inth l (Suc i)) σ))) )
    using cpl-projection assms by (simp add: cpl-projection)
  have 02: (inth (cpl f g (sub (inth l i) (inth l (Suc i)) σ))
    (ilen (cpl f g (sub (inth l i) (inth l (Suc i)) σ)))) =
    ilen (sub (inth l i) (inth l (Suc i)) σ)
  using 01 by auto
  have 03: ilast(cpl f g (sub (inth l i) (inth l (Suc i)) σ)) =
    (inth (cpl f g (sub (inth l i) (inth l (Suc i)) σ))
      (ilen (cpl f g (sub (inth l i) (inth l (Suc i)) σ))))
  by simp
  have 04: (inth l i) < (inth l (Suc i))
  by (simp add: assms)
  have 05: ilen (sub (inth l i) (inth l (Suc i)) σ) = (inth l (Suc i)) - (inth l i)
  by (simp add: assms less-imp-le-nat)
  have 06: ilast((imap (shift (inth l i)) (cpl f g (sub (inth l i) (inth l (Suc i)) σ)))) =
    ((inth l (Suc i)) - (inth l i)) + (inth l i)
  by (metis 02 05 shift-def ilen-imap inth-imap)
  show ?thesis using 04 06 by auto
qed

```

8.2.4 lcpl lemmas

```

lemma lcpl-inth:
  assumes index-sequence (inth l 0) l
    i < ilen l
  shows (inth (lcpl f g σ l) i) =
    (imap (shift (inth l i)) (cpl f g (sub (inth l i) (inth l (Suc i)) σ)))
using assms
proof (induct l arbitrary: i)
case (INil x)
then show ?case by simp
next
case (ICons x1a ls)
then show ?case
  proof (cases i)
  case 0
  then show ?thesis
    proof (cases ls)
    case (INil x1)
    then show ?thesis by (simp add: 0)
    next
    case (ICons x21 x22)
    then show ?thesis by (simp add: 0)
  qed
next
case (Suc nat)
then show ?thesis
  proof (cases ls)
  case (INil x1)

```

```

then show ?thesis
by (metis ICons.prem1(2) One-nat-def Suc ilen.simps(1) ilen.simps(2) leD le-add1
    plus-1-eq-Suc)
next
case (ICons x21 x22)
then show ?thesis
using ICons.hyps ICons.prem1(1) ICons.prem1(2) Suc idx-ICons by auto
qed
qed
qed

```

```

lemma lcpl-ilen:
assumes index-sequence (inth l 0) l
        ilen l > 0
shows    ilen(lcpl f g σ l) = ilen l - 1
using assms
proof
  (induct l)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a l)
  then show ?case
  proof (cases l)
  case (INil x1)
  then show ?thesis by simp
  next
  case (ICons x21 x22)
  then show ?thesis using ICons.hyps ICons.prem1(1) idx-ICons by auto
  qed
qed

```

```

lemma lcpl-ilen-zero:
assumes index-sequence (inth l 0) l
        ilen l = 0
shows    ilen(lcpl f g σ l) = 0
using assms
by (metis One-nat-def ilen.simps(1) ilen.simps(2) interval.exhaust lcpl.simps(1) leD le-add1
    le-imp-less-Suc order-refl)

```

```

lemma lcpl-last:
assumes index-sequence (inth l 0) l
        (inth l (ilen l)) = ilen σ
        ilen l > 0
shows    ilast (lcpl f g σ l) =
        (imap (shift (inth l (ilen l - 1)))
            (cpl f g (sub (inth l (ilen l - 1)) (inth l (ilen l)) σ)))
proof -

```

have 1: $ilast (lcpl\ f\ g\ \sigma\ l) = (inth\ (lcpl\ f\ g\ \sigma\ l)\ (ilen\ (lcpl\ f\ g\ \sigma\ l)))$
by *simp*
have 2: $(inth\ (lcpl\ f\ g\ \sigma\ l)\ (ilen\ (lcpl\ f\ g\ \sigma\ l))) =$
 $(imap\ (shift\ (inth\ l\ (ilen\ (lcpl\ f\ g\ \sigma\ l))))$
 $(cpl\ f\ g\ (sub\ (inth\ l\ (ilen\ (lcpl\ f\ g\ \sigma\ l)))$
 $(inth\ l\ (Suc\ (ilen\ (lcpl\ f\ g\ \sigma\ l))))$
 $\sigma)))$
using *assms lcpl-inth*
by (*metis One-nat-def Suc-pred diff-le-self lcpl-ilen le-eq-less-or-eq n-not-Suc-n*)
have 3: $(imap\ (shift\ (inth\ l\ (ilen\ (lcpl\ f\ g\ \sigma\ l))))$
 $(cpl\ f\ g\ (sub\ (inth\ l\ (ilen\ (lcpl\ f\ g\ \sigma\ l)))$
 $(inth\ l\ (Suc\ (ilen\ (lcpl\ f\ g\ \sigma\ l))))$
 $\sigma))) =$
 $(imap\ (shift\ (inth\ l\ (ilen\ l-1)))$
 $(cpl\ f\ g\ (sub\ (inth\ l\ (ilen\ l-1))\ (inth\ l\ (Suc\ (ilen\ l-1)))\ \sigma)))$
using *assms by (metis lcpl-ilen)*
show *?thesis* **by** (*simp add: 2 3 assms(3)*)
qed

lemma *lcpl-last-last:*

assumes *index-sequence* $(inth\ l\ 0)\ l$
 $(inth\ l\ (ilen\ l)) = ilen\ \sigma$
 $((sub\ (inth\ l\ (ilen\ l-1))\ (inth\ l\ (ilen\ l))\ \sigma) \models f\ \triangle\ g)$
 $ilen\ l > 0$

shows $ilast\ (ilast\ (lcpl\ f\ g\ \sigma\ l)) = ilen\ \sigma$

using *assms*

by (*metis (no-types, lifting) One-nat-def Suc-pred' cpl-ilast-i diff-Suc-less*
dual-order.order-iff-strict index-sequence-def lcpl-last)

lemma *lcpl-zero-zero:*

assumes *index-sequence* $(inth\ l\ 0)\ l$
 $(inth\ l\ (ilen\ l)) = ilen\ \sigma$
 $ilen\ l = 0$

shows $(inth\ (lcpl\ f\ g\ \sigma\ l)\ 0) =$
 $(imap\ (shift\ (inth\ l\ 0))\ (cpl\ f\ g\ (sub\ (inth\ l\ 0)\ (inth\ l\ 0)\ \sigma)))$

using *assms*

by *simp*

(*metis INil-ilen lcpl.simps(1)*)

lemma *lcpl-ifirst:*

assumes *index-sequence* $(inth\ l\ 0)\ l$
 $(inth\ l\ (ilen\ l)) = ilen\ \sigma$
 $(\forall\ i < ilen\ l. (sub\ (inth\ l\ i)\ (inth\ l\ (Suc\ i))\ \sigma) \models f\ \triangle\ g)$
 $ilen\ l > 0$

shows $ifirst(ifirst((lcpl\ f\ g\ \sigma\ l))) = ifirst\ l$

proof –

have 01: $(ifirst((lcpl\ f\ g\ \sigma\ l))) =$
 $(imap\ (shift\ (inth\ l\ 0))\ (cpl\ f\ g\ (sub\ (inth\ l\ 0)\ (inth\ l\ (Suc\ 0))\ \sigma)))$

using *assms by (metis lcpl-inth)*

have 02: $ilen\ l > 0 \longrightarrow$

```

    ifirst((imap (shift (inth l 0)) (cpl f g (sub (inth l 0) (inth l (Suc 0)) σ)))) =
      (inth l 0)
    using assms by (metis Suc-leI cpl-ifirst ilast-ifirst ilast-prefix)
  show ?thesis
  using 01 02 inth-zero-ifirst by (simp add: assms(4))
qed

```

lemma *lcpl-lfuse-lastfirst*:

```

assumes index-sequence (inth l 0) l
  (inth l (ilen l)) = ilen σ
  ((ilen l = 0 ∧ ((sub (inth l 0) (inth l 0) σ) ⊨ f Δ g)) ∨
   (ilen l > 0 ∧ (∀ i < ilen l. (sub (inth l i) (inth l (Suc i)) σ) ⊨ f Δ g)))
shows lastfirst (lcpl f g σ l)
using assms
proof
  (induct l)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a ls)
  then show ?case
  proof –
    have 1: ilen ls = 0 ⟶ lastfirst (lcpl f g σ (x1a ⊙ ls))
      using ICons.prem1 by (metis One-nat-def add-diff-cancel-left' INil-ilen
        ilen.simps(2) lastfirst.simps(1) lcpl-ilen plus-1-eq-Suc zero-less-one)
    have 2: ilen ls > 0 ⟶ lastfirst (lcpl f g σ (x1a ⊙ ls)) =
      lastfirst((imap (shift x1a) (cpl f g (sub x1a (ifirst ls) σ))) ⊙ (lcpl f g σ ls))
    by (metis (no-types, lifting) interval.simps(6) ilen-ICons-1 inth-zero
      lcpl.simps(2))
    have 3: ilen ls > 0 ⟶
      lastfirst((imap (shift x1a) (cpl f g (sub x1a (ifirst ls) σ))) ⊙ (lcpl f g σ ls)) =
      ( ilast( (imap (shift x1a) (cpl f g (sub x1a (ifirst ls) σ))) ) =
        ifirst(ifirst((lcpl f g σ ls))) ∧ lastfirst((lcpl f g σ ls)) )
      using lastfirst.simps(2) by blast
    have 4: x1a < (ifirst ls)
      using ICons.prem1 by (metis index-sequence-def inth-Suc inth-zero
        ilen.simps(2) plus-1-eq-Suc zero-less-Suc)
    have 5: (ifirst ls) ≤ ilen σ
      using ICons.prem1 by (metis Suc-lessI eq-iff idx-less-last-1
        inth-Suc ilen.simps(2) less-imp-le-nat plus-1-eq-Suc zero-less-Suc)
    have 6: ilen ls > 0 ⟶
      ilast( (imap (shift x1a) (cpl f g (sub x1a (ifirst ls) σ))) ) = ifirst ls
      using ICons.prem1 by (metis 4 5 cpl-ilast inth-Suc inth-zero less-le-not-le)
    have 7: index-sequence (inth ls 0) ls
      using assms ICons.prem1 index-sequence-def by auto
    have 8: inth ls (ilen ls) = ilen σ
      using ICons.prem1 by auto
    have 9: (∀ i < ilen (ls). (sub (inth (ls) i) (inth (ls) (Suc i)) σ) ⊨ f Δ g)
      using ICons.prem1 by auto
  end

```

```

have 10:  $ilen\ ls > 0 \longrightarrow ifirst(ifirst((lcpl\ f\ g\ \sigma\ ls))) = ifirst\ ls$ 
using 7 8 9
by (metis cpl-ifirst ilen-ICons-1 inth-Suc lcpl-inth)
have 11:  $ilen\ ls > 0 \longrightarrow$ 
       $lastfirst((lcpl\ f\ g\ \sigma\ ls))$ 
using 7 8 9 ICons.hyps by blast
show ?thesis using 1 10 11 2 6 by auto
qed
qed

lemma lcpl-lfuse-ilen:
assumes index-sequence (inth l 0) l
      (inth l (ilen l)) = ilen  $\sigma$ 
      ((ilen l = 0  $\wedge$  ((sub (inth l 0) (inth l 0)  $\sigma \models f \triangle g$ )))  $\vee$ 
      (ilen l > 0  $\wedge$  ( $\forall\ i < ilen\ l.$  (sub (inth l i) (inth l (Suc i))  $\sigma \models f \triangle g$ ))))
shows (ilen l = 0  $\longrightarrow$  ilen(lfuse (lcpl f g  $\sigma$  l)) = 0)  $\wedge$ 
      (ilen l > 0  $\longrightarrow$ 
      ilen(lfuse (lcpl f g  $\sigma$  l)) = ( $\sum k=0..ilen\ l-1.$  ilen (inth (lcpl f g  $\sigma$  l) k)))
proof -
have 1: lastfirst (lcpl f g  $\sigma$  l)
using assms lcpl-lfuse-lastfirst by blast
have 2: ilen l = 0  $\longrightarrow$ 
      lfuse (lcpl f g  $\sigma$  l) =
      ((imap (shift (inth l 0)) (cpl f g (sub (inth l 0) (inth l 0)  $\sigma$ ))))
by (metis assms(1) assms(2) INil-ilen lcpl-ilen-zero lcpl-zero-zero lfuse-INil)
have 3: ilen l = 0  $\longrightarrow$ 
      ilen ((imap (shift (inth l 0)) (cpl f g (sub (inth l 0) (inth l 0)  $\sigma$ )))) = 0
using assms cpl-empty ilen-sub by fastforce
have 4: ilen l = 0  $\longrightarrow$  ilen(lfuse (lcpl f g  $\sigma$  l)) = 0
using 2 3 by auto
have 5: ilen l > 0  $\longrightarrow$ 
      ilen(lfuse (lcpl f g  $\sigma$  l)) = ( $\sum k=0..ilen\ l-1.$  ilen (inth (lcpl f g  $\sigma$  l) k))
using lfuse-ilen
by (metis assms lcpl-ilen lcpl-lfuse-lastfirst)
from 4 5 show ?thesis by simp
qed

```

```

lemma lcpl-lfuse-idx:
assumes index-sequence 0 l
      (inth l (ilen l)) = ilen  $\sigma$ 
      ( $\forall\ i < ilen\ l.$  (sub (inth l i) (inth l (Suc i))  $\sigma \models f \triangle g$ )
      ilen l > 0)
shows index-sequence (ifirst (lfuse (lcpl f g  $\sigma$  l))) (lfuse (lcpl f g  $\sigma$  l))
proof -
have 0: ilen  $\sigma > 0 \longrightarrow$  ilen l > 0
using assms gr-zeroI index-sequence-def by fastforce
have 2: ilen  $\sigma > 0 \longrightarrow$  lastfirst (lcpl f g  $\sigma$  l)
using 0 assms index-sequence-def lcpl-lfuse-lastfirst by blast
have 3: ( $\forall\ i < ilen\ l.$  index-sequence 0 (cpl f g (sub (inth l i) (inth l (Suc i))  $\sigma$ )))
using assms cpl-projection by auto

```

have 4: $(\forall i < \text{ilen } l.$
 $\text{index-sequence } (\text{inth } l \ i)$
 $(\text{imap } (\text{shift } (\text{inth } l \ i)) (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma))))$
using 3 idx-link by blast
have 5: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l. \text{index-sequence } (\text{inth } l \ i) (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i))$
using assms by (metis 4 index-sequence-def lcpl-inth)
have 6: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l. \text{ifirst } (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i) = (\text{inth } l \ i))$
by (metis 5 index-sequence-def)
have 7: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $\text{index-sequence } (\text{ifirst } (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)) (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i))$
using 6 5 by simp
have 8: $\text{ilen } \sigma > 0 \longrightarrow$
 $\text{index-sequence } (\text{ifirst } (\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l))) (\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l)))$
using assms
by (metis (mono-tags, lifting) 2 7 Suc-diff-1 index-sequence-def idx-lfuse
 $\text{lcpl-ilen le-imp-less-Suc})$
from 8 show ?thesis
by (metis assms(1) assms(2) assms(4) index-sequence-def idx-less-last-1)
qed

lemma lcpl-ilen-inth-gr-zero:

assumes $\text{index-sequence } 0 \ l$

$(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$

$\text{ilen } \sigma > 0$

shows $(\forall j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l). \text{ilen}(\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ j) > 0)$

proof

fix j

show $j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) \longrightarrow 0 < \text{ilen } (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ j)$

proof –

have 1: $\text{ilen } \sigma > 0 \longrightarrow \text{lastfirst } (\text{lcpl } f \ g \ \sigma \ l)$

by (metis assms index-sequence-def lcpl-lfuse-lastfirst neq0-conv)

have 2: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } l > 0$

using assms gr-zeroI index-sequence-def by fastforce

have 3: $\text{ilen } \sigma > 0 \longrightarrow j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) \longrightarrow$

$(\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ j) =$

$(\text{imap } (\text{shift } (\text{inth } l \ j)) (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ j) (\text{inth } l \ (\text{Suc } j)) \ \sigma))))$

using assms lcpl-inth[of l j f g σ]

by (metis 2 One-nat-def Suc-pred index-sequence-def lcpl-ilen less-Suc-eq-le)

have 4: $\text{ilen } \sigma > 0 \longrightarrow j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) \longrightarrow$

$\text{ilen } (\text{imap } (\text{shift } (\text{inth } l \ j)) (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ j) (\text{inth } l \ (\text{Suc } j)) \ \sigma)))) > 0$

by (metis (no-types, lifting) 2 One-nat-def Suc-pred add commute assms cpl-sub-more gr0I

$\text{index-sequence-def idx-expand ilen-imap lcpl-ilen$

$\text{le-imp-less-Suc plus-1-eq-Suc})$

show ?thesis

using 3 4 by (simp add: assms(4))

qed

qed

lemma *lcpl-ilast-inth*:

assumes *index-sequence* 0 *l*

$(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$

$\text{ilen } \sigma > 0$

$j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l)$

shows $\text{ilast}(\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ j) = (\text{inth } l \ (\text{Suc } j))$

proof –

have 0: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } l > 0$

using *assms* *gr-zeroI* *index-sequence-def* **by** *fastforce*

have 1: $\text{ilen } \sigma > 0 \longrightarrow$

$j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) \longrightarrow$

$\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ j =$

$(\text{imap } (\text{shift } (\text{inth } l \ j)) \ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ j) \ (\text{inth } l \ (\text{Suc } j)) \ \sigma)))$

using *assms* *lcpl-inth*[*of l j f g σ*]

proof –

show *?thesis*

by (*metis* (*no-types*) 0 *Suc-diff-1*

$\langle \llbracket \text{index-sequence } (\text{inth } l \ 0) \ l; j < \text{ilen } l \rrbracket \Longrightarrow$

$\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ j =$

$\text{imap } (\text{shift } (\text{inth } l \ j)) \ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ j) \ (\text{inth } l \ (\text{Suc } j)) \ \sigma)) \rangle$

$\langle \text{index-sequence } 0 \ l \rangle \text{index-sequence-def lcpl-ilen le-imp-less-Suc}$)

qed

have 2: $\text{ilen } \sigma > 0 \longrightarrow$

$j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) \longrightarrow (\text{inth } l \ (\text{Suc } j)) \leq \text{ilen } \sigma$

using *assms*

by (*metis* 0 *Suc-diff-1* *Suc-eq-plus1* *index-sequence-def* *idx-expand* *lcpl-ilen* *leD* *not-less-eq*)

have 2: $\text{ilen } \sigma > 0 \longrightarrow$

$j \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) \longrightarrow$

$\text{ilast}(\text{imap } (\text{shift } (\text{inth } l \ j)) \ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ j) \ (\text{inth } l \ (\text{Suc } j)) \ \sigma))) =$

$(\text{inth } l \ (\text{Suc } j))$

using *assms* *cpl-ilast-i*[*of f g l j σ*]

by (*metis* 0 2 *One-nat-def* *Suc-pred* *index-sequence-def* *lcpl-ilen* *le-imp-less-Suc*)

show *?thesis* **using** 1 2 **by** (*simp* *add*: *assms*(4) *assms*(5))

qed

lemma *lcpl-lfuse-filt-power-help*:

assumes *index-sequence* 0 *l*

$(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$

$\text{ilen } \sigma > 0$

shows $(\forall i < \text{ilen } l. g \ (\text{filt } \sigma \ (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)))$

proof –

have 1: $(\forall i < \text{ilen } l. g (\text{filt } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma) \\ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \))$
using *assms cpl-projection* **by** *blast*
have 2: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)) =$
 $(\text{filt } \sigma (\text{imap } (\text{shift } (\text{inth } l \ i)) (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma))))))$
using *assms* **by** *(metis index-sequence-def lcpl-inth)*
have 3: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $\text{ilen } (\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)) =$
 $\text{ilen } (\text{filt } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma) \\ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \))$
 $)$
by *(simp add: 2 filt-ilen)*
have 4: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\forall j \leq \text{ilen } (\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)).$
 $(\text{inth } (\text{filt } \sigma (\text{imap } (\text{shift } (\text{inth } l \ i)) (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)))) \ j)$
 $=$
 $(\text{inth } \sigma ((\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ j) + (\text{inth } l \ i)) \))$
 $)$
using *inth-imap shift-def filt-imap* **by** *metis*
have 5: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\forall j \leq \text{ilen } (\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)).$
 $(\text{inth } (\text{filt } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma) \\ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \) \ j) =$
 $(\text{inth } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma) \\ (\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ j))$
 $))$
by *(simp add: filt-imap inth-imap)*
have 6: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\text{inth } l \ (\text{Suc } i)) \leq \text{ilen } \sigma$
 $)$
using *assms idx-expand* **by** *auto*
have 7: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\text{inth } l \ i) \leq (\text{inth } l \ (\text{Suc } i))$
 $)$
using *assms index-sequence-def less-imp-le* **by** *blast*
have 8: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\forall j \leq \text{ilen } (\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)).$
 $(\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ j) \leq (\text{inth } l \ (\text{Suc } i)) - (\text{inth } l \ i)$
 $))$

by (metis 3 6 7 assms(3) cpl-projection filt-ilen idx-less-last-1 ilen-sub
 le-eq-less-or-eq)
 have 9: $\text{ilen } \sigma > 0 \longrightarrow$
 ($\forall i < \text{ilen } l.$
 ($\forall j \leq \text{ilen } (\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)).$
 ($\text{inth } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)$
 ($\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ j$)) =
 ($\text{inth } \sigma ((\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ j) + (\text{inth } l \ i))$
))
 using 6
 by (simp add: 7 8 add commute)
 have 10: $\text{ilen } \sigma > 0 \longrightarrow$
 ($\forall i < \text{ilen } l.$
 ($\forall j \leq \text{ilen } (\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)).$
 ($\text{filt } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)$
 ($\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \) =$
 ($\text{filt } \sigma (\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ i)$
))
 by (simp add: filt-expand)
 (metis 2 3 4 9 filt-ilen filt-imap inth-imap)
 show ?thesis using 1 10 assms(4) by fastforce
 qed

8.2.5 lsum lemmas

lemma *lsum-INil*:

$\text{lsum } \langle xs \rangle \ a = \langle a + \text{ilen } xs \rangle$

by *simp*

lemma *lsum-addzero-INil*:

$\text{addzero } (\text{lsum } \langle xs \rangle \ 0) = (\text{if } \text{ilen } xs = 0 \text{ then } \langle 0 \rangle \text{ else } \langle 0, \text{ilen } xs \rangle)$

by (simp add: addzero-def)

lemma *lsum-addzero-ICons*:

$\text{addzero } (\text{lsum } (xs \odot xxs) \ 0) = 0 \odot (\text{lsum } (xs \odot xxs) \ 0)$

by (simp add: addzero-def)

lemma *lsum-ifirst*:

$\text{ifirst } (\text{lsum } xxs \ a) = a + \text{ilen}(\text{ifirst } xxs)$

by (case-tac xxs) simp-all

lemma *lsum-ilen*:

$\text{ilen } (\text{lsum } xxs \ a) = \text{ilen } xxs$

by (induct xxs arbitrary: a) simp-all

lemma *lsum-addzero-ifirst*:

$\text{ifirst } (\text{addzero } (\text{lsum } xxs \ 0)) = 0$

by (simp add: addzero-def lsum-ifirst)

lemma *lsum-addzero-ilen*:

$(ilen\ xxs = 0 \wedge ilen(ifirst\ xxs) = 0 \longrightarrow$
 $ilen\ (addzero\ (lsum\ xxs\ 0)) = 0)$
 \wedge
 $(ilen\ xxs = 0 \wedge ilen(ifirst\ xxs) > 0 \longrightarrow$
 $ilen\ (addzero\ (lsum\ xxs\ 0)) = 1)$
 \wedge
 $(ilen\ xxs > 0 \longrightarrow$
 $ilen\ (addzero\ (lsum\ xxs\ 0)) = (ilen\ xxs) + 1)$

by (*simp add: addzero-def*)

(*metis add.left-neutral lsum-ifirst lsum-ilen*)

lemma *lsum-inth-help*:

assumes $i > 0$

$i \leq ilen\ xxs + 1$

shows $(\sum k = 0..(i-1). ilen\ (inth\ (xxs)\ k)) =$
 $(\sum k = 1..i. ilen\ (inth\ (xxs)\ (k-1)))$

using *assms*

proof

(*induct i*)

case 0

then show ?case **by** *blast*

next

case (*Suc i*)

then show ?case

proof *simp-all*

assume *a1*: $0 < i \implies (\sum k = 0..i - Suc\ 0. ilen\ (inth\ xxs\ k)) =$
 $(\sum k = Suc\ 0..i. ilen\ (inth\ xxs\ (k - Suc\ 0)))$

have *f2*: $\forall f. sum\ f\ \{1::nat..0\} = (0::nat)$

by *auto*

have *f3*: $\forall n\ f. sum\ f\ \{n::nat..n\} = (f\ n::nat)$

by *simp*

have *f4*: $\forall f\ n. (sum\ f\ \{0::nat..n - 1\} + (f\ n::nat) = sum\ f\ \{0..n\} \vee 0 = n) \vee \neg 0 \leq n$

by (*metis (no-types) One-nat-def Suc-pred le-eq-less-or-eq sum.atLeast0-atMost-Suc*)

have *f5*: $\forall n. (0::nat) \leq n$

by *blast*

have $\forall n. (0::nat) + n = n$

by *linarith*

then show $(\sum n = 0..i. ilen\ (inth\ xxs\ n)) =$
 $(\sum n = Suc\ 0..i. ilen\ (inth\ xxs\ (n - Suc\ 0))) + ilen\ (inth\ xxs\ i)$

using *f5 f4 f3 f2 a1* **by** (*metis One-nat-def le-eq-less-or-eq*)

qed

qed

lemma *lsum-inth*:

assumes $i \leq ilen\ xxs$

shows $inth\ (lsum\ xxs\ a)\ i = a + (\sum k::nat = 0..i. ilen(inth\ xxs\ k))$

using *assms*

```

proof
  (induct xs arbitrary: a i)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof -
    have 1:  $\text{inth} (\text{lsum} (x1a \odot xs) a) i = \text{inth} ((a + \text{ilen } x1a) \odot (\text{lsum } xs (a + \text{ilen } x1a))) i$ 
      by simp
    have 2:  $i \leq \text{ilen } xs + 1$ 
      using ICons.premis by auto
    have 3:  $i = 0 \longrightarrow$ 
       $\text{inth} ((a + \text{ilen } x1a) \odot (\text{lsum } xs (a + \text{ilen } x1a))) i = (a + \text{ilen } x1a)$ 
      by simp
    have 4:  $a + \text{ilen } x1a = a + (\sum k = 0..0. \text{ilen} (\text{inth} (x1a \odot xs) k))$ 
      by simp
    have 5:  $i > 0 \wedge i \leq \text{ilen } xs + 1 \longrightarrow$ 
       $\text{inth} ((a + \text{ilen } x1a) \odot (\text{lsum } xs (a + \text{ilen } x1a))) i =$ 
       $\text{inth} (\text{lsum } xs (a + \text{ilen } x1a)) (i - 1)$ 
      by (metis One-nat-def Suc-pred inth-Suc)
    have 6:  $i > 0 \wedge i \leq \text{ilen } xs + 1 \longrightarrow$ 
       $\text{inth} (\text{lsum } xs (a + \text{ilen } x1a)) (i - 1) =$ 
       $(a + \text{ilen } x1a) + (\sum k = 0..(i - 1). \text{ilen} (\text{inth} (xs) k))$ 
      using ICons.hyps le-diff-conv by blast
    have 7:  $i > 0 \wedge i \leq \text{ilen } xs + 1 \longrightarrow$ 
       $(\sum k = 0..(i - 1). \text{ilen} (\text{inth} (xs) k)) =$ 
       $(\sum k = 1..i. \text{ilen} (\text{inth} (xs) (k - 1)))$ 
      using lsum-inth-help by blast
    have 8:  $i > 0 \wedge i \leq \text{ilen } xs + 1 \longrightarrow$ 
       $(\sum k = 1..i. \text{ilen} (\text{inth} (xs) (k - 1))) =$ 
       $(\sum k = 1..i. \text{ilen} (\text{inth} (x1a \odot xs) (k)))$ 
      by (metis (no-types, lifting) atLeastAtMost-iff inth-Suc
        ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc sum.cong)
    have 9:  $i > 0 \wedge i \leq \text{ilen } xs + 1 \longrightarrow$ 
       $(a + \text{ilen } x1a) + (\sum k = 1..i. \text{ilen} (\text{inth} (x1a \odot xs) (k))) =$ 
       $a + (\sum k = 0..i. \text{ilen} (\text{inth} (x1a \odot xs) (k)))$ 
      by (simp add: sum.atLeast-Suc-atMost)
    show ?thesis
    by (metis 1 2 3 4 5 6 7 8 9 not-gr-zero)
  qed
qed

```

lemma lsum-addzero-inth:

```

assumes  $i \leq \text{ilen} (\text{addzero} (\text{lsum } xs \ 0))$ 
shows  $(\text{ilen } xs = 0 \wedge \text{ilen}(\text{ifirst } xs) = 0 \longrightarrow$ 
   $\text{inth} (\text{addzero} (\text{lsum } xs \ 0)) i = (\text{inth} (\text{lsum } xs \ 0) i) )$ 
   $\wedge$ 

```

$$\begin{aligned}
& (\text{ilen } xxs = 0 \wedge \text{ilen}(\text{ifirst } xxs) > 0 \longrightarrow \\
& \quad \text{inth } (\text{addzero } (\text{lsum } xxs \ 0)) \ i = (\text{inth } (0 \odot (\text{lsum } xxs \ 0)) \ i)) \\
& \wedge \\
& (\text{ilen } xxs > 0 \longrightarrow \\
& \quad \text{inth } (\text{addzero } (\text{lsum } xxs \ 0)) \ i = (\text{inth } (0 \odot (\text{lsum } xxs \ 0)) \ i))
\end{aligned}$$

using *assms*

by (*metis add.left-neutral addzero-def less-numeral-extra*(3) *lsum-ifirst lsum-ilen*)

lemma *lsum-ilast*:

ilast (*lsum* *xxs* *a*) = *a* + ($\sum k::\text{nat} = 0..(\text{ilen } xxs). \text{ilen}(\text{inth } xxs \ k)$)

by (*metis le-refl lsum-ilen lsum-inth*)

lemma *lsum-addzero-ilast*:

ilast (*addzero* (*lsum* *xxs* 0)) = *ilast*(*lsum* *xxs* 0)

by (*simp add: addzero-def*)

lemma *lsum-inth-leq-Suc*:

assumes *i* < *ilen* *xxs*

($\forall j \leq \text{ilen } xxs. \text{ilen}(\text{inth } xxs \ j) > 0$)

shows *inth* (*lsum* *xxs* *a*) *i* < *inth* (*lsum* *xxs* *a*) (*Suc* *i*)

proof –

have 1: *inth* (*lsum* *xxs* *a*) *i* = *a* + ($\sum k::\text{nat} = 0..i. \text{ilen}(\text{inth } xxs \ k)$)

using *assms less-imp-le-nat lsum-inth* **by** *blast*

have 2: *inth* (*lsum* *xxs* *a*) (*Suc* *i*) = *a* + ($\sum k::\text{nat} = 0..(\text{Suc } i). \text{ilen}(\text{inth } xxs \ k)$)

using *assms* **by** (*simp add: lsum-inth Suc-leI*)

have 3: ($\sum k::\text{nat} = 0..(\text{Suc } i). \text{ilen}(\text{inth } xxs \ k)$) =

($\sum k::\text{nat} = 0..i. \text{ilen}(\text{inth } xxs \ k)$) + *ilen*(*inth* *xxs* (*Suc* *i*))

using *sum.atLeast0-atMost-Suc* **by** *blast*

have 4: *ilen*(*inth* *xxs* (*Suc* *i*)) > 0

using *assms* **by** *auto*

show ?thesis **using** 1 2 3 4 **by** *linarith*

qed

lemma *lsum-addzero-inth-leq-Suc*:

assumes *i* < *ilen*(*addzero* (*lsum* *xxs* 0))

($\forall j \leq \text{ilen } xxs. \text{ilen}(\text{inth } xxs \ j) > 0$)

shows *inth* (*addzero* (*lsum* *xxs* 0)) *i* < *inth* (*addzero* (*lsum* *xxs* 0)) (*Suc* *i*)

proof (*cases* *i*)

case 0

then show ?thesis

by (*metis add.left-neutral addzero-def assms*(2) *ilen-gr-zero inth-Suc*

less-numeral-extra(3) *lsum-addzero-ifirst lsum-ifirst*)

next

case (*Suc* *nat*)

then show ?thesis

by (*metis* (*no-types, lifting*) *Suc-less-eq2 add-diff-cancel-left' add-diff-cancel-right' addzero-def*

*assms(1) assms(2) inth-Suc lsum-addzero-ilen lsum-inth-leq-Suc neq0-conv not-add-less1
plus-1-eq-Suc)*

qed

lemma *lsum-idr*:

assumes $(\forall j \leq \text{ilen } xxs. \text{ilen}(\text{inth } xxs \ j) > 0)$

shows *index-sequence* $(\text{inth } (\text{lsum } xxs \ a) \ 0) (\text{lsum } xxs \ a)$

by (*simp add: assms index-sequence-def lsum-ilen lsum-inth-leq-Suc*)

lemma *lsum-addzero-idr*:

assumes $(\forall j \leq \text{ilen } xxs. \text{ilen}(\text{inth } xxs \ j) > 0)$

shows *index-sequence* $0 (\text{addzero } (\text{lsum } xxs \ 0))$

by (*metis idx-expand1 add.left-neutral addzero-def assms le0
lsum-idr lsum-ifirst*)

lemma *filt-lfuse-lsum-a*:

assumes *lastfirst* $(xs \odot xxs)$

$(\forall j \leq \text{ilen } xxs. \text{ilen}(\text{inth } xxs \ j) > 0)$

ilen xs > 0

shows $(\text{filt } (\text{fuse } xs \ (\text{lfuse } xxs)) (\text{lsum } xxs \ (\text{ilen } xs))) =$
 $(\text{filt } (\text{lfuse } xxs) (\text{lsum } xxs \ 0))$

using *assms*

proof

(*induct xxs arbitrary: xs*)

case (*INil x*)

then show *?case*

proof –

have 1: $\text{filt } (\text{fuse } xs \ (\text{lfuse } \langle x \rangle)) (\text{lsum } \langle x \rangle (\text{ilen } xs)) =$
 $\text{filt } (\text{fuse } xs \ x) (\text{lsum } \langle x \rangle (\text{ilen } xs))$

by *simp*

have 2: $\text{filt } (\text{fuse } xs \ x) (\text{lsum } \langle x \rangle (\text{ilen } xs)) =$
 $\text{filt } (\text{fuse } xs \ x) (\langle \text{ilen } xs + \text{ilen } x \rangle)$

by *simp*

have 3: $\text{filt } (\text{fuse } xs \ x) (\langle \text{ilen } xs + \text{ilen } x \rangle) =$
 $\langle (\text{inth } (\text{fuse } xs \ x) (\text{ilen } xs + \text{ilen } x)) \rangle$

by *simp*

have 4: $\langle (\text{inth } (\text{fuse } xs \ x) (\text{ilen } xs + \text{ilen } x)) \rangle =$
 $\langle (\text{inth } x (\text{ilen } x)) \rangle$

using *INil.premis fuse-inth-a* **by** *auto*

have 5: $\text{filt } (\text{lfuse } \langle x \rangle) (\text{lsum } \langle x \rangle \ 0) =$
 $\text{filt } (x) (\langle \text{ilen } x \rangle)$

by *simp*

have 6: $\text{filt } (x) (\langle \text{ilen } x \rangle) = \langle (\text{inth } x (\text{ilen } x)) \rangle$

by *auto*

show *?thesis*

by (*simp add: 4*)

qed

next

```

case (ICons x1a xxs)
then show ?case
proof –
  have 01: filt (fuse xs (lfuse (x1a  $\odot$  xxs))) (lsum (x1a  $\odot$  xxs) (ilen xs)) =
    filt (fuse xs (fuse x1a (lfuse xxs))) (lsum (x1a  $\odot$  xxs) (ilen xs))
    by simp
  have 02: filt (fuse xs (fuse x1a (lfuse xxs)))
    (lsum (x1a  $\odot$  xxs) (ilen xs)) =
    filt (fuse xs (fuse x1a (lfuse xxs)))
    ((ilen xs + ilen x1a)  $\odot$  lsum (xxs) (ilen xs + ilen x1a))
    by simp
  have 03: filt (fuse xs (fuse x1a (lfuse xxs)))
    ((ilen xs + ilen x1a)  $\odot$  lsum (xxs) (ilen xs + ilen x1a)) =
    (inth (fuse xs (fuse x1a (lfuse xxs))) (ilen xs + ilen x1a))  $\odot$ 
    filt (fuse xs (fuse x1a (lfuse xxs))) (lsum (xxs) (ilen xs + ilen x1a))
    by simp
  have 04: (inth (fuse xs (fuse x1a (lfuse xxs))) (ilen xs + ilen x1a)) =
    (inth (fuse xs x1a) (ilen xs + ilen x1a))
    proof –
      have f1: inth xs (ilen xs) = inth x1a 0
      using ICons.prem(1) by force
      have f2: lastfirst (x1a  $\odot$  xxs)
      using ICons.prem(1) lastfirst.simp(2) by blast
      then have inth x1a (ilen x1a) = inth ( inth xxs 0) 0
      by simp
      then show ?thesis
      using f2 f1
      by (metis ICons.prem(1) add.right-neutral fuse-ilen-a
        fuse-inth-a ifirst-lfuse-ifirst
        inth-zero le0 le-add1 lfuse-ICons)
    qed
  have 05: (inth (fuse xs x1a) (ilen xs + ilen x1a)) = (inth x1a (ilen x1a))
    using ICons.prem(1) fuse-inth-a by force
  have 06: filt (fuse xs (fuse x1a (lfuse xxs))) (lsum (xxs) (ilen xs + ilen x1a)) =
    filt (fuse (fuse xs x1a) (lfuse xxs)) (lsum (xxs) (ilen(fuse xs x1a)))
    using FusionAssoc[of xs x1a (lfuse xxs)]
    by (metis ICons.prem(1) fuse-ilen-a inth-zero lastfirst.simp(2) lastfirst-lfuse)
  have 07: lastfirst ( (fuse xs x1a)  $\odot$  xxs )
    by (metis 05 ICons.prem(1) fuse-ilen-a lastfirst.simp(2))
  have 08: ilen (fuse xs x1a) > 0
    by (simp add: ICons.prem fuse-ilen-a)
  have 09: filt (fuse (fuse xs x1a) (lfuse xxs)) (lsum (xxs) (ilen(fuse xs x1a))) =
    filt (lfuse xxs) (lsum (xxs) 0)
    by (metis 07 08 ICons.prem(2) Suc-leI inth-Suc ilen.simp(2) less-Suc-eq-le
      ICons(1) plus-1-eq-Suc)
  have 10: filt (fuse xs (fuse x1a (lfuse xxs)))
    ((ilen xs + ilen x1a)  $\odot$  lsum (xxs) (ilen xs + ilen x1a)) =
    (inth x1a (ilen x1a))  $\odot$  filt (lfuse xxs) (lsum (xxs) 0)
    by (simp add: 04 05 06 09)
  have 11: filt (lfuse (x1a  $\odot$  xxs)) (lsum (x1a  $\odot$  xxs) 0) =

```



```

      filt (fuse x1a (lfuse xxs)) ((ilen x1a)  $\odot$  (lsum xxs (ilen x1a)))
    by simp
  have 12: filt (fuse x1a (lfuse xxs)) ((ilen x1a)  $\odot$  (lsum xxs (ilen x1a))) =
    (inth (fuse x1a (lfuse xxs)) (ilen x1a))  $\odot$ 
    (filt (fuse x1a (lfuse xxs)) (lsum xxs (ilen x1a)))
    by simp
  have 13: (inth (fuse x1a (lfuse xxs)) (ilen x1a)) = (inth x1a (ilen x1a))
    by (metis ICons.prem1 eq-iff fuse-ilen-a fuse-inth
      ifirst-lfuse-ifirst lastfirst.simps(2) le-add1)
  have 14: (filt (fuse x1a (lfuse xxs)) (lsum xxs (ilen x1a))) =
    filt (lfuse xxs) (lsum xxs 0)
    using ICons.prem1 ICons.prem2 inth-zero ilen.simps(2) lastfirst.simps(2)
    ICons(1) by fastforce
  have 15: filt (fuse x1a (lfuse xxs)) ((ilen x1a)  $\odot$  (lsum xxs (ilen x1a))) =
    (inth x1a (ilen x1a))  $\odot$  (filt (lfuse xxs) (lsum xxs 0))
    by (simp add: 13 14)
  show ?thesis using 10 15 by auto
qed
qed

```

lemma *filt-lfuse-lsum*:

```

assumes lastfirst (xs  $\odot$  xxs)
  ( $\forall j \leq \text{ilen } xxs. \text{ilen}(\text{inth } xxs j) > 0$ )
   $\text{ilen } xs > 0$ 
shows (filt (lfuse (xs  $\odot$  xxs)) (addzero (lsum (xs  $\odot$  xxs) 0))) =
  (ifirst xs)  $\odot$  (ilast xs)  $\odot$  (filt (lfuse xxs) (lsum xxs 0))
proof -
  have 1: lfuse (xs  $\odot$  xxs) = fuse xs (lfuse xxs)
    by simp
  have 2: (filt (lfuse (xs  $\odot$  xxs)) (addzero (lsum (xs  $\odot$  xxs) 0))) =
    (filt (fuse xs (lfuse xxs)) (addzero (lsum (xs  $\odot$  xxs) 0)))
    using 1 by simp
  have 3: addzero (lsum (xs  $\odot$  xxs) 0) =
    0  $\odot$  (ilen xs)  $\odot$  (lsum xxs (ilen xs))
    using lsum-addzero-ICons by auto
  have 4: (filt (fuse xs (lfuse xxs)) (addzero (lsum (xs  $\odot$  xxs) 0))) =
    (filt (fuse xs (lfuse xxs)) (0  $\odot$  (ilen xs)  $\odot$  (lsum xxs (ilen xs))))
    using 3 by auto
  have 5: (filt (fuse xs (lfuse xxs)) (0  $\odot$  (ilen xs)  $\odot$  (lsum xxs (ilen xs)))) =
    (inth (fuse xs (lfuse xxs)) 0)  $\odot$ 
    (inth (fuse xs (lfuse xxs)) (ilen xs))  $\odot$ 
    (filt (fuse xs (lfuse xxs)) (lsum xxs (ilen xs)))
    by simp
  have 6: (inth (fuse xs (lfuse xxs)) 0) = (inth xs 0)
    using assms by (metis fuse-inth ifirst-lfuse-ifirst ilen-gr-zero
      lastfirst.simps(2))
  have 7: (inth (fuse xs (lfuse xxs)) (ilen xs)) = (inth xs (ilen xs))
    using assms by (metis fuse-ilen-a fuse-inth ifirst-lfuse-ifirst
      lastfirst.simps(2) le-add1 order-refl)
  have 8: index-sequence 0 (addzero (lsum xxs 0))

```

```

using assms lsum-addzero-idx by blast
have 9: (filt (fuse xs (lfuse xs)) (lsum xs (ilen xs))) =
  (filt (lfuse xs) (lsum xs 0))
using assms filt-lfuse-lsum-a by blast
show ?thesis
using 3 6 7 9 inth-ilen-ilast inth-zero-ifirst by force
qed

```

```

lemma filt-lfuse-lsum-1:
assumes lastfirst (xs  $\odot$  xs)
  ( $\forall j \leq \text{ilen } xs. \text{ilen}(\text{inth } xs \ j) > 0$ )
  ilen xs > 0
shows (filt (lfuse (xs  $\odot$  xs)) ((lsum (xs  $\odot$  xs) 0))) =
  (ilast xs)  $\odot$  (filt (lfuse xs) (lsum xs 0))
using assms by (simp,
  metis assms(1) eq-iff filt-lfuse-lsum-a fuse-ilen-a
  fuse-inth ifirst-lfuse-ifirst lastfirst.simps(2) le-add1)

```

```

lemma filt-lfuse-lsum-2:
assumes lastfirst (xs)
  ( $\forall j \leq \text{ilen } xs. \text{ilen}(\text{inth } xs \ j) > 0$ )
  j  $\leq \text{ilen } xs$ 
shows (inth (filt (lfuse (xs)) ((lsum (xs) 0))) j) = ilast(inth xs j)
using assms
proof (induct xs arbitrary: j)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases j)
  case 0
then show ?thesis using ICons.hyps ICons.prems filt-expand ifirst-lfuse-ifirst[of x1a xs]
  by simp
  (metis 0 add.right-neutral fuse-inth-a le0)
next
case (Suc nat)
then show ?thesis using ICons.hyps ICons.prems by simp-all
  (metis ICons.prems(1) Suc-le-mono filt-lfuse-lsum-a le-eq-less-or-eq old.nat.simps(4)
  old.nat.simps(5) zero-less-Suc)
qed
qed

```

```

lemma filt-lfuse-lsum-3:
assumes lastfirst (xs)
  ( $\forall j \leq \text{ilen } xs. \text{ilen}(\text{inth } xs \ j) > 0$ )
  j  $\leq \text{ilen } (\text{addzero } (\text{lsum } xs \ 0))$ 
shows ( j=0  $\longrightarrow$  (inth (filt (lfuse (xs)) (addzero(lsum (xs) 0))) j) = ifirst(ifirst xs)
   $\wedge$ 
  (j>0  $\longrightarrow$  (inth (filt (lfuse (xs)) (addzero(lsum (xs) 0))) j) = ilast(inth xs (j-1)))

```

```

using assms
proof –
  have 1:  $j \leq \text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)) \wedge j=0 \longrightarrow$ 
     $(\text{inth} (\text{filt} (\text{lfuse } (\text{xxs})) (\text{addzero} (\text{lsum } (\text{xxs}) 0))) j) = \text{ifirst}(\text{ifirst } \text{xxs})$ 
    using assms by (metis filt-ilen filt-inth lastfirst-lfuse
      lsum-addzero-ifirst)
  have 2:  $j \leq \text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)) \wedge j>0 \longrightarrow$ 
     $(\text{inth} (\text{filt} (\text{lfuse } \text{xxs}) (\text{addzero} (\text{lsum } (\text{xxs}) 0))) j) =$ 
     $(\text{inth} (\text{lfuse } \text{xxs}) (\text{inth} (\text{addzero} (\text{lsum } (\text{xxs}) 0)) j))$ 

    by (simp add: filt-ilen filt-inth)
  have 3:  $j \leq \text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)) \wedge j>0 \longrightarrow$ 
     $\text{inth} (\text{addzero} (\text{lsum } (\text{xxs}) 0)) j = \text{inth} (\text{lsum } \text{xxs } 0) (j-1)$ 

    by (metis One-nat-def Suc-pred inth-Suc leD lsum-addzero-ilen lsum-addzero-inth neq0-conv)
  have 4:  $j \leq \text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)) \wedge j>0 \longrightarrow$ 
     $(\text{inth} (\text{lfuse } \text{xxs}) (\text{inth} (\text{lsum } \text{xxs } 0) (j-1))) = \text{ilast}(\text{inth } \text{xxs } (j-1))$ 
    by (metis assms diff-is-0-eq' filt-ilen filt-lfuse-lsum-2 filt-inth
      le0 le-diff-conv lsum-addzero-ilen lsum-ilen neq0-conv)
  show ?thesis
  using 1 2 3 4 by (simp add: assms(3))
qed

```

```

lemma filt-lfuse-lsum-4:
  assumes lastfirst (xxs)
     $(\forall j \leq \text{ilen } \text{xxs}. \text{ilen}(\text{inth } \text{xxs } j) > 0)$ 
  shows  $(\text{inth} (\text{addzero} (\text{lsum } \text{xxs } 0)) (\text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)))) \leq \text{ilen} (\text{lfuse } \text{xxs})$ 
  using assms
  by (metis add-cancel-right-left eq-iff lfuse-ilen
    lsum-addzero-ilast lsum-ilen lsum-inth)

```

```

lemma filt-lfuse-lsum-5:
  assumes lastfirst (xxs)
     $(\forall j \leq \text{ilen } \text{xxs}. \text{ilen}(\text{inth } \text{xxs } j) > 0)$ 
     $i \leq \text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0))$ 
  shows  $(\text{inth} (\text{addzero} (\text{lsum } \text{xxs } 0)) i) \leq \text{ilen} (\text{lfuse } \text{xxs})$ 
  using assms filt-lfuse-lsum-4[of xxs] lsum-addzero-idx[of xxs]
  proof –
  have f1:  $\text{inth} (\text{addzero} (\text{lsum } \text{xxs } 0)) (\text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0))) \leq \text{ilen} (\text{lfuse } \text{xxs})$ 
  using  $\langle \forall j \leq \text{ilen } \text{xxs}. 0 < \text{ilen} (\text{inth } \text{xxs } j) \rangle$ 
     $\langle \llbracket \text{lastfirst } \text{xxs}; \forall j \leq \text{ilen } \text{xxs}. 0 < \text{ilen} (\text{inth } \text{xxs } j) \rrbracket \implies$ 
     $\text{inth} (\text{addzero} (\text{lsum } \text{xxs } 0)) (\text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0))) \leq \text{ilen} (\text{lfuse } \text{xxs}) \rangle$ 
     $\langle \text{lastfirst } \text{xxs} \rangle$  by blast
  have  $\neg i < \text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)) \vee$ 
     $\text{inth} (\text{addzero} (\text{lsum } \text{xxs } 0)) i < \text{inth} (\text{addzero} (\text{lsum } \text{xxs } 0)) (\text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)))$ 
  using  $\langle \forall j \leq \text{ilen } \text{xxs}. 0 < \text{ilen} (\text{inth } \text{xxs } j) \implies \text{index-sequence } 0 (\text{addzero} (\text{lsum } \text{xxs } 0)) \rangle$ 
     $\langle \forall j \leq \text{ilen } \text{xxs}. 0 < \text{ilen} (\text{inth } \text{xxs } j) \rangle$  idx-less-last-1 by blast

```

then show *?thesis*
using *f1* $\langle i \leq \text{ilen} (\text{addzero} (\text{lsum } \text{xxs } 0)) \rangle$ **by** *force*
qed

lemma *lfuse-ilen-b:*

assumes *lastfirst xxs*

shows $(\forall i. 1 \leq i \wedge i \leq \text{ilen} (\text{xxs}) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (\text{xxs}) (i)).$
 $\text{inth} ((\text{lsum } \text{xxs } 0)) (i-1) + j \leq \text{ilen} (\text{lfuse } \text{xxs}))$

proof –

have *0*: $(\forall i. 1 \leq i \wedge i \leq \text{ilen} (\text{xxs}) \longrightarrow$
 $(i-1) \leq \text{ilen } \text{xxs}$

)

by *linarith*

have *1*: $(\forall i. 1 \leq i \wedge i \leq \text{ilen} (\text{xxs}) \longrightarrow$
 $\text{inth} ((\text{lsum } \text{xxs } 0)) (i-1) =$
 $(\sum k::\text{nat} = 0..(i-1). \text{ilen}(\text{inth } \text{xxs } k))$

)

by *(metis 0 add.left-neutral lsum-inth)*

have *2*: $\text{ilen} (\text{lfuse } \text{xxs}) = (\sum k::\text{nat} = 0..(\text{ilen } \text{xxs}). \text{ilen}(\text{inth } \text{xxs } k))$

using *assms lfuse-ilen* **by** *blast*

have *3*: $(\forall i. 1 \leq i \wedge i \leq \text{ilen} (\text{xxs}) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (\text{xxs}) (i)).$
 $(\sum k::\text{nat} = 0..(i-1). \text{ilen}(\text{inth } \text{xxs } k)) + j \leq$
 $(\sum k::\text{nat} = 0..(i). \text{ilen}(\text{inth } \text{xxs } k))$
 $))$

by *(metis (no-types, lifting) add-le-cancel-left le-add-diff-inverse plus-1-eq-Suc sum.atLeast0-atMost-Suc)*

have *4*: $(\forall i. 1 \leq i \wedge i \leq \text{ilen} (\text{xxs}) \longrightarrow$
 $(\sum k::\text{nat} = 0..(i). \text{ilen}(\text{inth } \text{xxs } k)) \leq$
 $(\sum k::\text{nat} = 0..(\text{ilen } \text{xxs}). \text{ilen}(\text{inth } \text{xxs } k))$
 $)$

using *sum.ub-add-nat[of - - $\lambda k. \text{ilen}(\text{inth } \text{xxs } k)$]*

by *(metis Suc-eq-plus1 le-add1 le-add-diff-inverse le-simps(1) zero-less-Suc)*

show *?thesis*

using *1 2 3 4* **by** *fastforce*

qed

lemma *lsum-shift:*

assumes *lastfirst xxs*

$(\forall j \leq \text{ilen } \text{xxs}. \text{ilen}(\text{inth } \text{xxs } j) > 0)$
 $i \leq \text{ilen } \text{xxs}$

shows $\text{inth} (\text{lsum } \text{xxs } a) i = a + \text{inth} (\text{lsum } \text{xxs } 0) i$

using *assms* **by** *(simp add: lsum-inth)*

lemma *lsum-lfuse-inth-lsum-inth:*

assumes *lastfirst xxs*

$(\forall j \leq \text{ilen } xxs. \text{ilen}(\text{inth } xxs \ j) > 0)$
shows $(\forall i \leq \text{ilen } xxs.$
 $(\forall j \leq \text{ilen}(\text{inth } xxs \ i).$
 $(\text{inth}(\text{lfuse } xxs) ((\text{inth}(\text{addzero}(\text{lsum } xxs \ 0)) \ i) + j)) =$
 $(\text{inth}(\text{inth } xxs \ i) \ j))$
using *assms*
proof
(induct xxs)
case (*INil x*)
then show ?*case*
by (*metis inth.simps(1) add-cancel-right-left ilen.simps(1)*
le-zero-eq lfuse-INil lsum-addzero-ifirst)
next
case (*ICons x1a xxs*)
then show ?*case*
proof –
have 1: $(\forall i \leq \text{ilen} (x1a \odot xxs).$
 $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xxs) \ i).$
 $\text{inth}(\text{lfuse} (x1a \odot xxs)) (\text{inth}(\text{addzero}(\text{lsum} (x1a \odot xxs) \ 0)) \ i + j) =$
 $\text{inth}(\text{inth} (x1a \odot xxs) \ i) \ j))$
 $=$
 $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xxs) \ 0).$
 $\text{inth}(\text{lfuse} (x1a \odot xxs)) (\text{inth}(\text{addzero}(\text{lsum} (x1a \odot xxs) \ 0)) \ 0 + j) =$
 $\text{inth}(\text{inth} (x1a \odot xxs) \ 0) \ j) \wedge$
 $(\forall i. 1 \leq i \wedge i \leq \text{ilen} (x1a \odot xxs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xxs) \ i).$
 $\text{inth}(\text{lfuse} (x1a \odot xxs)) (\text{inth}(\text{addzero}(\text{lsum} (x1a \odot xxs) \ 0)) \ i + j) =$
 $\text{inth}(\text{inth} (x1a \odot xxs) \ i) \ j))$

by (*metis One-nat-def Suc-leI ilen-gr-zero not-gr-zero*)
have 2: $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xxs) \ 0).$
 $\text{inth}(\text{lfuse} (x1a \odot xxs)) (\text{inth}(\text{addzero}(\text{lsum} (x1a \odot xxs) \ 0)) \ 0 + j) =$
 $\text{inth}(\text{inth} (x1a \odot xxs) \ 0) \ j)$
 $=$
 $(\forall j \leq \text{ilen} (x1a).$
 $\text{inth}(\text{lfuse} (x1a \odot xxs)) (j) = \text{inth} (x1a) \ j)$

by (*metis add-cancel-right-left inth-zero*
lsum-addzero-ifirst)
have 3: *ilast x1a = ifirst(ifirst xxs)*
using *ICons.prem lastfirst.simps(2)* **by** *blast*
have 4: $(\forall j \leq \text{ilen} (x1a).$
 $\text{inth}(\text{lfuse} (x1a \odot xxs)) (j) = \text{inth} (x1a) \ j)$
by (*metis ICons.prem(1) fuse-ilen-a fuse-inth*
ifirst-lfuse-ifirst lastfirst.simps(2) le-add1 le-trans lfuse-ICons)
have 41: $(\forall j \leq \text{ilen} (\text{inth} (x1a \odot xxs) \ 0).$
 $\text{inth}(\text{lfuse} (x1a \odot xxs)) (\text{inth}(\text{addzero}(\text{lsum} (x1a \odot xxs) \ 0)) \ 0 + j) =$
 $\text{inth}(\text{inth} (x1a \odot xxs) \ 0) \ j)$
using 2 4 **by** *blast*
have 5: $(\forall i. 1 \leq i \wedge i \leq \text{ilen} (x1a \odot xxs) \longrightarrow$

$(\forall j \leq \text{ilen} \ (\text{inth} \ (x1a \odot xs) \ i).$
 $\text{inth} \ (\text{lfuse} \ (x1a \odot xs)) \ (\text{inth} \ (\text{addzero} \ (\text{lsum} \ (x1a \odot xs) \ 0)) \ i + j) =$
 $\text{inth} \ (\text{inth} \ (x1a \odot xs) \ i \ j)) =$
 $(\forall i. \ 0 \leq i-1 \wedge i-1 \leq \text{ilen} \ (xs) \longrightarrow$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (x1a \odot xs) \ i).$
 $\text{inth} \ (\text{lfuse} \ (x1a \odot xs)) \ (\text{inth} \ (\text{addzero} \ (\text{lsum} \ (x1a \odot xs) \ 0)) \ i + j) =$
 $\text{inth} \ (\text{inth} \ (x1a \odot xs) \ i \ j))$
using 1 2 4 *le-diff-conv* **by** *auto*
have 6: $(\forall i. \ 0 \leq i-1 \wedge i-1 \leq \text{ilen} \ (xs) \longrightarrow$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (x1a \odot xs) \ i).$
 $\text{inth} \ (\text{lfuse} \ (x1a \odot xs)) \ (\text{inth} \ (\text{addzero} \ (\text{lsum} \ (x1a \odot xs) \ 0)) \ i + j) =$
 $\text{inth} \ (\text{inth} \ (x1a \odot xs) \ i \ j)) =$
 $(\forall i \leq \text{ilen} \ (xs)).$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (x1a \odot xs) \ (\text{Suc} \ i)).$
 $\text{inth} \ (\text{lfuse} \ (x1a \odot xs)) \ (\text{inth} \ (\text{addzero} \ (\text{lsum} \ (x1a \odot xs) \ 0)) \ (\text{Suc} \ i) + j) =$
 $\text{inth} \ (\text{inth} \ (x1a \odot xs) \ (\text{Suc} \ i) \ j)) \text{ (is } ?L = ?R)$
proof
show $?L \Longrightarrow ?R$
using 2 4 **by** *simp*
show $?R \Longrightarrow ?L$
using 2 4 **by** (*metis One-nat-def Suc-pred gr0I*)
qed
have 7: $(\forall i \leq \text{ilen} \ (xs)).$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (x1a \odot xs) \ (\text{Suc} \ i)).$
 $\text{inth} \ (\text{lfuse} \ (x1a \odot xs)) \ (\text{inth} \ (\text{addzero} \ (\text{lsum} \ (x1a \odot xs) \ 0)) \ (\text{Suc} \ i) + j) =$
 $\text{inth} \ (\text{inth} \ (x1a \odot xs) \ (\text{Suc} \ i) \ j)) =$
 $(\forall i \leq \text{ilen} \ (xs)).$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (xs) \ (i)).$
 $\text{inth} \ (\text{lfuse} \ (x1a \odot xs)) \ (\text{inth} \ (\text{addzero} \ (\text{lsum} \ (x1a \odot xs) \ 0)) \ (\text{Suc} \ i) + j) =$
 $\text{inth} \ (\text{inth} \ (xs) \ (i) \ j))$
by *simp*
have 8: $(\forall i \leq \text{ilen} \ (xs)).$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (xs) \ (i)).$
 $\text{inth} \ (\text{lfuse} \ (x1a \odot xs)) \ (\text{inth} \ (\text{addzero} \ (\text{lsum} \ (x1a \odot xs) \ 0)) \ (\text{Suc} \ i) + j) =$
 $\text{inth} \ (\text{inth} \ (xs) \ (i) \ j)) =$
 $(\forall i \leq \text{ilen} \ (xs)).$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (xs) \ (i)).$
 $\text{inth} \ (\text{fuse} \ x1a \ (\text{lfuse} \ xs)) \ (\text{inth} \ ((\text{lsum} \ (x1a \odot xs) \ 0)) \ (i) + j) =$
 $\text{inth} \ (\text{inth} \ (xs) \ (i) \ j))$
by (*metis inth-Suc lfuse-ICons lsum-addzero-ICons*)
have 9: $(\forall i \leq \text{ilen} \ (xs)).$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (xs) \ (i)).$
 $\text{inth} \ (\text{fuse} \ x1a \ (\text{lfuse} \ xs)) \ (\text{inth} \ ((\text{lsum} \ (x1a \odot xs) \ 0)) \ (i) + j) =$
 $\text{inth} \ (\text{inth} \ (xs) \ (i) \ j))$
 $=$
 $(\forall i \leq \text{ilen} \ (xs)).$
 $(\forall j \leq \text{ilen} \ (\text{inth} \ (xs) \ (i)).$
 $\text{inth} \ (\text{fuse} \ x1a \ (\text{lfuse} \ xs)) \ (\text{inth} \ ((\text{ilen}(x1a) \odot (\text{lsum} \ xs) \ (\text{ilen} \ x1a)) \ i + j) =$
 $\text{inth} \ (\text{inth} \ (xs) \ (i) \ j))$

by simp

have 11: $(\forall i \leq \text{ilen } (xs)).$

$$\begin{aligned}
& (\forall j \leq \text{ilen } (\text{inth } (xs) (i))). \\
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{inth } ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (i) + j) = \\
& \text{inth } (\text{inth } (xs) (i)) j) \\
& = \\
& ((\forall i. i=0 \longrightarrow \\
& (\forall j \leq \text{ilen } (\text{inth } (xs) (i))). \\
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{inth } ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (i) + j) = \\
& \text{inth } (\text{inth } (xs) (i)) j)) \\
& \wedge \\
& (\forall i. 1 \leq i \wedge i \leq \text{ilen } (xs) \longrightarrow \\
& (\forall j \leq \text{ilen } (\text{inth } (xs) (i))). \\
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{inth } ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (i) + j) = \\
& \text{inth } (\text{inth } (xs) (i)) j))
\end{aligned}$$

by (metis One-nat-def Suc-leI gr-zeroI ilen-gr-zero)

have 12: $(\forall i. i=0 \longrightarrow$

$$\begin{aligned}
& (\forall j \leq \text{ilen } (\text{inth } (xs) (i))). \\
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{inth } ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (i) + j) = \\
& \text{inth } (\text{inth } (xs) (i)) j) = \\
& (\forall j \leq \text{ilen } (\text{inth } (xs) (0))). \\
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{inth } ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (0) + j) = \\
& \text{inth } (\text{inth } (xs) (0)) j)
\end{aligned}$$

by blast

have 13: $(\forall j \leq \text{ilen } (\text{inth } (xs) (0))).$

$$\begin{aligned}
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{inth } ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (0) + j) = \\
& \text{inth } (\text{inth } (xs) (0)) j) = \\
& (\forall j \leq \text{ilen } (\text{inth } (xs) (0))). \\
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{ilen}(x1a) + j) = \\
& \text{inth } (\text{inth } (xs) (0)) j)
\end{aligned}$$

by auto

have 14: $(\forall j \leq \text{ilen } (\text{inth } (xs) (0))).$

$$\begin{aligned}
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{ilen}(x1a) + j) = \\
& \text{inth } (\text{inth } (xs) (0)) j) \\
& = \\
& (\forall j \leq \text{ilen } (\text{inth } (xs) (0))). \\
& \text{inth } (\text{ (lfuse } xs)) (j) = \\
& \text{inth } (\text{inth } (xs) (0)) j)
\end{aligned}$$

using ICons.premis fuse-inth-a ifirst-lfuse-ifirst lfuse-ilen-a

by (metis lastfirst.simps(2) le0)

have 15: $(\forall i. 1 \leq i \wedge i \leq \text{ilen } (xs) \longrightarrow$

$$\begin{aligned}
& (\forall j \leq \text{ilen } (\text{inth } (xs) (i))). \\
& \text{inth } (\text{fuse } x1a \text{ (lfuse } xs)) (\text{inth } ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (i) + j) = \\
& \text{inth } (\text{inth } (xs) (i)) j) = \\
& (\forall i. 1 \leq i \wedge i \leq \text{ilen } (xs) \longrightarrow \\
& (\forall j \leq \text{ilen } (\text{inth } (xs) (i))).
\end{aligned}$$

$$\text{inth} (\text{fuse } x1a (\text{lfuse } xs)) (\text{inth} ((\text{lsum } xs (\text{ilen } x1a))) (i-1) + j) = \\ \text{inth} (\text{inth} (xs) (i)) j)$$

by (*metis inth-Suc ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc*)

have 16: $(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $\text{inth} ((\text{lsum } xs (\text{ilen } x1a))) (i-1) =$
 $\text{ilen } x1a + \text{inth} (\text{lsum } xs 0) (i-1))$

by (*simp add: le-diff-conv lsum-inth*)

have 17: $(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} (\text{fuse } x1a (\text{lfuse } xs)) (\text{inth} ((\text{lsum } xs (\text{ilen } x1a))) (i-1) + j) =$
 $\text{inth} (\text{inth} (xs) (i)) j))$
 $=$

$(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} (\text{fuse } x1a (\text{lfuse } xs)) (\text{ilen } x1a + \text{inth} ((\text{lsum } xs 0)) (i-1) + j) =$
 $\text{inth} (\text{inth} (xs) (i)) j))$

using 16 by auto

have 18: $(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} ((\text{lsum } xs 0)) (i-1) + j \leq \text{ilen} (\text{lfuse } xs)))$

using *ICons.premis lastfirst.simps(2) lfuse-ilen-b* **by blast**

have 19: $(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} (\text{fuse } x1a (\text{lfuse } xs)) (\text{ilen } x1a + \text{inth} ((\text{lsum } xs 0)) (i-1) + j) =$
 $\text{inth} ((\text{lfuse } xs)) (\text{inth} ((\text{lsum } xs 0)) (i-1) + j)))$

by (*metis ICons.premis(1) ab-semigroup-add-class.add-ac(1) fuse-inth-a*
ifirst-lfuse-ifirst lastfirst.simps(2) lfuse-ilen-b)

have 20: $(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} (\text{fuse } x1a (\text{lfuse } xs)) (\text{ilen } x1a + \text{inth} ((\text{lsum } xs 0)) (i-1) + j) =$
 $\text{inth} (\text{inth} (xs) (i)) j)) =$
 $(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} ((\text{lfuse } xs)) (\text{inth} ((\text{lsum } xs 0)) (i-1) + j) =$
 $\text{inth} (\text{inth} (xs) (i)) j))$

using 19 by auto

have 201: $(\forall i. i=0 \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} (\text{fuse } x1a (\text{lfuse } xs)) (\text{inth} ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (i) + j) =$
 $\text{inth} (\text{inth} (xs) (i)) j))$
 \wedge
 $(\forall i. 1 \leq i \wedge i \leq \text{ilen } xs) \longrightarrow$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (i)).$
 $\text{inth} (\text{fuse } x1a (\text{lfuse } xs)) (\text{inth} ((\text{ilen}(x1a) \odot (\text{lsum } xs) (\text{ilen } x1a))) (i) + j) =$
 $\text{inth} (\text{inth} (xs) (i)) j)) =$
 $(\forall j \leq \text{ilen} (\text{inth} (xs) (0)).$


```

      inth ( (lfuse xxs)) (j) =
      inth (inth (xxs) (0)) j) ∧
    (∀ i. 1 ≤ i ∧ i ≤ ilen (xxs) →
      (∀ j ≤ ilen (inth (xxs) (i)).
        inth ( (lfuse xxs)) ( inth ((lsum xxs 0) ) (i-1) + j) =
        inth (inth (xxs) (i)) j)) )
  using 14 15 17 20 by auto
have 21: lastfirst xxs
  using ICons.premis lastfirst.simps(2) by blast
have 22: (∀ j ≤ ilen xxs. ilen(inth xxs j) > 0)
  using ICons.premis by auto
have 23: (∀ i ≤ ilen xxs.
  (∀ j ≤ ilen (inth xxs i).
    inth (lfuse xxs) (inth (addzero (lsum xxs 0)) i + j) =
    inth (inth xxs i) j))
  using 21 22 ICons.hyps by blast
have 24: ((∀ j ≤ ilen (inth xxs 0).
  inth (lfuse xxs) (inth (addzero (lsum xxs 0)) 0 + j) =
  inth ( inth xxs 0) j) ∧
  (∀ i. 1 ≤ i ∧ i ≤ ilen xxs →
    (∀ j ≤ ilen (inth xxs i).
      inth (lfuse xxs) (inth (addzero (lsum xxs 0)) i + j) =
      inth (inth xxs i) j)) )
  using 23 by blast
have 25: (∀ j ≤ ilen (inth (xxs) (0)).
  inth ( (lfuse xxs)) (j) =
  inth (inth (xxs) (0)) j)
  by (metis 24 add-cancel-right-left lsum-addzero-ifirst)
have 26: (∀ i. 1 ≤ i ∧ i ≤ ilen (xxs) →
  inth (addzero (lsum xxs 0)) i = inth ((lsum xxs 0) ) (i-1)
)
  by (metis addzero-def inth-Suc less-eq-Suc-le less-le-trans lsum-ilen
    not-less-eq-eq ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc
    zero-less-one)
have 27: (∀ i. 1 ≤ i ∧ i ≤ ilen (xxs) →
  (∀ j ≤ ilen (inth (xxs) (i)).
    inth ( (lfuse xxs)) ( inth ((lsum xxs 0) ) (i-1) + j) =
    inth (inth (xxs) (i)) j)) =
  (∀ i. 1 ≤ i ∧ i ≤ ilen xxs →
    (∀ j ≤ ilen (inth xxs i).
      inth (lfuse xxs) (inth (addzero (lsum xxs 0)) i + j) =
      inth (inth xxs i) j))
  by (simp add: 26)
show ?thesis
using 11 201 24 25 27 6 8 by auto
qed

qed

```

lemma *lcpl-lsum-less-th-equal*:

assumes *index-sequence* 0 *l*

$(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall \ i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$

$\text{ilen } \sigma > 0$

$i < \text{ilen } (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0))$

shows $(\text{inth } (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0)) \ (\text{Suc } i)) \leq$
 $\text{ilen}(\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l))$

using *assms*

by (*metis* (*no-types*, *lifting*) *Suc-leI* *filt-lfuse-lsum-5* *index-sequence-def*

lcpl-ilen-inth-gr-zero *lcpl-lfuse-lastfirst* *neg0-conv*)

lemma *lcpl-lsum-ilen*:

assumes *index-sequence* 0 *l*

$(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall \ i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$

$\text{ilen } \sigma > 0$

shows $\text{ilen } (\text{addzero } (\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0)) = \text{ilen } l$

proof –

have 1: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) = \text{ilen } l - 1$

using *assms* *index-sequence-def* *lcpl-ilen* **by** *fastforce*

have 2: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } l > 0$

using *assms* *gr-zeroI* *index-sequence-def* **by** *fastforce*

have 3: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } (\text{ifirst } (\text{lcpl } f \ g \ \sigma \ l)) > 0$

by (*metis* *assms* *ilen-gr-zero* *lcpl-ilen-inth-gr-zero*)

have 4: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) = 0 \longrightarrow$

$\text{ilen } (\text{addzero } (\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0)) = \text{ilen } l$

using 1 2 3 *lsum-addzero-ilen* **by** *fastforce*

have 5: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } (\text{lcpl } f \ g \ \sigma \ l) > 0 \longrightarrow$

$\text{ilen } (\text{addzero } (\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0)) = \text{ilen } l$

by (*simp* *add*: 1 *lsum-addzero-ilen*)

show ?thesis **using** 4 5 **using** *assms*(4) **by** *blast*

qed

lemma *lcpl-lsum-inth*:

assumes *index-sequence* 0 *l*

$(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall \ i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$

$\text{ilen } \sigma > 0$

$j \leq \text{ilen } (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0))$

shows $(j = 0 \longrightarrow$

$(\text{inth } (\text{filt } (\text{lfuse } ((\text{lcpl } f \ g \ \sigma \ l)))) (\text{addzero}(\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0))) j =$

$\text{ifirst}(\text{ifirst } (\text{lcpl } f \ g \ \sigma \ l))$

\wedge

$(j > 0 \longrightarrow (\text{inth } (\text{filt } (\text{lfuse } ((\text{lcpl } f \ g \ \sigma \ l)))) (\text{addzero}(\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0))) j =$

$ilast(inth (lcpl f g \sigma l) (j-1)))$

proof –

have 0: $ilen \sigma > 0 \longrightarrow ilen l > 0$

using *assms gr-zeroI index-sequence-def* **by** *fastforce*

have 1: $ilen \sigma > 0 \longrightarrow lastfirst (lcpl f g \sigma l)$

using 0 *assms index-sequence-def lcpl-lfuse-lastfirst* **by** *blast*

have 2: $ilen \sigma > 0 \longrightarrow$

$(\forall j \leq ilen (lcpl f g \sigma l). ilen(inth (lcpl f g \sigma l) j) > 0)$

by (*simp add: assms lcpl-ilen-inth-gr-zero*)

have 3: $ilen \sigma > 0 \longrightarrow$

$ilen (addzero (lsum (lcpl f g \sigma l) 0)) =$

$ilen (lcpl f g \sigma l) + 1$

by (*metis 2 One-nat-def Suc-eq-plus1 ilen-gr-zero*

le-imp-less-or-eq lsum-addzero-ilen)

have 4: $ilen \sigma > 0 \longrightarrow$

$(inth (filt (lfuse ((lcpl f g \sigma l))) (addzero(lsum ((lcpl f g \sigma l) 0))) 0) =$
 $ifirst(ifirst (lcpl f g \sigma l))$

by (*simp add: 1 2 filt-lfuse-lsum-3*)

have 5: $ilen \sigma > 0 \longrightarrow$

$j \leq ilen (addzero (lsum (lcpl f g \sigma l) 0)) \wedge j > 0 \longrightarrow$

$(inth (filt (lfuse ((lcpl f g \sigma l))) (addzero(lsum ((lcpl f g \sigma l) 0))) (j))$

$= ilast(inth (lcpl f g \sigma l) (j-1))$

by (*simp add: 1 2 filt-lfuse-lsum-3*)

from 4 5 **show** *?thesis* **using** *assms(4) assms(5)* **by** *blast*

qed

lemma *lcpl-lsum-inth-a:*

assumes *index-sequence 0 l*

$(inth l (ilen l)) = ilen \sigma$

$(\forall i < ilen l. (sub (inth l i) (inth l (Suc i)) \sigma) \models f \triangle g)$

$ilen \sigma > 0$

$j \leq ilen l$

shows $(inth (filt (lfuse ((lcpl f g \sigma l))) (addzero(lsum ((lcpl f g \sigma l) 0))) j) =$
 $(inth l j)$

proof –

have 1: $ilen \sigma > 0 \longrightarrow ilen l > 0$

using *assms gr-zeroI index-sequence-def* **by** *fastforce*

have 2: $ilen \sigma > 0 \longrightarrow ilen (addzero (lsum (lcpl f g \sigma l) 0)) = ilen l$

by (*simp add: assms lcpl-lsum-ilen*)

have 3: $ilen \sigma > 0 \longrightarrow$

$j \leq ilen l \longrightarrow$

$(j = 0 \longrightarrow$

$(inth (filt (lfuse ((lcpl f g \sigma l))) (addzero(lsum ((lcpl f g \sigma l) 0))) j) =$

$ifirst(ifirst (lcpl f g \sigma l))$

\wedge

$(j > 0 \longrightarrow (inth (filt (lfuse ((lcpl f g \sigma l))) (addzero(lsum ((lcpl f g \sigma l) 0))) j) =$

$ilast(inth (lcpl f g \sigma l) (j-1)))$

by (metis 2 assms lcpl-lsum-inth)
 have 4: $\text{ilen } \sigma > 0 \longrightarrow$
 $j \leq \text{ilen } l \wedge j = 0 \longrightarrow \text{ifirst}(\text{ifirst } (\text{lcpl } f \ g \ \sigma \ l)) = (\text{inth } l \ j)$
 using assms lcpl-ifirst[of $l \ \sigma \ f \ g$]
 by (metis 1 index-sequence-def)
 have 5: $\text{ilen } \sigma > 0 \longrightarrow$
 $j \leq \text{ilen } l \wedge j > 0 \longrightarrow j - 1 \leq \text{ilen } (\text{lcpl } f \ g \ \sigma \ l)$
 using assms by (metis 1 diff-le-mono index-sequence-def lcpl-ilen)
 have 6: $\text{ilen } \sigma > 0 \longrightarrow$
 $j \leq \text{ilen } l \wedge j > 0 \longrightarrow \text{ilast}(\text{inth } (\text{lcpl } f \ g \ \sigma \ l) \ (j - 1)) = (\text{inth } l \ j)$
 using lcpl-ilast-inth
 by (metis 5 One-nat-def Suc-pred assms)
 show ?thesis
 using 3 4 6 using assms(4) assms(5) by auto
 qed

lemma lcpl-filt-lfuse-lsum:
 assumes index-sequence 0 l
 $(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$
 $(\forall \ i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$
 $\text{ilen } \sigma > 0$
 shows $(\text{filt } (\text{lfuse } ((\text{lcpl } f \ g \ \sigma \ l))) \ (\text{addzero}(\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0))) = l$
 using assms
 proof –
 have 0: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } l > 0$
 using assms gr-zeroI index-sequence-def by fastforce
 have 1: $\text{ilen } \sigma > 0 \longrightarrow$
 $\text{ilen } (\text{filt } (\text{lfuse } ((\text{lcpl } f \ g \ \sigma \ l))) \ (\text{addzero}(\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0))) = \text{ilen } l$
 by (simp add: assms filt-ilen lcpl-lsum-ilen)
 have 2: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall \ j. j \leq \text{ilen } l \longrightarrow$
 $(\text{inth } (\text{filt } (\text{lfuse } ((\text{lcpl } f \ g \ \sigma \ l))) \ (\text{addzero}(\text{lsum } ((\text{lcpl } f \ g \ \sigma \ l)) \ 0))) \ j) =$
 $(\text{inth } l \ j))$
 by (simp add: assms lcpl-lsum-inth-a)
 from 1 2 show ?thesis by (simp add: assms(4) interval-eq-inth-eq)
 qed

lemma lcpl-lfuse-filt-power:
 assumes index-sequence 0 l
 $(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$
 $(\forall \ i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$
 $\text{ilen } \sigma > 0$
 shows $\text{powerinterval } g \ (\text{filt } \sigma \ (\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l))) \ (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0))$
 proof –
 have 0: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } l > 0$
 using assms gr-zeroI index-sequence-def by fastforce
 have 01: $(\forall \ i < \text{ilen } l.$
 $g \ (\text{filt } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma))))$
 using assms cpl-projection by auto

have 02: $ilen \sigma > 0 \longrightarrow (\forall i < ilen \ l. \ g \ (filt \ \sigma \ (inth \ (lcpl \ f \ g \ \sigma \ l) \ i)) \)$
using $0 \ assms \ index-sequence-def \ lcpl-lfuse-filt-power-help$ **by** $blast$
have 03: $powerinterval \ g \ (filt \ \sigma \ (lfuse \ (lcpl \ f \ g \ \sigma \ l))) \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) =$
 $(\forall i < ilen \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0))).$
 $g \ (sub \ (inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ i)$
 $(inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ (Suc \ i))$
 $(filt \ \sigma \ (lfuse \ (lcpl \ f \ g \ \sigma \ l))))$
by $(simp \ add: \ powerinterval-def)$
have 04: $ilen \sigma > 0 \longrightarrow \ ilen \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) = \ ilen \ l$
by $(simp \ add: \ assms \ lcpl-lsum-ilen)$
have 05: $ilen \sigma > 0 \longrightarrow$
 $(\forall i < ilen \ l.$
 $(filt \ \sigma \ (inth \ (lcpl \ f \ g \ \sigma \ l) \ i)) =$
 $(sub \ (inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ i)$
 $(inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ (Suc \ i))$
 $(filt \ \sigma \ (lfuse \ (lcpl \ f \ g \ \sigma \ l)))) \)$

proof –
have 06: $ilen \sigma > 0 \longrightarrow$
 $(\forall i < ilen \ l. \ ilen \ (filt \ \sigma \ (inth \ (lcpl \ f \ g \ \sigma \ l) \ i)) =$
 $ilen \ (inth \ (lcpl \ f \ g \ \sigma \ l) \ i))$
using $filt-ilen$ **by** $blast$
have 07: $ilen \sigma > 0 \longrightarrow$
 $(\forall i < ilen \ l. \ ilen \ (inth \ (lcpl \ f \ g \ \sigma \ l) \ i) =$
 $ilen \ (imap \ (shift \ (inth \ l \ i)) \ (cpl \ f \ g \ (sub \ (inth \ l \ i) \ (inth \ l \ (Suc \ i)) \ \sigma))))$

using $assms$ **by** $(metis \ index-sequence-def \ lcpl-inth)$
have 08: $ilen \sigma > 0 \longrightarrow$
 $(\forall i < ilen \ l.$
 $ilen \ (imap \ (shift \ (inth \ l \ i)) \ (cpl \ f \ g \ (sub \ (inth \ l \ i) \ (inth \ l \ (Suc \ i)) \ \sigma))) =$
 $ilen \ (cpl \ f \ g \ (sub \ (inth \ l \ i) \ (inth \ l \ (Suc \ i)) \ \sigma)))$

using $ilen-imap$ **by** $blast$
have 09: $ilen \sigma > 0 \longrightarrow$
 $(\forall i < ilen \ l.$
 $ilen \ (sub \ (inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ i)$
 $(inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ (Suc \ i))$
 $(filt \ \sigma \ (lfuse \ (lcpl \ f \ g \ \sigma \ l)))) =$
 $(inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ (Suc \ i)) -$
 $(inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ i) \)$

by $(metis \ 04 \ assms \ filt-ilen \ ilen-sub \ lcpl-lsum-less-th-equal$
 $lcpl-ilen-inth-gr-zero \ le-eq-less-or-eq \ lsum-addzero-inth-leq-Suc)$
have 10: $ilen \sigma > 0 \longrightarrow$
 $(\forall i < ilen \ l. \ (inth \ (addzero \ (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ (Suc \ i)) =$
 $(inth \ (0 \odot (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ (Suc \ i)))$

using $04 \ addzero-def$ **by** $auto$
have 11: $ilen \sigma > 0 \longrightarrow$
 $(\forall i < ilen \ l. \ (inth \ (0 \odot (lsum \ (lcpl \ f \ g \ \sigma \ l) \ 0)) \ (Suc \ i)) =$

```

      (∑ k::nat= 0..(i). ilen(inth (lcpl f g σ l) k)) )
    using 04
  proof simp-all
    assume 0 < ilen σ → ilen (addzero (lsum (lcpl f g σ l) 0)) = ilen l
    show 0 < ilen σ →
      (∀ i<ilen l. inth (lsum (lcpl f g σ l) 0) i =
        (∑ k = 0..i. ilen (inth (lcpl f g σ l) k)))
    proof
      assume 0 < ilen σ
      show ∀ i<ilen l. inth (lsum (lcpl f g σ l) 0) i =
        (∑ k = 0..i. ilen (inth (lcpl f g σ l) k))
    proof
      fix i
      show i < ilen l →
        inth (lsum (lcpl f g σ l) 0) i = (∑ k = 0..i. ilen (inth (lcpl f g σ l) k))
      by (metis (no-types, lifting) 0 One-nat-def Suc-leI Suc-le-mono Suc-pred
        add.left-neutral assms(1) assms(4) index-sequence-def lcpl-ilen lsum-inth sum.cong)
    qed
  qed
  qed
  have 12: ilen σ > 0 →
    (∀ i<ilen l. (inth (addzero (lsum (lcpl f g σ l) 0)) i) =
      (inth (0⊙(lsum (lcpl f g σ l) 0)) i) )

    using 04 addzero-def by auto
  have 13: ilen σ > 0 →
    (∀ i<ilen l. (inth (0⊙(lsum (lcpl f g σ l) 0)) i) =
      (case i of 0 ⇒ 0
        | Suc j ⇒ (∑ k::nat= 0..(j). ilen(inth (lcpl f g σ l) k)) ))
    using 11 by (simp-all add: Nitpick.case-nat-unfold)
  have 14: ilen σ > 0 →
    (∀ i<ilen l.
      (inth (addzero (lsum (lcpl f g σ l) 0)) (Suc i)) -
      (inth (addzero (lsum (lcpl f g σ l) 0)) i) =
      (∑ k::nat= 0..(i). ilen(inth (lcpl f g σ l) k)) -
      (case i of 0 ⇒ 0
        | Suc j ⇒ (∑ k::nat= 0..(j). ilen(inth (lcpl f g σ l) k)) ))

    using 10 11 12 13 by auto
  have 15: ilen σ > 0 →
    (∀ i<ilen l.
      (∑ k::nat= 0..(i). ilen(inth (lcpl f g σ l) k)) -
      (case i of 0 ⇒ 0
        | Suc j ⇒ (∑ k::nat= 0..(j). ilen(inth (lcpl f g σ l) k)) )) =
      (case i of 0 ⇒ ilen(inth (lcpl f g σ l) 0)
        | Suc j ⇒ (∑ k::nat= 0..(Suc j). ilen(inth (lcpl f g σ l) k)) -
          (∑ k::nat= 0..(j). ilen(inth (lcpl f g σ l) k)) ))

    by (simp add: Nitpick.case-nat-unfold)
  have 16: ilen σ > 0 →

```

$(\forall i < \text{ilen } l.$
 $(\text{case } i \text{ of } 0 \Rightarrow \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ 0)$
 $\mid \text{Suc } j \Rightarrow (\sum k::\text{nat} = 0..(\text{Suc } j). \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ k)) -$
 $(\sum k::\text{nat} = 0..j). \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ k))) =$
 $(\text{case } i \text{ of } 0 \Rightarrow \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ 0)$
 $\mid \text{Suc } j \Rightarrow \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) (\text{Suc } j))))$

by (*metis (no-types, lifting) Nitpick.case-nat-unfold add-diff-cancel-left'*
sum.atLeast0-atMost-Suc)

have 17: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\text{case } i \text{ of } 0 \Rightarrow \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ 0)$
 $\mid \text{Suc } j \Rightarrow \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) (\text{Suc } j))) =$
 $\text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ i))$

by (*simp add: Nitpick.case-nat-unfold*)

have 18: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l. \text{ilen} (\text{filt } \sigma (\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ i)) =$
 $\text{ilen} (\text{sub} (\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) \ i)$
 $(\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) (\text{Suc } i))$
 $(\text{filt } \sigma (\text{lfuse} (\text{lcpl } f \ g \ \sigma \ l)))))$

by (*simp add: 09 14 15 16 17 filt-ilen*)

have 19: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\forall j \leq \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ i).$
 $(\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) (\text{Suc } i)) \leq$
 $\text{ilen} (\text{filt } \sigma (\text{lfuse} (\text{lcpl } f \ g \ \sigma \ l)))))$

by (*simp add: 04 assms filt-ilen lcpl-lsum-less-th-equal*)

have 22: $\text{ilen } \sigma > 0 \longrightarrow \text{lastfirst} (\text{lcpl } f \ g \ \sigma \ l)$

using 0 *assms index-sequence-def lcpl-lfuse-lastfirst* **by** *blast*

have 23: $\text{ilen } \sigma > 0 \longrightarrow (\forall j \leq \text{ilen} (\text{lcpl } f \ g \ \sigma \ l). \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ j) > 0)$

by (*simp add: assms lcpl-ilen-inth-gr-zero*)

have 190: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\forall j \leq \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ i).$
 $(\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) \ i) \leq$
 $(\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) (\text{Suc } i))$
 $))$

by (*simp add: 04 23 less-imp-le-nat lsum-addzero-inth-leq-Suc*)

have 20: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\forall i < \text{ilen } l.$
 $(\forall j \leq \text{ilen}(\text{inth} (\text{lcpl } f \ g \ \sigma \ l) \ i).$
 $(\text{inth} (\text{sub} (\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) \ i)$
 $(\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) (\text{Suc } i))$
 $(\text{filt } \sigma (\text{lfuse} (\text{lcpl } f \ g \ \sigma \ l)))) \ j) =$
 $(\text{inth} (\text{filt } \sigma (\text{lfuse} (\text{lcpl } f \ g \ \sigma \ l))))$
 $((\text{inth} (\text{addzero} (\text{lsum} (\text{lcpl } f \ g \ \sigma \ l) \ 0)) \ i) + j)))$

```

by (simp add: 14 15 16 17 19 190)
have 21:  $ilen\ \sigma > 0 \longrightarrow$ 
  ( $\forall\ i < ilen\ l.$ 
    ( $\forall\ j \leq ilen(inth\ (lcpl\ f\ g\ \sigma\ l)\ i).$ 
      ( $inth\ (filt\ \sigma\ (lfuse\ (lcpl\ f\ g\ \sigma\ l)))$ 
        ( $(inth\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0))\ i) + j$ ) ) =
        ( $inth\ \sigma\ (inth\ (lfuse\ (lcpl\ f\ g\ \sigma\ l))$ 
          ( $((inth\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0))\ i) + j)$  ) ) ) )

by (simp add: filt-imap inth-imap)
have 24:  $ilen\ \sigma > 0 \longrightarrow$ 
  ( $\forall\ i \leq ilen\ (lcpl\ f\ g\ \sigma\ l).$ 
    ( $\forall\ j \leq ilen(inth\ (lcpl\ f\ g\ \sigma\ l)\ i).$ 
      (  $(inth\ (lfuse\ (lcpl\ f\ g\ \sigma\ l))\ ((inth\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0))\ i) + j)$  ) ) =
      (  $(inth\ (inth\ (lcpl\ f\ g\ \sigma\ l)\ i)\ j)$  ) ) )

by (simp add: 22 23 lsum-lfuse-inth-lsum-inth)
have 241:  $ilen\ \sigma > 0 \longrightarrow ilen\ (lcpl\ f\ g\ \sigma\ l) = ilen\ l - 1$ 
using 0 assms index-sequence-def lcpl-ilen by blast
have 25:  $ilen\ \sigma > 0 \longrightarrow$ 
  ( $\forall\ i < ilen\ l.$ 
    ( $\forall\ j \leq ilen(inth\ (lcpl\ f\ g\ \sigma\ l)\ i).$ 
      ( $inth\ \sigma\ (inth\ (lfuse\ (lcpl\ f\ g\ \sigma\ l))$ 
        ( $((inth\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0))\ i) + j)$  ) ) =
        ( $inth\ (filt\ \sigma\ (inth\ (lcpl\ f\ g\ \sigma\ l)\ i))\ j)$  ) ) )

by (simp add: 06 24 241 assms filt-inth)
have 26:  $ilen\ \sigma > 0 \longrightarrow$ 
  ( $\forall\ i < ilen\ l.$ 
    ( $\forall\ j \leq ilen(inth\ (lcpl\ f\ g\ \sigma\ l)\ i).$ 
      ( $inth\ (filt\ \sigma\ (inth\ (lcpl\ f\ g\ \sigma\ l)\ i))\ j$ ) =
      ( $inth\ (sub\ (inth\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0))\ i)$ 
        ( $inth\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0))\ (Suc\ i)$ )
        ( $filt\ \sigma\ (lfuse\ (lcpl\ f\ g\ \sigma\ l)))\ j$ ) ) ) )

by (simp add: 20 21 25)
from 18 26 show ?thesis
by (simp add: filt-ilen interval-eq-inth-eq)
qed
show ?thesis
using 02 03 04 05 by (simp add: assms(4))
qed

lemma lcpl-lfuse-filt-ilen:
assumes index-sequence 0 l
  ( $inth\ l\ (ilen\ l) = ilen\ \sigma$ 
    ( $\forall\ i < ilen\ l. (sub\ (inth\ l\ i)\ (inth\ l\ (Suc\ i))\ \sigma) \models f\ \Delta\ g$ )
     $ilen\ \sigma > 0$ 
shows ( $inth\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0))\ (ilen\ (addzero\ (lsum\ (lcpl\ f\ g\ \sigma\ l)\ 0)))$ ) =

```



```

      ilen (filt  $\sigma$  (lfuse (lcpl f g  $\sigma$  l)))
proof -
  have 0: ilen  $\sigma > 0 \longrightarrow$  ilen l > 0
    using assms gr-zeroI index-sequence-def by fastforce
  have 1: ilen  $\sigma > 0 \longrightarrow$ 
    ilen (filt  $\sigma$  (lfuse (lcpl f g  $\sigma$  l))) = ilen ( (lfuse (lcpl f g  $\sigma$  l)))
    using filt-ilen by blast
  have 2: ilen  $\sigma > 0 \longrightarrow$ 
    ilen ( (lfuse (lcpl f g  $\sigma$  l))) =
    ( $\sum k::nat= 0..(ilen (lcpl f g \sigma l)).$  ilen(inth (lcpl f g  $\sigma$  l) k))
    using 0 assms index-sequence-def lfuse-ilen lcpl-lfuse-lastfirst by blast
  have 3: ilen  $\sigma > 0 \longrightarrow$ 
    ilast( addzero (lsum (lcpl f g  $\sigma$  l) 0)) = ilast( (lsum (lcpl f g  $\sigma$  l) 0))
    using lsum-addzero-ilast by blast
  have 4: ilen  $\sigma > 0 \longrightarrow$ 
    ilast( (lsum (lcpl f g  $\sigma$  l) 0)) =
    ( $\sum k::nat= 0..(ilen (lcpl f g \sigma l)).$  ilen(inth (lcpl f g  $\sigma$  l) k))
    by (metis add-cancel-right-left lsum-ilast)
  show ?thesis using 1 2 3 4 by (simp add: assms(4))
qed

```

8.3 Soundness of Projection Axioms

8.3.1 PJ1

lemma *PJ1sem*:

($\sigma \models f \triangle (g \vee h) \longrightarrow f \triangle g \vee f \triangle h$)
 by (simp add: projection-d-def) blast

lemma *PJ1sema*:

($\sigma \models f \triangle (g \vee h) = (f \triangle g \vee f \triangle h)$)
 by (simp add: projection-d-def) blast

8.3.2 PJ2

lemma *PJ2sem*:

($\sigma \models f \triangle \text{empty} = \text{empty}$)

proof *auto*

show ($\sigma \models f \triangle \text{empty} \implies \sigma \models \text{empty}$)

unfolding *projection-d-def empty-defs index-sequence-def*

by (*metis filt.simps(2) ilen-ICons-1 neq0-conv*)

show $\sigma \models \text{empty} \implies (\sigma \models f \triangle \text{empty})$

unfolding *projection-d-def empty-defs index-sequence-def powerinterval-def*

by (*metis inth.simps(1) filt.simps(1) index-sequence-def ilen.simps(1)*
not-less-zero)

qed

8.3.3 PJ3

lemma *PJ3help*:

sub 0 (ilen σ) $\sigma = \sigma$

by (simp add: sub-zero-prefix)

lemma *PJ3help1*:

assumes $f \sigma \wedge 0 < \text{ilen } \sigma$

shows $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. f \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge$
 $(\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (\text{inth } l \ i)) \wedge \text{ilen } s1 = \text{Suc } 0))$

proof –

have 1: $\text{index-sequence } 0 \ \langle 0, \text{ilen } \sigma \rangle$

by (simp add: assms index-sequence-def)

have 2: $\text{inth } \langle 0, \text{ilen } \sigma \rangle \ (\text{ilen } \langle 0, \text{ilen } \sigma \rangle) = \text{ilen } \sigma$

by auto

have 3: $(\forall i < \text{ilen } \langle 0, \text{ilen } \sigma \rangle. f \ (\text{sub } (\text{inth } \langle 0, \text{ilen } \sigma \rangle \ i) \ (\text{inth } \langle 0, \text{ilen } \sigma \rangle \ (\text{Suc } i)) \ \sigma))$

by (simp add: PJ3help assms)

have 4: $\text{ilen } \langle \text{inth } \sigma \ (0), \text{inth } \sigma \ (\text{ilen } \sigma) \rangle = \text{ilen } \langle 0, \text{ilen } \sigma \rangle$

by simp

have 5: $(\forall i \leq \text{ilen } \langle \text{inth } \sigma \ (0), \text{inth } \sigma \ (\text{ilen } \sigma) \rangle.$

$\text{inth } \langle \text{inth } \sigma \ (0), \text{inth } \sigma \ (\text{ilen } \sigma) \rangle \ i = \text{inth } \sigma \ (\text{inth } \langle 0, \text{ilen } \sigma \rangle \ i))$

using antisym-conv2 **by** fastforce

have 6: $\text{ilen } \langle \text{inth } \sigma \ (0), \text{inth } \sigma \ (\text{ilen } \sigma) \rangle = \text{Suc } 0$

by simp

show ?thesis

using 1 2 3 4 5 6 **by** blast

qed

lemma *PJ3sem*:

$(\sigma \models f \ \Delta \ \text{skip} = (f \wedge \text{more}))$

proof –

have 1: $(\sigma \models f \ \Delta \ \text{skip}) \implies (\sigma \models f \wedge \text{more})$

by (metis (mono-tags, lifting) One-nat-def PJ3help cpl-projection filt-expand index-sequence-def
more-defs powerinterval-def skip-defs unl-lift2 zero-less-one)

have 2: $(\sigma \models f \wedge \text{more}) \implies (\sigma \models f \ \Delta \ \text{skip})$

by (simp add: projection-d-def skip-defs more-defs powerinterval-def,
metis PJ3help1 filt-ilen)

show ?thesis

using 1 2 unl-lift2 **by** blast

qed

8.3.4 PJ4

lemma *PJ4semchaina*:

assumes $(\sigma \models f \ \Delta \ (g;h))$

shows $(\sigma \models (f \ \Delta \ g) ; (f \ \Delta \ h))$

proof –

have 1: $(\sigma \models f \ \Delta \ (g;h))$

using assms **by** auto

have 2: $(\exists l. \text{index-sequence } 0 \ l \wedge$

```

    inth l (ilen l) = ilen σ ∧
    powerinterval f σ l ∧
    (∃ n ≤ ilen (filt σ l). g (prefix n (filt σ l)) ∧ h (suffix n (filt σ l))))
  by (metis assms chop-defs projection-d-def)
  obtain l where 3: index-sequence 0 l ∧
    inth l (ilen l) = ilen σ ∧
    powerinterval f σ l ∧
    (∃ n ≤ ilen (filt σ l). g (prefix n (filt σ l)) ∧ h (suffix n (filt σ l)))
  using 2 by auto
  have 4: index-sequence 0 l
  using 3 by auto
  have 5: powerinterval f σ l
  using 3 by auto
  have 6: inth l (ilen l) = ilen σ
  using 3 by auto
  have 7: (∃ n ≤ ilen (filt σ l). g (prefix n (filt σ l)) ∧ h (suffix n (filt σ l)))
  using 3 by auto
  obtain n where 8: n ≤ ilen (filt σ l) ∧ g (prefix n (filt σ l)) ∧ h (suffix n (filt σ l))
  using 7 by auto
  have 9: n ≤ ilen (filt σ l)
  using 8 by auto
  have 10: g (prefix n (filt σ l))
  using 8 by auto
  have 11: h (suffix n (filt σ l))
  using 8 by auto
  have 12: index-sequence 0 (prefix n l)
  by (metis 4 8 filt-ilen idx-split)
  have 13: index-sequence (inth l n) (suffix n l)
  by (metis 4 9 filt-ilen idx-split)
  have 14: index-sequence 0 ((imap (shiftn (inth l n)) (suffix n l)))
  using 13 idx-shiftn by blast
  have 15: g (filt σ (prefix n l))
  by (metis 8 filt-ilen filt-prefix)
  have 16: h (filt σ (suffix n l))
  by (metis 11 9 filt-ilen filt-suffix)
  have 17: g (filt (prefix (inth l n) σ) (prefix n l))
  by (metis (no-types, lifting) 12 15 4 6 8 filt-ilen filt-prefix-idx
    idx-less-equal ilast-prefix order-refl)
  have 18: h (filt (suffix (inth l n) σ) ((imap (shiftn (inth l n)) (suffix n l))))
  proof -
    have 181: ilen((filt σ (suffix n l))) =
      ilen(filt (suffix (inth l n) σ) ((imap (shiftn (inth l n)) (suffix n l))))
    by (simp add: filt-ilen)
    have 182: (∀ j ≤ ilen((filt σ (suffix n l))).
      (inth (filt σ (suffix n l)) j) =
      (inth σ ((inth l (n+j)))))
    by (metis 9 filt-ilen filt-imap inth-imap inth-suffix
      suffix-ilen-good)
    have 183: (∀ j ≤ ilen((filt σ (suffix n l))).

```

```

    (inth (filt (suffix (inth l n) σ) ((imap (shiftn (inth l n)) (suffix n l)))) j) =
    (inth (suffix (inth l n) σ) (inth (imap (shiftn (inth l n)) (suffix n l)) j))
  )
  by (simp add: filt-imap inth-imap)
have 184: (inth l n) ≤ ilen σ
  by (metis 4 6 9 filt-ilen idx-less-equal order-refl)
have 185: (∀ j ≤ ilen (filt σ (suffix n l))).
  (inth (imap (shiftn (inth l n)) (suffix n l)) j) =
  (inth (suffix n l) j) - (inth l n)
  )
  by (metis 4 6 9 filt-ilen inth-suffix idx-shiftn-suffix-inth
    suffix-ilen-good)
have 186: (∀ j ≤ ilen (filt σ (suffix n l))).
  (inth (suffix n l) j) - (inth l n) = (inth l (j+n)) - (inth l n)
  )
  by (metis 9 add.commute filt-ilen inth-suffix suffix-ilen-good)
have 187: (∀ j ≤ ilen (filt σ (suffix n l))).
  (inth l (j+n)) - (inth l n) ≤ ilen σ - (inth l n)
  )
  by (metis 4 6 9 add.commute diff-le-mono eq-imp-le filt-ilen
    idx-less-equal suffix-ilen-good nat-add-left-cancel-le
    ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 188: (∀ j ≤ ilen (filt σ (suffix n l))).
  (inth (suffix (inth l n) σ) (inth (imap (shiftn (inth l n)) (suffix n l)) j)) =
  (inth σ ((inth (imap (shiftn (inth l n)) (suffix n l)) j) + (inth l n)))
  )
  using inth-suffix
  by (simp add: 184 185 186 187 add.commute)
have 189: (∀ j ≤ ilen (filt σ (suffix n l))).
  (inth σ ((inth (imap (shiftn (inth l n)) (suffix n l)) j) + (inth l n))) =
  (inth σ (inth (suffix n l) j))
  )
  by (metis 13 185 filt-ilen idx-greater
    ordered-cancel-comm-monoid-diff-class.le-imp-diff-is-add)
have 190: (∀ j ≤ ilen (filt σ (suffix n l))).
  (inth (filt σ (suffix n l)) j) =
  (inth (filt (suffix (inth l n) σ) ((imap (shiftn (inth l n)) (suffix n l)))) j)
  )
  using 183 188 189 filt-expand by fastforce
show ?thesis
  by (metis 16 181 190 interval-eq-inth-eq)
qed
have 19: powerinterval f (prefix (inth l n) σ) (prefix n l)
  by (metis 4 5 6 8 filt-ilen powerinterval-splita)
have 20: powerinterval f (suffix (inth l n) σ) ((imap (shiftn (inth l n)) (suffix n l)))
  by (metis 4 5 6 9 filt-ilen powerinterval-split)
have 21: (inth l n) ≤ ilen σ
  by (metis 3 9 filt-ilen idx-less-equal order-refl)
have 22: inth (prefix n l) (ilen (prefix n l)) = (inth l n)
  by (metis 8 filt-ilen ilast-prefix)

```

have 23: $\text{inth } ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))$
 $(\text{ilen } ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l)))) = \text{ilen } \sigma - (\text{inth } l \ n)$
by (metis 4 6 9 eq-imp-le filt-ilen ilen-imap idx-shiftn-suffix-inth
suffix-ilen-good ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 24: $l = \text{fuse } (\text{prefix } n \ l)$
 $(\text{imap } (\text{shift } (\text{inth } l \ n)) \ ((\text{imap } (\text{shiftn } (\text{inth } l \ n)) \ (\text{suffix } n \ l))))$
using fuse-prefix-suffix
by (metis 13 14 8 filt-ilen lsk-ls)
have 25: $(\exists l1. \text{index-sequence } 0 \ l1 \wedge$
 $\text{inth } l1 \ (\text{ilen } l1) = \text{ilen } (\text{prefix } (\text{inth } l \ n) \ \sigma) \wedge$
 $\text{powerinterval } f \ (\text{prefix } (\text{inth } l \ n) \ \sigma) \ l1 \wedge g \ (\text{filt } (\text{prefix } (\text{inth } l \ n) \ \sigma) \ l1))$
by (metis 12 17 19 21 22 prefix-ilen-good)
have 26: $(\exists l2. \text{index-sequence } 0 \ l2 \wedge$
 $\text{inth } l2 \ (\text{ilen } l2) = \text{ilen } (\text{suffix } (\text{inth } l \ n) \ \sigma) \wedge$
 $\text{powerinterval } f \ (\text{suffix } (\text{inth } l \ n) \ \sigma) \ l2 \wedge h \ (\text{filt } (\text{suffix } (\text{inth } l \ n) \ \sigma) \ l2))$
using 18 14 20 23 21 **by** auto
have 27: $(\text{prefix } (\text{inth } l \ n) \ \sigma) \models f \triangle g$
by (metis 25 projection-d-def)
have 28: $(\text{suffix } (\text{inth } l \ n) \ \sigma) \models f \triangle h$
by (metis 26 projection-d-def)
show ?thesis
using 21 27 28 chop-defs **by** auto
qed

lemma PJ4semchainb:

assumes $(\sigma \models (f \triangle g) ; (f \triangle h))$

shows $(\sigma \models f \triangle (g;h))$

proof –

have 1: $(\exists n \leq \text{ilen } \sigma.$

$(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } (\text{prefix } n \ \sigma) \wedge$
 $\text{powerinterval } f \ (\text{prefix } n \ \sigma) \ l \wedge g \ (\text{filt } (\text{prefix } n \ \sigma) \ l)) \wedge$
 $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } (\text{suffix } n \ \sigma) \wedge$
 $\text{powerinterval } f \ (\text{suffix } n \ \sigma) \ l \wedge h \ (\text{filt } (\text{suffix } n \ \sigma) \ l)))$

using assms **by** (metis chop-defs cpl-projection)

obtain cp **where** 2: $cp \leq \text{ilen } \sigma \wedge$

$(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } (\text{prefix } cp \ \sigma) \wedge$
 $\text{powerinterval } f \ (\text{prefix } cp \ \sigma) \ l \wedge g \ (\text{filt } (\text{prefix } cp \ \sigma) \ l)) \wedge$
 $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } (\text{suffix } cp \ \sigma) \wedge$
 $\text{powerinterval } f \ (\text{suffix } cp \ \sigma) \ l \wedge h \ (\text{filt } (\text{suffix } cp \ \sigma) \ l))$

using 1 **by** auto

have 3: $cp \leq \text{ilen } \sigma$

using 2 **by** auto

have 4: $(\exists l. \text{index-sequence } 0 \ l \wedge$

$\text{inth } l \ (\text{ilen } l) = \text{ilen } (\text{prefix } cp \ \sigma) \wedge$
 $\text{powerinterval } f \ (\text{prefix } cp \ \sigma) \ l \wedge g \ (\text{filt } (\text{prefix } cp \ \sigma) \ l))$

using 2 **by** auto

```

obtain l1 where 5: index-sequence 0 l1  $\wedge$ 
    inth l1 (ilen l1) = ilen (prefix cp  $\sigma$ )  $\wedge$ 
    powerinterval f (prefix cp  $\sigma$ ) l1  $\wedge$  g (filt (prefix cp  $\sigma$ ) l1)
    using 4 by auto
have 6: index-sequence 0 l1
    using 5 by auto
have 7: inth l1 (ilen l1) = ilen (prefix cp  $\sigma$ )
    using 5 by auto
have 8: powerinterval f (prefix cp  $\sigma$ ) l1
    using 5 by auto
have 9: g (filt (prefix cp  $\sigma$ ) l1)
    using 5 by auto
have 10: ( $\exists$  l. index-sequence 0 l  $\wedge$ 
    inth l (ilen l) = ilen (suffix cp  $\sigma$ )  $\wedge$ 
    powerinterval f (suffix cp  $\sigma$ ) l  $\wedge$  h (filt (suffix cp  $\sigma$ ) l))
    using 2 by auto
obtain l2 where 11: index-sequence 0 l2  $\wedge$ 
    inth l2 (ilen l2) = ilen (suffix cp  $\sigma$ )  $\wedge$ 
    powerinterval f (suffix cp  $\sigma$ ) l2  $\wedge$  h (filt (suffix cp  $\sigma$ ) l2)
    using 10 by auto
have 12: index-sequence 0 l2
    using 11 by auto
have 13: inth l2 (ilen l2) = ilen (suffix cp  $\sigma$ )
    using 11 by auto
have 14: powerinterval f (suffix cp  $\sigma$ ) l2
    using 11 by auto
have 15: h (filt (suffix cp  $\sigma$ ) l2)
    using 11 by auto
have 16: index-sequence 0 (fuse l1 (imap (shift cp) l2))
    by (metis 11 12 2 5 6 eq-imp-le idx-fuse-idx prefix-ilen-good
    suffix-ilen-good)
have 17: ilast l1 = ifirst (imap (shift cp) l2)
    by (metis 12 13 3 6 7 idx-fuse-ifirst-ilast prefix-ilen-good
    suffix-ilen-good)
have 18: inth (fuse l1 (imap (shift cp) l2)) (ilen l1) = cp
    by (metis 17 3 7 eq-imp-le fuse-ilen-a fuse-inth
    prefix-ilen-good le-add1)
have 19: ilast (fuse l1 (imap (shift cp) l2)) = ilen  $\sigma$ 
    proof –
    have 191: ilast (fuse l1 (imap (shift cp) l2)) = ilast (imap (shift cp) l2)
    by (metis 17 eq-imp-le fuse-inth-a fuse-ilen-a)
    have 192: ilast (imap (shift cp) l2) = ilast l2 + cp
    by (metis shift-def ilen-imap inth-imap)
    have 193: ilast l2 + cp = ilen  $\sigma$ 
    by (metis 13 3 Nat.le-imp-diff-is-add suffix-ilen-good)
    show ?thesis using 191 192 193 by auto
    qed
have 20: powerinterval f  $\sigma$  (fuse l1 (imap (shift cp) l2))
    using powerinterval-fuse[of l1 (l2) cp  $\sigma$  f]
    using 12 13 14 3 5 by auto

```

```

have 21:  $\sigma = \text{fuse } (\text{prefix } cp \ \sigma) \ (\text{suffix } cp \ \sigma)$ 
  by (simp add: 3 fuse-prefix-suffix)
have 22:  $\text{inth } ((\text{fuse } l1 \ (\text{imap } (\text{shift } cp) \ l2))) \ (\text{ilen } l1) = cp$ 
  using 18 by blast
have 23:  $(\text{prefix } (\text{ilen } l1) \ (\text{filt } \sigma \ (\text{fuse } l1 \ (\text{imap } (\text{shift } cp) \ l2))) =$ 
   $(\text{filt } (\text{prefix } cp \ \sigma) \ l1)$ 
  by (metis 17 2 5 filt-prefix filt-prefix-idx fuse-ilen-a
    prefix-fuse prefix-ilen-good le-add1)
have 24:  $g \ (\text{prefix } (\text{ilen } l1) \ (\text{filt } \sigma \ (\text{fuse } l1 \ (\text{imap } (\text{shift } cp) \ l2))))$ 
  by (simp add: 23 9)
have 25:  $(\text{suffix } (\text{ilen } l1) \ (\text{filt } \sigma \ (\text{fuse } l1 \ (\text{imap } (\text{shift } cp) \ l2)))) =$ 
   $(\text{filt } (\text{suffix } cp \ \sigma) \ l2)$ 
  by (metis 12 13 17 2 23 filt-ilen filt-suffix filt-suffix-idx
    prefix-ilen-bound suffix-fuse suffix-ilen-good)
have 26:  $\text{ilen } l1 \leq \text{ilen } (\text{filt } \sigma \ (\text{fuse } l1 \ (\text{imap } (\text{shift } cp) \ l2)))$ 
  by (metis 23 filt-ilen prefix-ilen-bound)
have 27:  $(\exists l. \text{index-sequence } 0 \ l \wedge$ 
   $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$ 
   $\text{powerinterval } f \ \sigma \ l \wedge$ 
   $(\exists n \leq \text{ilen } (\text{filt } \sigma \ l). \ g \ (\text{prefix } n \ (\text{filt } \sigma \ l)) \wedge h \ (\text{suffix } n \ (\text{filt } \sigma \ l))))$ 
  by (metis 11 16 19 20 24 25 26)
show ?thesis
by (metis 27 chop-fuse fuse-prefix-suffix ilast-ifirst projection-d-def)
qed

```

lemma *PJ4sem*:

$(\sigma \models f \triangle (g;h) = (f \triangle g) ; (f \triangle h))$

using *PJ4semchaina PJ4semchainb unl-lift2* **by** *blast*

8.3.5 PJ5

lemma *PJ5sem*:

$(\sigma \models f \triangle \text{init}(g) \longrightarrow \text{init}(g))$

by (*simp add: projection-d-def init-defs*)

(*metis filt-inth filt-ilen index-sequence-def ilen-gr-zero*)

8.3.6 PJ6

lemma *PJ6help1*:

assumes *index-sequence* 0 *l*

$(\text{inth } l \ (\text{ilen } l)) = (\text{ilen } \sigma)$

shows $(\forall i. 0 \leq i \wedge i < \text{ilen } l \longrightarrow \text{ilen } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)$
 $= (\text{inth } l \ (\text{Suc } i)) - (\text{inth } l \ i))$

proof

fix *i*

show $0 \leq i \wedge i < \text{ilen } l \longrightarrow$

$\text{ilen } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) = \text{inth } l \ (\text{Suc } i) - \text{inth } l \ i$

using *assms*

by (*simp add: index-sequence-def sub-def*)

(*metis Suc-lessI assms(1) idx-less-last-1 le-diff-iff le-eq-less-or-eq min.orderE*)

qed

lemma *PJ6help2*:
assumes *index-sequence 0 l*
 $\text{inth } l \text{ (ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. \text{inth } l \text{ (Suc } i) - \text{inth } l \text{ } i = \text{Suc } 0)$
shows $(\forall i \leq \text{ilen } l. \text{inth } l \text{ } i = i)$
proof
fix *i*
show $i \leq \text{ilen } l \longrightarrow \text{inth } l \text{ } i = i$
proof
(induct i)
case 0
then show ?case **using** *assms index-sequence-def by blast*
next
case (Suc *i*)
then show ?case
by (*metis One-nat-def Suc-eq-plus1 Suc-leD Suc-le-lessD assms idx-expand*
le-add-diff-inverse2 plus-1-eq-Suc)
qed
qed

lemma *PJ6help3*:
assumes *index-sequence 0 l*
 $\text{inth } l \text{ (ilen } l) = \text{ilen } \sigma$
 $(\forall i \leq \text{ilen } l. \text{inth } l \text{ } i = i)$
shows $(\forall i < \text{ilen } l. \text{inth } l \text{ (Suc } i) - \text{inth } l \text{ } i = \text{Suc } 0)$
proof
fix *i*
show $i < \text{ilen } l \longrightarrow \text{inth } l \text{ (Suc } i) - \text{inth } l \text{ } i = \text{Suc } 0$
by (*simp add: assms*)
qed

lemma *PJ6help4*:
 $(\exists l. \text{index-sequence } 0 \text{ } l \wedge l = [0.. \leq \text{ilen } \sigma] \wedge$
 $\text{inth } l \text{ (ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i \leq \text{ilen } l. \text{inth } l \text{ } i = i) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \text{ } i = \text{inth } \sigma \text{ (} i)) \wedge s1 = \sigma))$
by (*simp add: index-sequence-def upt-ilen upt-inth*)

lemma *PJ6help5*:
 $(\exists l. \text{index-sequence } 0 \text{ } l \wedge$
 $\text{inth } l \text{ (ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. \text{ilen } (\text{sub } (\text{inth } l \text{ } i) (\text{inth } l \text{ (Suc } i)) \sigma) = \text{Suc } 0) \wedge$

$(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (\text{inth } l \ i)) \wedge g \ s1))$
 $= g \ \sigma$
proof –
have $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. \text{ilen } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) = \text{Suc } 0) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (\text{inth } l \ i)) \wedge g \ s1))$
 $=$
 $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. \text{inth } l \ (\text{Suc } i) - \text{inth } l \ i = \text{Suc } 0) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (\text{inth } l \ i)) \wedge g \ s1))$
using *PJ6help1* **by** (*metis zero-order(1)*)
also have ... =
 $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i \leq \text{ilen } l. \text{inth } l \ i = i) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (\text{inth } l \ i)) \wedge g \ s1))$

using *PJ6help2 PJ6help3* **by** *blast*
also have ... =
 $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i \leq \text{ilen } l. \text{inth } l \ i = i) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (i)) \wedge g \ s1))$

by *metis*
also have ... =
 $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i \leq \text{ilen } l. \text{inth } l \ i = i) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (i)) \wedge s1 = \sigma \wedge g \ \sigma))$

by (*metis interval-eq-inth-eq le-eq-less-or-eq*)
also have ... =
 $g \ \sigma$

using *PJ6help4* **by** *blast*
finally show $(\exists l. \text{index-sequence } 0 \ l \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. \text{ilen } (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) = \text{Suc } 0) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1 \ i = \text{inth } \sigma \ (\text{inth } l \ i)) \wedge g \ s1))$
 $= g \ \sigma$
 \cdot
qed

lemma *PJ6sem*:
 $(\sigma \models \text{skip} \triangle g = g)$
proof –
have 1: $(\sigma \models \text{skip} \triangle g = g) =$

$((\exists l. \text{index-sequence } 0\ l \wedge \text{inth } l\ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l\ i) (\text{inth } l\ (\text{Suc } i))\ \sigma) \models \text{skip}) \wedge g\ (\text{filt } \sigma\ l)) =$
 $g\ \sigma)$
by (*simp add: projection-d-def powerinterval-def*)
have 2: $(\exists l. \text{index-sequence } 0\ l \wedge \text{inth } l\ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l\ i) (\text{inth } l\ (\text{Suc } i))\ \sigma) \models \text{skip}) \wedge g\ (\text{filt } \sigma\ l) =$
 $(\exists l\ s1. \text{index-sequence } 0\ l \wedge \text{inth } l\ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l\ i) (\text{inth } l\ (\text{Suc } i))\ \sigma) \models \text{skip}) \wedge$
 $\text{ilen } s1 = \text{ilen } l \wedge$
 $(\forall i \leq \text{ilen } s1. (\text{inth } s1\ i) = (\text{inth } \sigma\ (\text{inth } l\ i))) \wedge$
 $g\ s1)$
using *filt-expand* **by** *metis*
have 3: $(\exists l\ s1. \text{index-sequence } 0\ l \wedge \text{inth } l\ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l\ i) (\text{inth } l\ (\text{Suc } i))\ \sigma) \models \text{skip}) \wedge$
 $\text{ilen } s1 = \text{ilen } l \wedge$
 $(\forall i \leq \text{ilen } s1. (\text{inth } s1\ i) = (\text{inth } \sigma\ (\text{inth } l\ i))) \wedge$
 $g\ s1) =$
 $(\exists l. \text{index-sequence } 0\ l \wedge \text{inth } l\ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. \text{ilen } (\text{sub } (\text{inth } l\ i) (\text{inth } l\ (\text{Suc } i))\ \sigma) = \text{Suc } 0) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1\ i = \text{inth } \sigma\ (\text{inth } l\ i)) \wedge g\ s1))$
by (*simp add: skip-defs*)
have 4: $(\exists l. \text{index-sequence } 0\ l \wedge \text{inth } l\ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall i < \text{ilen } l. \text{ilen } (\text{sub } (\text{inth } l\ i) (\text{inth } l\ (\text{Suc } i))\ \sigma) = \text{Suc } 0) \wedge$
 $(\exists s1. \text{ilen } s1 = \text{ilen } l \wedge (\forall i \leq \text{ilen } s1. \text{inth } s1\ i = \text{inth } \sigma\ (\text{inth } l\ i)) \wedge g\ s1)) =$
 $(g\ \sigma)$
by (*simp add: PJ6help5*)
from 1 2 3 4 **show** *?thesis*
by *simp*
qed

8.3.7 PJ7

lemma *PJemptyImp:*

assumes $\text{ilen } \sigma = 0$

shows $(\sigma \models (f \triangle g) = g)$

using *assms*

by (*simp add: projection-d-def index-sequence-def powerinterval-def,*

auto,

metis filt.simps(1) INil-ilen lessI not-less-zero old.nat.exhaust,

metis inth.simps(1) filt.simps(1) suffix-ilast suffix-zero

ilen.simps(1) not-less-zero)

lemma *PJ7empty:*

assumes $\text{ilen } \sigma = 0$

shows $(\sigma \models f \triangle (g \triangle h) = (f \triangle g) \triangle h)$

proof –

have 1: $(\sigma \models f \triangle (g \triangle h) = (g \triangle h))$

using *PJemptyImp assms* **by** *blast*

have 2: $(\sigma \models (g \triangle h) = h)$

using *PJemptyImp assms* **by** *blast*

have 3: $(\sigma \models (f \triangle g) \triangle h = h)$
using *PJemptyImp* *assms* **by** *blast*
from 1 2 3 **show** *?thesis* **by** *simp*
qed

lemma *PJ7helpchain1a-help-1:*

assumes *index-sequence* 0 *l*

$(\text{inth } l (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma) \models f \triangle g)$

$\text{ilen } \sigma > 0$

shows *index-sequence* 0 (*lfuse* (*lcpl* *f g* σ *l*)))

proof –

have 0: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } l > 0$

using *assms* *gr-zeroI* *index-sequence-def* **by** *fastforce*

have 01: $\text{ifirst } l = 0$

using *assms* *index-sequence-def* **by** *auto*

have 02: *lastfirst* (*lcpl* *f g* σ *l*)

using *assms* *lcpl-lfuse-lastfirst* **by** (*simp* *add: projection-d-def*)

(*metis* 0 01 *assms*(3) *lcpl-lfuse-lastfirst*)

have 1: $\text{ilen } \sigma > 0 \longrightarrow \text{ifirst } (\text{lfuse } (\text{lcpl } f g \sigma l)) = 0$

using *assms* 0 01 02

lcpl-ifirst[*of* *l* σ *f g*]

by (*metis* (*mono-tags*, *lifting*) *lastfirst-lfuse*)

from 1 0 **show** *?thesis* **using** *assms* *lcpl-lfuse-id_x*[*of* *l* σ *f g*] **by** (*simp* *add: projection-d-def*)

qed

lemma *PJ7helpchain1a-help-2:*

assumes *index-sequence* 0 *l*

$(\text{inth } l (\text{ilen } l)) = \text{ilen } \sigma$

$(\forall i < \text{ilen } l. (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma) \models f \triangle g)$

$\text{ilen } \sigma > 0$

shows *powerinterval* *f* σ (*lfuse* (*lcpl* *f g* σ *l*)))

proof –

have 1: $(\forall i < \text{ilen } l.$

$\text{powerinterval } f (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)$

$(\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)))$

using *assms* *cpl-projection* **by** *blast*

have 2: $\forall i < \text{ilen } l.$

$\forall ia < \text{ilen } (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)).$

$f (\text{sub } (\text{inth } (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)) ia)$

$(\text{inth } (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)) (\text{Suc } ia))$

$(\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma))$

using 1 **by** (*simp* *add: powerinterval-def*)

have 3: $\forall i < \text{ilen } l.$

$\forall ia < \text{ilen } (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)).$

$(\text{sub } (\text{inth } (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)) ia)$

$(\text{inth } (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)) (\text{Suc } ia))$

$(\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)) =$

$(\text{sub } (\text{inth } (\text{imap } (\text{shift } (\text{inth } l i)) (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)))) ia)$

$(\text{inth } (\text{imap } (\text{shift } (\text{inth } l i)) (\text{cpl } f g (\text{sub } (\text{inth } l i) (\text{inth } l (\text{Suc } i)) \sigma)))) (\text{Suc } ia))$

$\sigma)$
proof –
have 30: $\forall i < \text{ilen } l.$
 $\forall ia < \text{ilen } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)).$
 $\forall j \leq \text{ilen } (\text{sub } (\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ ia)$
 $(\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ (\text{Suc } ia))$
 $(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)).$
 $(\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ (\text{Suc } ia)) \leq$
 $(\text{inth } l \ (\text{Suc } i)) - (\text{inth } l \ (i))$
using *assms* **by** (*metis add.commute cpl-projection idx-expand ilen-sub*
plus-1-eq-Suc)
have 31: $\forall i < \text{ilen } l.$
 $\forall ia < \text{ilen } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)).$
 $\text{ilen } (\text{sub } (\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ ia)$
 $(\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ (\text{Suc } ia))$
 $(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) =$
 $\text{ilen } (\text{sub } (\text{inth } (\text{imap } (\text{shift } (\text{inth } l \ i))$
 $(\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma))) \ ia)$
 $(\text{inth } (\text{imap } (\text{shift } (\text{inth } l \ i))$
 $(\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma))) \ (\text{Suc } ia))$
 $\sigma)$
using *assms* **unfolding** *shift-def inth-imap*
by (*metis (no-types, lifting) PJ6help1 add.commute cpl-projection idx-expand*
sub-sub-1 le-add1 le-add-same-cancel1 plus-1-eq-Suc)
have 32: $\forall i < \text{ilen } l.$
 $\forall ia < \text{ilen } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)).$
 $\forall j \leq \text{ilen } (\text{sub } (\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ ia)$
 $(\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ (\text{Suc } ia))$
 $(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)).$
 $\text{inth } (\text{sub } (\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ ia)$
 $(\text{inth } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ (\text{Suc } ia))$
 $(\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)) \ j =$
 $\text{inth } (\text{sub } (\text{inth } (\text{imap } (\text{shift } (\text{inth } l \ i))$
 $(\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma))) \ ia)$
 $(\text{inth } (\text{imap } (\text{shift } (\text{inth } l \ i))$
 $(\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma))) \ (\text{Suc } ia))$
 $\sigma) \ j$
using *assms* 30 *sub-sub-1[of - - - σ]*
by (*auto simp add: shift-def inth-imap cpl-projection idx-expand*)
 $(\text{metis add.commute idx-expand plus-1-eq-Suc})$
show ?thesis **using** 31 32 *interval-eq-inth-eq* **by** (*simp add: interval-eq-inth-eq*)
qed
have 4: $\forall i < \text{ilen } l.$
 $\forall ia < \text{ilen } (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma)).$
 $f \ (\text{sub } (\text{inth } (\text{imap } (\text{shift } (\text{inth } l \ i)) \ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma))) \ ia)$
 $(\text{inth } (\text{imap } (\text{shift } (\text{inth } l \ i)) \ (\text{cpl } f \ g \ (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma))) \ (\text{Suc } ia))$
 $\sigma)$
using 2 3 **by** *auto*
have 5: $\text{ilen}(\text{lcpl } f \ g \ \sigma \ l) = \text{ilen } l - 1$
using *assms index-sequence-def lcpl-ilen lcpl-ilen-zero* **by** *fastforce*

have 6: $ilen \sigma > 0 \longrightarrow ilen l > 0$
using *assms gr-zeroI index-sequence-def* **by** *fastforce*
have 7: $\forall i < ilen l.$
 $\quad \forall ia < ilen ((inth (lcpl f g \sigma l) i)).$
 $\quad f (sub (inth (inth (lcpl f g \sigma l) i) ia)$
 $\quad \quad (inth (inth (lcpl f g \sigma l) i) (Suc ia))$
 $\quad \quad \sigma)$
using *assms* **by** (*metis (no-types, lifting) 4 index-sequence-def ilen-imap lcpl-inth*)
have 8: $ilen \sigma > 0 \longrightarrow$
 $\quad (\forall i \leq ilen(lcpl f g \sigma l).$
 $\quad \quad \forall ia < ilen ((inth (lcpl f g \sigma l) i)).$
 $\quad \quad f (sub (inth (inth (lcpl f g \sigma l) i) ia)$
 $\quad \quad \quad (inth (inth (lcpl f g \sigma l) i) (Suc ia))$
 $\quad \quad \quad \sigma))$
by (*metis 5 6 7 One-nat-def Suc-pred le-imp-less-Suc*)
have 9: $ilen \sigma > 0 \longrightarrow$
 $\quad powerinterval f \sigma (lfuse (lcpl f g \sigma l)) =$
 $\quad (\forall i < ilen (lfuse (lcpl f g \sigma l)).$
 $\quad \quad f (sub (inth (lfuse (lcpl f g \sigma l) i) (inth (lfuse (lcpl f g \sigma l) (Suc i)) \sigma))$
by (*simp add: powerinterval-def*)
have 10: $ilen \sigma > 0 \longrightarrow lastfirst (lcpl f g \sigma l)$
using *6 assms index-sequence-def lcpl-lfuse-lastfirst* **by** *blast*
have 11: $ilen \sigma > 0 \longrightarrow$
 $\quad (\forall j \leq ilen (lcpl f g \sigma l). ilen(inth (lcpl f g \sigma l) j) > 0)$

by (*simp add: assms lcpl-ilen-inth-gr-zero*)
have 12: $ilen \sigma > 0 \longrightarrow$
 $\quad (\forall i \leq ilen(lcpl f g \sigma l).$
 $\quad \quad (\forall ia < ilen ((inth (lcpl f g \sigma l) i)).$
 $\quad \quad \quad f (sub (inth (inth (lcpl f g \sigma l) i) ia)$
 $\quad \quad \quad \quad (inth (inth (lcpl f g \sigma l) i) (Suc ia))$
 $\quad \quad \quad \quad \sigma))) =$
 $\quad \quad (\forall j < ilen (lfuse (lcpl f g \sigma l)).$
 $\quad \quad \quad f (sub (inth (lfuse (lcpl f g \sigma l) j)$
 $\quad \quad \quad \quad (inth (lfuse (lcpl f g \sigma l) (Suc j))$
 $\quad \quad \quad \quad \sigma)))$

using *lfuse-split* [*of (lcpl f g \sigma l) f \sigma*] *10 11* **by** *auto*
show *?thesis* **using** *12 8 9* **using** *assms(4)* **by** *blast*
qed

lemma *PJ7helpchain1a-help-3:*

assumes *index-sequence 0 l*

$(inth l (ilen l)) = ilen \sigma$

$(\forall i < ilen l. (sub (inth l i) (inth l (Suc i)) \sigma) \models f \triangle g)$

$h (filt \sigma l)$

$ilen \sigma > 0$

shows $ilast (lfuse (lcpl f g \sigma l)) = ilen \sigma$

proof –

have 0: $ilen \sigma > 0 \longrightarrow ilen l > 0$

```

using assms gr-zeroI index-sequence-def by fastforce
have 1: ilen σ > 0  $\longrightarrow$ 
  
$$\text{ilast} (\text{lfuse} (\text{lcpl } f \ g \ \sigma \ l)) = \text{ilast}(\text{ilast} (\ (\text{lcpl } f \ g \ \sigma \ l)))$$

  using assms
  using 0 index-sequence-def lastfirst-lfuse-ilast lcpl-lfuse-lastfirst by blast
have 2: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & \text{ilast} (\ (\text{lcpl } f \ g \ \sigma \ l)) \\ &= (\text{inth} (\text{lcpl } f \ g \ \sigma \ l) (\text{ilen } l - 1)) \end{aligned}$$

  using assms by (simp add: 0 index-sequence-def lcpl-ilen)
have 3: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & (\text{inth} (\text{lcpl } f \ g \ \sigma \ l) (\text{ilen } l - 1)) = \\ & (\text{imap} (\text{shift} (\text{inth } l (\text{ilen } l - 1))) \\ & \quad (\text{cpl } f \ g (\text{sub} (\text{inth } l (\text{ilen } l - 1)) (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \ \sigma))) \end{aligned}$$


  using assms by (simp add: 0 index-sequence-def lcpl-inth)
have 4: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & \text{ilast}(\ (\ (\text{cpl } f \ g (\text{sub} (\text{inth } l (\text{ilen } l - 1)) (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \ \sigma))) = \\ & \text{ilen} ((\text{sub} (\text{inth } l (\text{ilen } l - 1)) (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \ \sigma)) \\ & \quad \text{using } 0 \text{ assms} \\ & \quad \text{by } (\text{metis } \text{cpl-projection diff-less zero-less-one}) \end{aligned}$$

have 5: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & \text{ilen} ((\text{sub} (\text{inth } l (\text{ilen } l - 1)) (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \ \sigma)) = \\ & (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) - (\text{inth } l (\text{ilen } l - 1)) \end{aligned}$$


  using 0 PJ6help1 assms diff-less zero-less-one by blast
have 6: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & \text{ilast} (\ (\text{imap} (\text{shift} (\text{inth } l (\text{ilen } l - 1))) \\ & \quad (\text{cpl } f \ g (\text{sub} (\text{inth } l (\text{ilen } l - 1)) (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \ \sigma)))) = \\ & (\text{shift} (\text{inth } l (\text{ilen } l - 1))) \\ & (\text{ilast} (\text{cpl } f \ g (\text{sub} (\text{inth } l (\text{ilen } l - 1)) (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \ \sigma))) \end{aligned}$$


  by (metis ilen-imap inth-imap)
have 7: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & (\text{shift} (\text{inth } l (\text{ilen } l - 1))) \\ & (\text{ilast} (\text{cpl } f \ g (\text{sub} (\text{inth } l (\text{ilen } l - 1)) (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \ \sigma))) = \\ & (\text{shift} (\text{inth } l (\text{ilen } l - 1))) ((\text{inth } l (\text{Suc } (\text{ilen } l - 1))) - (\text{inth } l (\text{ilen } l - 1))) \end{aligned}$$


  using 4 5 by auto
have 8: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & (\text{shift} (\text{inth } l (\text{ilen } l - 1))) ((\text{inth } l (\text{Suc } (\text{ilen } l - 1))) - (\text{inth } l (\text{ilen } l - 1))) = \\ & ((\text{inth } l (\text{Suc } (\text{ilen } l - 1))) - (\text{inth } l (\text{ilen } l - 1))) + (\text{inth } l (\text{ilen } l - 1)) \end{aligned}$$


  by (simp add: shift-def)
have 9: ilen σ > 0  $\longrightarrow$ 
  
$$\begin{aligned} & ((\text{inth } l (\text{Suc } (\text{ilen } l - 1))) - (\text{inth } l (\text{ilen } l - 1))) + (\text{inth } l (\text{ilen } l - 1)) \\ &= (\text{inth } l (\text{Suc } (\text{ilen } l - 1))) \end{aligned}$$


  using assms
  by (metis 0 add commute diff-add diff-less idx-expand plus-1-eq-Suc zero-less-one)
have 10: ilen σ > 0  $\longrightarrow$   $(\text{inth } l (\text{Suc } (\text{ilen } l - 1))) = \text{ilen } \sigma$ 

```

by (simp add: 0 assms)
 show ?thesis
 using 1 10 2 3 6 7 8 9 assms(5) by presburger
 qed

lemma *PJ7helpchain1a-help-4:*

assumes *index-sequence 0 l*
 $(\text{inth } l \ (\text{ilen } l)) = \text{ilen } \sigma$
 $(\forall \ i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g)$
 $h \ (\text{filt } \sigma \ l)$
 $\text{ilen } \sigma > 0$
shows $((\text{filt } \sigma \ (\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l))) \models g \ \Delta \ h)$
proof –
have 0: $\text{ilen } \sigma > 0 \longrightarrow \text{ilen } l > 0$
using *assms gr-zeroI index-sequence-def* **by** *fastforce*
have 1: $\text{ilen } \sigma > 0 \longrightarrow \text{index-sequence } 0 \ (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0))$
by (simp add: *assms lcpl-ilen-inth-gr-zero lsum-addzero-idx*)
have 2: $\text{ilen } \sigma > 0 \longrightarrow$
 $(\text{inth } (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0)) \ (\text{ilen}(\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0)))) =$
 $\text{ilen } (\text{filt } \sigma \ (\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l)))$
by (simp add: *assms lcpl-lfuse-filt-ilen*)
have 3: $\text{ilen } \sigma > 0 \longrightarrow$
 $\text{powerinterval } g \ (\text{filt } \sigma \ (\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l))) \ (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0))$

by (simp add: *assms lcpl-lfuse-filt-power*)
have 4: $\text{ilen } \sigma > 0 \longrightarrow$
 $h \ (\text{filt } (\text{filt } \sigma \ (\text{lfuse } (\text{lcpl } f \ g \ \sigma \ l))) \ (\text{addzero } (\text{lsum } (\text{lcpl } f \ g \ \sigma \ l) \ 0)))$
by (simp add: *assms filt-imap-filt lcpl-filt-lfuse-lsum*)
show ?thesis **by** (metis 1 2 3 4 assms(5) *projection-d-def*)
 qed

lemma *PJ7helpchain1a:*

assumes $\text{ilen } \sigma > 0$
 $(\sigma \models (f \ \Delta \ g) \ \Delta \ h)$
shows $(\sigma \models f \ \Delta \ (g \ \Delta \ h))$
proof –
have 1: $\text{ilen } \sigma > 0$
using *assms* **by** *auto*
have 2: $(\exists \ l. \text{index-sequence } 0 \ l \wedge \text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $\text{powerinterval } (\text{LIFT}(f \ \Delta \ g)) \ \sigma \ l \wedge$
 $h \ (\text{filt } \sigma \ l))$
using *assms* **using** *cpl-projection* **by** *blast*
obtain *l* **where** 3: $\text{index-sequence } 0 \ l \wedge \text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $\text{powerinterval } (\text{LIFT}(f \ \Delta \ g)) \ \sigma \ l \wedge$
 $h \ (\text{filt } \sigma \ l)$
using 2 **by** *blast*
have 4: $\text{index-sequence } 0 \ l \wedge \text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $(\forall \ i < \text{ilen } l. (\text{sub } (\text{inth } l \ i) \ (\text{inth } l \ (\text{Suc } i)) \ \sigma) \models f \ \Delta \ g) \wedge$
 $h \ (\text{filt } \sigma \ l)$

```

using 3 by (simp add: powerinterval-def)
have 6:  $\text{ilen } l > 0$ 
  using 1 4 gr-zeroI index-sequence-def by fastforce
have 7: index-sequence 0 (lfuse (lcpl f g  $\sigma$  l))
by (metis (no-types, lifting) 1 4 PJ7helpchain1a-help-1)
have 8: powerinterval f  $\sigma$  (lfuse (lcpl f g  $\sigma$  l))
  using 4 6 PJ7helpchain1a-help-2 assms index-sequence-def by auto
have 9: (inth (lfuse (lcpl f g  $\sigma$  l)) (ilen (lfuse (lcpl f g  $\sigma$  l)))) =  $\text{ilen } \sigma$ 
  by (metis 1 4 PJ7helpchain1a-help-3)
have 10: ((filt  $\sigma$  (lfuse (lcpl f g  $\sigma$  l)))  $\models g \triangle h$ )
  by (simp add: 1 4 6 PJ7helpchain1a-help-4)
show ?thesis
using 10 7 8 9 by (metis projection-d-def)
qed

```

lemma PJ7helpchain1b:

```

assumes  $\text{ilen } \sigma > 0$ 
  ( $\sigma \models f \triangle (g \triangle h)$ )
shows ( $\sigma \models (f \triangle g) \triangle h$ )
proof -
  have 1:  $\text{ilen } \sigma > 0$ 
  using assms by auto
  have 2: ( $\exists l.$  index-sequence 0 l  $\wedge$  inth l ( $\text{ilen } l$ ) =  $\text{ilen } \sigma \wedge$ 
    powerinterval f  $\sigma$  l  $\wedge$ 
    ((filt  $\sigma$  l)  $\models g \triangle h$ ))
  using assms by (simp add: projection-d-def)
  obtain l where 3: index-sequence 0 l  $\wedge$  inth l ( $\text{ilen } l$ ) =  $\text{ilen } \sigma \wedge$ 
    powerinterval f  $\sigma$  l  $\wedge$ 
    ((filt  $\sigma$  l)  $\models g \triangle h$ )
  using 2 by blast
  have 4: ( $\exists la.$  index-sequence 0 la  $\wedge$  inth la ( $\text{ilen } la$ ) =  $\text{ilen}(\text{filt } \sigma l) \wedge$ 
    powerinterval g (filt  $\sigma$  l) la  $\wedge$ 
    ((filt (filt  $\sigma$  l) la)  $\models h$ ))
  using 3 using cpl-projection by blast
  obtain la where 5: index-sequence 0 la  $\wedge$  inth la ( $\text{ilen } la$ ) =  $\text{ilen}(\text{filt } \sigma l) \wedge$ 
    powerinterval g (filt  $\sigma$  l) la  $\wedge$ 
    ((filt (filt  $\sigma$  l) la)  $\models h$ )
  using 4 by blast
  have 6:  $\text{ilen } l > 0$ 
  using 1 3 gr0I index-sequence-def by force
  have 7:  $\text{ilen}(\text{filt } \sigma l) = \text{ilen } l$ 
  by (simp add: filt-ilen)
  have 8: (filt (filt  $\sigma$  l) la) = (filt  $\sigma$  (filt l la))
  using filt-imap-filt by blast
  have 9: (inth (filt l la) (ilen (filt l la))) =  $\text{ilen } \sigma$ 
  by (metis 3 5 filt-expand order-refl)
  have 10:  $\text{ilen } la > 0$ 
  using 5 6 7 gr0I index-sequence-def by force
  have 11:  $\text{ilen } (\text{filt } l la) > 0$ 

```



```

by (simp add: 10 filt-ilen)
have 12: index-sequence 0 (filt l la)
proof -
  have 111: inth (filt l la) 0 = 0
  by (metis 3 5 filt-inth index-sequence-def ilen-gr-zero)
  have 112: ilen(filt l la) = ilen la
  using filt-expand by blast
  have 113: ( $\forall i < \text{ilen la. } (\text{inth } (\text{filt l la}) i) = (\text{inth l } (\text{inth la } i)))$ )
  by (simp add: filt-imap inth-imap)
  have 114: ( $\forall i < \text{ilen la. } (\text{inth } (\text{filt l la}) (\text{Suc } i)) = (\text{inth l } (\text{inth la } (\text{Suc } i))))$ )
  by (simp add: filt-imap inth-imap)
  have 115: ( $\forall i < \text{ilen la. } (\text{inth l } (\text{inth la } i)) < (\text{inth l } (\text{inth la } (\text{Suc } i)))$ )
  by (metis 3 5 7 Suc-lessI idx-less-than idx-less-last-1 lessI
    less-imp-le-nat)
  show ?thesis by (simp add: 111 113 114 115 filt-ilen index-sequence-def)
qed
have 20: powerinterval (LIFT( $f \triangle g$ ))  $\sigma$  (filt l la)
proof -
  have 201: powerinterval (LIFT( $f \triangle g$ ))  $\sigma$  (filt l la) =
    ( $\forall i < \text{ilen la. } (\text{sub } (\text{inth l } (\text{inth la } i)) (\text{inth l } (\text{inth la } (\text{Suc } i))) \sigma) \models f \triangle g$ )
  by (simp add: filt-imap inth-imap powerinterval-def)
  have 202: ( $\forall i < \text{ilen la. } (\text{sub } (\text{inth l } (\text{inth la } i)) (\text{inth l } (\text{inth la } (\text{Suc } i))) \sigma) \models f \triangle g =$ 
    ( $\forall i < \text{ilen la.}$ 
      ( $\exists ll. \text{index-sequence } 0 ll$ 
         $\wedge \text{ilast } ll = \text{ilen } (\text{sub } (\text{inth l } (\text{inth la } i)) (\text{inth l } (\text{inth la } (\text{Suc } i))) \sigma) \wedge$ 
         $\text{powerinterval } f (\text{sub } (\text{inth l } (\text{inth la } i)) (\text{inth l } (\text{inth la } (\text{Suc } i))) \sigma) ll \wedge$ 
        ( $(\text{filt } (\text{sub } (\text{inth l } (\text{inth la } i)) (\text{inth l } (\text{inth la } (\text{Suc } i))) \sigma) ll) \models g$ )
      ))
  by (simp add: projection-d-def)
  have 203: ( $\forall i < \text{ilen la. } \text{ilen } (\text{sub } (\text{inth l } (\text{inth la } i)) (\text{inth l } (\text{inth la } (\text{Suc } i))) \sigma) =$ 
    ( $\text{inth l } (\text{inth la } (\text{Suc } i))) - (\text{inth l } (\text{inth la } i))$ )
  by (metis 3 5 7 Suc-leI Suc-lessI idx-less-equal idx-less-last-1
    ilen-sub lessI less-imp-le-nat)
  have 2041: ( $\forall i < \text{ilen la.}$ 
    ( $\text{inth la } (\text{Suc } i)) \leq \text{ilen l}$ 
  )
  using 5 7 Suc-lessI idx-less-last-1 by fastforce
  have 204: ( $\forall i < \text{ilen la.}$ 
     $\text{ilen } (\text{imap } (\text{shiftn } (\text{inth l } (\text{inth la } i))) (\text{sub } (\text{inth la } i) (\text{inth la } (\text{Suc } i)) l)) =$ 
    ( $\text{inth la } (\text{Suc } i)) - (\text{inth la } i)$ )
  by (simp add: 5 PJ6help1 filt-ilen)
  have 205: ( $\forall i < \text{ilen la.}$ 
    ( $\forall j \leq (\text{inth la } (\text{Suc } i)) - (\text{inth la } i).$ 
      ( $\text{inth } (\text{sub } (\text{inth la } i) (\text{inth la } (\text{Suc } i)) l) j =$ 
        ( $\text{inth l } ((\text{inth la } i) + j)$ )
      )
    )
  )

```

using 2041 5 *index-sequence-def inth-sub order.strict-implies-order* **by** blast
have 2060: $(\forall i < \text{ilen } la. \text{ inth } la \ i \leq \text{ inth } la \ (\text{Suc } i))$)

using 5 *idx-expand* **by** fastforce
have 206: $(\forall i < \text{ilen } la. (\forall j \leq \text{ inth } la \ (\text{Suc } i)) - \text{ inth } la \ i. (\text{ inth } l \ (\text{ inth } la \ i)) \leq \text{ inth } l \ ((\text{ inth } la \ i) + j))$)

using 2041 3 2060 *idx-less-equal* **by** fastforce
have 207: $(\forall i < \text{ilen } la. (\forall j \leq \text{ inth } la \ (\text{Suc } i)) - \text{ inth } la \ i. (\text{ inth } (\text{imap } (\text{shiftn } (\text{ inth } l \ (\text{ inth } la \ i))) (\text{sub } (\text{ inth } la \ i) (\text{ inth } la \ (\text{Suc } i)) \ l)) \ j) = (\text{ inth } l \ ((\text{ inth } la \ i) + j)) - \text{ inth } l \ (\text{ inth } la \ i))$)

by (*simp add: 205 inth-imap shiftn-def*)
have 208: $(\forall i < \text{ilen } la. (\text{ inth } (\text{imap } (\text{shiftn } (\text{ inth } l \ (\text{ inth } la \ i))) (\text{sub } (\text{ inth } la \ i) (\text{ inth } la \ (\text{Suc } i)) \ l)) \ 0) = 0$)

by (*simp add: 207*)
have 209: $(\forall i < \text{ilen } la. \text{ ilast } (\text{imap } (\text{shiftn } (\text{ inth } l \ (\text{ inth } la \ i))) (\text{sub } (\text{ inth } la \ i) (\text{ inth } la \ (\text{Suc } i)) \ l)) = (\text{ inth } l \ (\text{ inth } la \ (\text{Suc } i))) - \text{ inth } l \ (\text{ inth } la \ i)$)

using 204 207 5 2060 **by** *simp-all*
have 210: $(\forall i < \text{ilen } la. \text{ index-sequence } 0 \ (\text{imap } (\text{shiftn } (\text{ inth } l \ (\text{ inth } la \ i))) (\text{sub } (\text{ inth } la \ i) (\text{ inth } la \ (\text{Suc } i)) \ l))$)

by (*metis 3 5 7 add commute idx-expand idx-shiftn idx-sub plus-1-eq-Suc*)
have 211: $(\forall i < \text{ilen } la. \text{ powerinterval } f \ (\text{sub } (\text{ inth } l \ (\text{ inth } la \ i)) (\text{ inth } l \ (\text{ inth } la \ (\text{Suc } i))) \ \sigma) (\text{imap } (\text{shiftn } (\text{ inth } l \ (\text{ inth } la \ i))) (\text{sub } (\text{ inth } la \ i) (\text{ inth } la \ (\text{Suc } i)) \ l))$)

proof –
have 2111: $(\forall i < \text{ilen } la. \text{ powerinterval } f \ (\text{sub } (\text{ inth } l \ (\text{ inth } la \ i)) (\text{ inth } l \ (\text{ inth } la \ (\text{Suc } i))) \ \sigma) (\text{imap } (\text{shiftn } (\text{ inth } l \ (\text{ inth } la \ i))) (\text{sub } (\text{ inth } la \ i) (\text{ inth } la \ (\text{Suc } i)) \ l))$) =
 $(\forall i < \text{ilen } la. (\forall ia < \text{ilen } (\text{sub } (\text{ inth } la \ i) (\text{ inth } la \ (\text{Suc } i)) \ l). f \ (\text{sub } (\text{ inth } (\text{imap } (\text{shiftn } (\text{ inth } l \ (\text{ inth } la \ i)))$

```

      (sub (inth la i) (inth la (Suc i) l)) ia)
    (inth (imap (shifm (inth l ( inth la i)))
      (sub (inth la i) (inth la (Suc i) l)) (Suc ia))
    (sub (inth l (inth la i)) (inth l (inth la (Suc i)) σ) )
  )
)
)
by (simp add: powerinterval-def)
have 2112: ... =
  (∀ i < ien la.
    (∀ ia < (inth la (Suc i)) - (inth la i).
      f (sub (inth (imap (shifm (inth l (inth la i)))
        (sub (inth la i) (inth la (Suc i) l)) ia)
      (inth (imap (shifm (inth l ( inth la i)))
        (sub (inth la i) (inth la (Suc i) l)) (Suc ia))
      (sub (inth l (inth la i)) (inth l (inth la (Suc i)) σ))
    )
  )
)
using 204 by auto
have 2113: ... =
  (∀ i < ien la.
    (∀ ia < (inth la (Suc i)) - (inth la i).
      f (sub ((inth l ((inth la i) + ia)) - (inth l (inth la i)))
        ((inth l ((inth la i) + (Suc ia))) - (inth l (inth la i)))
        (sub (inth l (inth la i)) (inth l (inth la (Suc i)) σ))
    )
  )
)
using 207 by auto
have 2114:
  (∀ i < ien la.
    (let m = (inth la (Suc i)); n = (inth la i) in
      (∀ ia < m - n.
        (inth l (n + ia)) ≤ ((inth l (n + (Suc ia))) )
      )
    )
  )
)
using 2041 3 2060 idx-less-equal by simp-all fastforce
have 2115:
  (∀ i < ien la.
    (let m = (inth la (Suc i)); n = (inth la i) in
      (∀ ia < m - n.
        (inth l (n + ia)) - (inth l n) ≤ ((inth l ((Suc (n + ia)))) - (inth l n))
      )
    )
  )
)
by (metis 2114 add-Suc-right diff-le-mono)
have 2116:
  (∀ i < ien la.
    (let m = (inth la (Suc i)); n = (inth la i) in
      (∀ ia < m - n.
        ((inth l ((Suc (n + ia)))) - (inth l n)) ≤ (inth l m) - (inth l n)
      )
    )
  )

```

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    )
  )
)
by (metis 3 5 7 idx-expand Suc-leI add.commute diff-le-mono
  idx-less-equal less-diff-conv plus-1-eq-Suc)
have 2117: (∀ i < iLen la.
  (let m = (inth la (Suc i)); n = (inth la i) in
    (inth l m) ≤ iLen σ ))
by (metis 12 9 Suc-lessI filt-inth filt-iLen idx-less-last-1
  less-or-eq-imp-le)
have 2118: (∀ i < iLen la.
  (let m = (inth la (Suc i)); n = (inth la i) in
    (∀ ia < m - n.
      (n + (Suc ia)) ≤ iLen l )
    )
  )
)
using 5 7 idx-expand less-diff-conv by fastforce
have 21190: (∀ i < iLen la.
  (let m = (inth la (Suc i)); n = (inth la i) in
    (∀ ia < m - n.
      ((inth l (n + ia)) - (inth l n)) ≤ ((inth l (n + (Suc ia))) - (inth l n))
    ) ))

by (meson 2114 diff-le-mono)
have 21191: (∀ i < iLen la.
  (let m = (inth la (Suc i)); n = (inth la i) in
    (∀ ia < m - n.
      ((inth l (n + (Suc ia))) - (inth l n)) ≤
        iLen (sub (inth l n) (inth l m) σ)
    )))

by (simp add: 203 2116)
have 2119: (∀ i < iLen la.
  (let m = (inth la (Suc i)); n = (inth la i) in
    (∀ ia < m - n.
      iLen (sub ((inth l (n + ia)) - (inth l n))
        ((inth l (n + (Suc ia))) - (inth l n))
        (sub (inth l n) (inth l m) σ)) =
        ((inth l (n + (Suc ia)))) - ((inth l (n + ia)))
    )
  )
)

by (metis (no-types, lifting) 206 21190 21191
  diff-diff-add iLen-sub le-add-diff-inverse less-imp-le)
have 2120: (∀ i < iLen la.
  (let m = (inth la (Suc i)); n = (inth la i) in
    (∀ ia < m - n.
      (∀ j ≤ ((inth l (n + (Suc ia)))) - ((inth l (n + ia))))
        inth (sub ((inth l (n + ia)) - (inth l n))

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      ((inth l (n+(Suc ia))) - (inth l n))
      (sub (inth l n) (inth l m) σ) j =
      inth (sub (inth l n) (inth l m) σ) (((inth l (n +ia)) - (inth l n))+j)
    ) )))
  by (metis 2119 21190 21191 ilen-sub inth-sub)
have 212111: (∀ i<ilen la.
  (let m = (inth la (Suc i)); n= (inth la i) in
  (∀ ia< m - n.
    ((inth l (n +ia))) ≤ ((inth l (n+(Suc ia))))))
  )
  )
  )

  using 2114 by blast
have 212112: (∀ i<ilen la.
  (let m = (inth la (Suc i)); n= (inth la i) in
  (∀ ia< m - n.
    ((inth l (n))) ≤ ((inth l (n+(ia))))))
  )
  )
  )
  by (simp add: 206)
have 212113: (∀ i<ilen la.
  (let m = (inth la (Suc i)); n= (inth la i) in
  (∀ ia< m - n.
    (∀ j ≤ ((inth l (n+(Suc ia)))) - ((inth l (n +ia))).
      ((inth l (n +ia)) )+j ≤ (inth l m)
    )
  )
  )
  )
  using 212111 212112 206 2060 2116
  unfolding Let-def by fastforce
have 21211: (∀ i<ilen la.
  (let m = (inth la (Suc i)); n= (inth la i) in
  (∀ ia< m - n.
    (∀ j ≤ ((inth l (n+(Suc ia)))) - ((inth l (n +ia))).
      ((inth l (n +ia)) - (inth l n))+j ≤ (inth l m) - (inth l n)
    )
  )
  )
  )

  by (metis 212112 212113 Nat.add-diff-assoc2 diff-le-mono)
have 2121: (∀ i<ilen la.
  (let m = (inth la (Suc i)); n= (inth la i) in
  (∀ ia< m - n.
    (∀ j ≤ ((inth l (n+(Suc ia)))) - ((inth l (n +ia))).
      (inth (sub (inth l n) (inth l m) σ) (((inth l (n +ia)) - (inth l n))+j) ) =
      (inth σ (((inth l n) + (((inth l (n +ia)) - (inth l n))+j)) )
    )
  )
  )

```

)
)
)
)

by (*metis 206 2117 21211 add-diff-inverse-nat inth-sub nat-diff-split not-less-zero order-refl*)

have 2122: ($\forall i < \text{ilen } la.$

(*let* $m = (\text{inth } la \text{ (Suc } i)); n = (\text{inth } la \text{ } i)$ *in*
 $(\forall ia < m - n.$
 $(\forall j \leq ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) - ((\text{inth } l \text{ (} n + ia))))).$
 $(\text{inth } \sigma ((\text{inth } l \text{ } n) + (((\text{inth } l \text{ (} n + ia)) - (\text{inth } l \text{ } n)) + j)) \text{)} =$
 $(\text{inth } \sigma (((\text{inth } l \text{ (} n + ia)) + j)) \text{)}$
 $)$
 $)$
 $)$
 $)$

by (*metis 206 add.assoc le-add-diff-inverse less-imp-le-nat*)

have 2123: ($\forall i < \text{ilen } la.$

(*let* $m = (\text{inth } la \text{ (Suc } i)); n = (\text{inth } la \text{ } i)$ *in*
 $(\forall ia < m - n.$
 $(\forall j \leq ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) - ((\text{inth } l \text{ (} n + ia))))).$
 $\text{inth (sub (inth } l \text{ (} n + ia)) ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) \sigma) j =$
 $(\text{inth } \sigma ((((\text{inth } l \text{ (} n + ia)) + j)) \text{)}$
 $)$
 $)$
 $)$
 $)$

by (*metis 2114 2118 3 eq-imp-le idx-less-equal inth-sub*)

have 2124: ($\forall i < \text{ilen } la.$

(*let* $m = (\text{inth } la \text{ (Suc } i)); n = (\text{inth } la \text{ } i)$ *in*
 $(\forall ia < m - n.$
 $\text{ilen (sub (inth } l \text{ (} n + ia)) ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) \sigma) =$
 $((\text{inth } l \text{ (} n + (\text{Suc } ia)))) - ((\text{inth } l \text{ (} n + ia))))$
 $)$
 $)$
 $)$

by (*metis 2118 3 Suc-eq-plus1 Suc-le-lessD add-Suc-right idx-expand ilen-sub*)

have 2125: ($\forall i < \text{ilen } la.$

(*let* $m = (\text{inth } la \text{ (Suc } i)); n = (\text{inth } la \text{ } i)$ *in*
 $(\forall ia < m - n.$
 $(\forall j \leq ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) - ((\text{inth } l \text{ (} n + ia))))).$
 $\text{inth (sub ((\text{inth } l \text{ (} n + ia)) - (\text{inth } l \text{ } n))$
 $((\text{inth } l \text{ (} n + (\text{Suc } ia)) - (\text{inth } l \text{ } n))$
 $(\text{sub (inth } l \text{ } n) (\text{inth } l \text{ } m) \sigma)) j =$
 $(\text{inth (sub (inth } l \text{ (} n + ia)) ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) \sigma) j)$
 $)$

)
)
)

by (*metis 2120 2121 2122 2123*)

have 2126: ($\forall i < \text{ilen } la.$

(let $m = (\text{inth } la \text{ (Suc } i)); n = (\text{inth } la \text{ } i)$ in
 $(\forall ia < m - n.$
 $\text{ilen } (\text{sub } ((\text{inth } l \text{ (} n + ia)) - (\text{inth } l \text{ } n))$
 $((\text{inth } l \text{ (} n + (\text{Suc } ia))) - (\text{inth } l \text{ } n))$
 $(\text{sub } (\text{inth } l \text{ } n) (\text{inth } l \text{ } m) \sigma)) =$
 $\text{ilen } (\text{sub } (\text{inth } l \text{ (} n + ia)) ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) \sigma)$
 $)$
 $)$
 $)$

by (*metis 2119 2124*)

have 2127: ($\forall i < \text{ilen } la.$

(let $m = (\text{inth } la \text{ (Suc } i)); n = (\text{inth } la \text{ } i)$ in
 $(\forall ia < m - n.$
 $(\text{sub } ((\text{inth } l \text{ (} n + ia)) - (\text{inth } l \text{ } n))$
 $((\text{inth } l \text{ (} n + (\text{Suc } ia))) - (\text{inth } l \text{ } n))$
 $(\text{sub } (\text{inth } l \text{ } n) (\text{inth } l \text{ } m) \sigma)) =$
 $(\text{sub } (\text{inth } l \text{ (} n + ia)) ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) \sigma)$
 $)$
 $)$
 $)$

using *interval-eq-inth-eq 2125 2126*

by (*metis 2119*)

have 2128: ($\forall i < \text{ilen } la.$

(let $m = (\text{inth } la \text{ (Suc } i)); n = (\text{inth } la \text{ } i)$ in
 $(\forall ia < m - n.$
 $f (\text{sub } (\text{inth } l \text{ (} n + ia)) ((\text{inth } l \text{ (} n + (\text{Suc } ia)))) \sigma)$
 $)$
 $)$
 $)$

by (*metis 2041 3 add.commute add-Suc-right less-diff-conv less-le-trans*
powerinterval-def)

show ?thesis using 2111 2112 2113 2127 2128 unfolding Let-def by (*simp add: 2113*)

qed

have 220: ($\forall i < \text{ilen } la.$

$((\text{filt } (\text{sub } (\text{inth } l \text{ (} \text{inth } la \text{ } i)) (\text{inth } l \text{ (} \text{inth } la \text{ (Suc } i)))) \sigma)$
 $(\text{imap } (\text{shiftm } (\text{inth } l \text{ (} \text{inth } la \text{ } i))) (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) l)))$
 $\models g)$
 $)$

proof –

have 2201: ($\forall i < \text{ilen } la.$

$\text{ilen } (\text{filt } (\text{sub } (\text{inth } l \text{ (} \text{inth } la \text{ } i)) (\text{inth } l \text{ (} \text{inth } la \text{ (Suc } i)))) \sigma)$

$$\begin{aligned}
& (\text{imap } (\text{shifm } (\text{inth } l \ (\text{inth } la \ i))) \\
& \quad (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l)) \) = \\
& \text{ilen } (\text{imap } (\text{shifm } (\text{inth } l \ (\text{inth } la \ i))) \ (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l)) \\
&)
\end{aligned}$$

using *filt-ilen by blast*
have 2202: $(\forall \ i < \text{ilen } la.$

$$\begin{aligned}
& \text{ilen } (\text{imap } (\text{shifm } (\text{inth } l \ (\text{inth } la \ i))) \\
& \quad (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l)) = \\
& \quad (\text{inth } la \ (\text{Suc } i)) - (\text{inth } la \ i)
\end{aligned}$$

$$\left. \begin{aligned}
& \right)
\end{aligned}$$

using 204 **by** *blast*
have 2203: $(\forall \ i < \text{ilen } la.$

$$\begin{aligned}
& (\forall \ j \leq (\text{inth } la \ (\text{Suc } i)) - (\text{inth } la \ i) . \\
& \text{inth } ((\text{filt } (\text{sub } (\text{inth } l \ (\text{inth } la \ i)) \ (\text{inth } l \ (\text{inth } la \ (\text{Suc } i)))) \ \sigma) \\
& \quad (\text{imap } (\text{shifm } (\text{inth } l \ (\text{inth } la \ i))) \\
& \quad \quad (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l)) \) \) \ j = \\
& \text{inth } (\text{sub } (\text{inth } l \ (\text{inth } la \ i)) \ (\text{inth } l \ (\text{inth } la \ (\text{Suc } i)))) \ \sigma) \\
& \quad (\text{inth } (\text{imap } (\text{shifm } (\text{inth } l \ (\text{inth } la \ i))) \\
& \quad \quad (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l)) \ j) \\
&))
\end{aligned}$$

by (*simp add: filt-imap inth-imap*)
have 2204: $(\forall \ i < \text{ilen } la.$

$$\begin{aligned}
& (\forall \ j \leq (\text{inth } la \ (\text{Suc } i)) - (\text{inth } la \ i) . \\
& \text{inth } (\text{sub } (\text{inth } l \ (\text{inth } la \ i)) \ (\text{inth } l \ (\text{inth } la \ (\text{Suc } i)))) \ \sigma) \\
& \quad (\text{inth } (\text{imap } (\text{shifm } (\text{inth } l \ (\text{inth } la \ i))) \\
& \quad \quad (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l)) \ j) = \\
& \text{inth } (\text{sub } (\text{inth } l \ (\text{inth } la \ i)) \ (\text{inth } l \ (\text{inth } la \ (\text{Suc } i)))) \ \sigma) \\
& \quad ((\text{inth } (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l) \ j) - (\text{inth } l \ (\text{inth } la \ i))) \\
&))
\end{aligned}$$

by (*simp add: inth-imap shifm-def*)
have 2205: $(\forall \ i < \text{ilen } la.$

$$\begin{aligned}
& (\\
& \quad (\text{inth } l \ (\text{inth } la \ i)) \leq (\text{inth } l \ (\text{inth } la \ (\text{Suc } i))) \wedge \\
& \quad (\text{inth } l \ (\text{inth } la \ (\text{Suc } i))) \leq \text{ilen } \sigma \\
&)
\end{aligned}$$

by (*metis 12 9 add.commute filt-ilen filt-imap idx-expand inth-imap plus-1-eq-Suc*)
have 2206 : $(\forall \ i < \text{ilen } la.$

$$\begin{aligned}
& (\forall \ j \leq (\text{inth } la \ (\text{Suc } i)) - (\text{inth } la \ i) . \\
& \quad (\text{inth } (\text{sub } (\text{inth } la \ i) \ (\text{inth } la \ (\text{Suc } i)) \ l) \ j) = \\
& \quad (\text{inth } l \ ((\text{inth } la \ i) + j)) \\
&))
\end{aligned}$$

using 205 **by** *blast*
have 2207: $(\forall \ i < \text{ilen } la.$

$$\begin{aligned}
& (\forall j \leq (\text{inth } la \text{ (Suc } i)) - (\text{inth } la \text{ } i)) . \\
& ((\text{inth } l \text{ (inth } la \text{ } i)) + ((\text{inth } (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) \text{ } l) \text{ } j) \\
& \quad - (\text{inth } l \text{ (inth } la \text{ } i)))) = \\
& \quad (\text{inth } l \text{ ((inth } la \text{ } i) + j)) \\
&))
\end{aligned}$$

by (*simp add: 206 2206*)

have 2208: $(\forall i < \text{ilen } la.$

$$\begin{aligned}
& (\forall j \leq (\text{inth } la \text{ (Suc } i)) - (\text{inth } la \text{ } i)) . \\
& \quad \text{inth } (\text{sub } (\text{inth } l \text{ (inth } la \text{ } i)) (\text{inth } l \text{ (inth } la \text{ (Suc } i))}) \sigma) \\
& \quad ((\text{inth } (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) \text{ } l) \text{ } j) - (\text{inth } l \text{ (inth } la \text{ } i))) = \\
& \quad \text{inth } \sigma \text{ (inth } l \text{ ((inth } la \text{ } i) + j)) \\
&))
\end{aligned}$$
using 2041 206 2060 2205 3 idx-less-equal[*of l σ*]
using Nat.le-diff-conv2 by auto

have 2209: $(\forall i < \text{ilen } la.$

$$\begin{aligned}
& (\forall j \leq (\text{inth } la \text{ (Suc } i)) - (\text{inth } la \text{ } i)) . \\
& \quad (\text{inth } (\text{filt } \sigma \text{ } l) ((\text{inth } la \text{ } i) + j)) = (\text{inth } \sigma \text{ (inth } l \text{ ((inth } la \text{ } i) + j))) \\
&))
\end{aligned}$$

by (*simp add: filt-imap inth-imap*)

have 2210: $(\forall i < \text{ilen } la.$

$$\begin{aligned}
& \text{ilen } (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) (\text{filt } \sigma \text{ } l)) = \\
& \quad (\text{inth } la \text{ (Suc } i)) - (\text{inth } la \text{ } i) \\
&)
\end{aligned}$$

using 5 PJ6help1 by blast

have 2211: $(\forall i < \text{ilen } la.$

$$\begin{aligned}
& (\forall j \leq (\text{inth } la \text{ (Suc } i)) - (\text{inth } la \text{ } i)) . \\
& \quad (\text{inth } (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) (\text{filt } \sigma \text{ } l)) \text{ } j) = \\
& \quad (\text{inth } \sigma \text{ (inth } l \text{ ((inth } la \text{ } i) + j))) \\
&))
\end{aligned}$$

using 2209 5 idx-expand inth-sub by fastforce

have 2212: $(\forall i < \text{ilen } la.$

$$\begin{aligned}
& (\forall j \leq (\text{inth } la \text{ (Suc } i)) - (\text{inth } la \text{ } i)) . \\
& \quad \text{inth } ((\text{filt } (\text{sub } (\text{inth } l \text{ (inth } la \text{ } i)) (\text{inth } l \text{ (inth } la \text{ (Suc } i))}) \sigma) \\
& \quad \quad (\text{imap } (\text{shiftn } (\text{inth } l \text{ (inth } la \text{ } i)))) \\
& \quad \quad (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) \text{ } l)) \text{ }) \text{ } j = \\
& \quad (\text{inth } (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) (\text{filt } \sigma \text{ } l)) \text{ } j) \\
&))
\end{aligned}$$

by (*simp add: 2203 2204 2208 2211*)

have 2213: $(\forall i < \text{ilen } la.$

$$\begin{aligned}
& \text{ilen } ((\text{filt } (\text{sub } (\text{inth } l \text{ (inth } la \text{ } i)) (\text{inth } l \text{ (inth } la \text{ (Suc } i))}) \sigma) \\
& \quad \quad (\text{imap } (\text{shiftn } (\text{inth } l \text{ (inth } la \text{ } i)))) \\
& \quad \quad (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) \text{ } l)) \text{ }) \text{ }) = \\
& \text{ilen } (\text{sub } (\text{inth } la \text{ } i) (\text{inth } la \text{ (Suc } i)) (\text{filt } \sigma \text{ } l)) \\
&)
\end{aligned}$$

```

    by (metis 2202 2210 filt-ilen)
  have 2214: (∀ i < ilen la.
    (filt (sub (inth l (inth la i)) (inth l (inth la (Suc i))) σ)
      (imap (shiftn (inth l (inth la i))) (sub (inth la i) (inth la (Suc i) l))) =
      (sub (inth la i) (inth la (Suc i)) (filt σ l))
    )

  using interval-eq-inth-eq 2212 2213 using 2201 2202 by fastforce
  have 2215: (∀ i < ilen la.
    (sub (inth la i) (inth la (Suc i)) (filt σ l)) ⊨ g
    )
  using 5 powerinterval-def by blast
  show ?thesis by (simp add: 2214 2215)
qed
show ?thesis
using 201 202 203 209 210 211 220 by metis
qed
show ?thesis
by (metis 12 20 5 8 9 projection-d-def)
qed

```

```

lemma PJ7sem:
(σ ⊨ f Δ (g Δ h) = (f Δ g) Δ h)
proof -
  have 1: ilen σ > 0 ⟶ (σ ⊨ f Δ (g Δ h) = (f Δ g) Δ h)
    using PJ7helpchain1a PJ7helpchain1b unl-lift2 by blast
  have 2: ilen σ = 0 ⟶ (σ ⊨ f Δ (g Δ h) = (f Δ g) Δ h)
    using PJ7empty by blast
  from 1 2 show ?thesis by auto
qed

```

8.3.8 PJ8

```

lemma PJ8semhelp:
  assumes index-sequence 0 l
    inth l (ilen l) = ilen σ
    (∀ n na. na + n ≤ ilen σ ⟶ f (sub n (n+ na) σ) ⟶ g (sub n (n + na) σ))
  shows
    (∀ i < ilen l. f (sub ( inth l i) ( inth l (Suc i)) σ)
      ⟶ g (sub ( inth l i) ( inth l (Suc i)) σ)
    )
  by (metis add.commute assms idx-expand
    ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc)

```

```

lemma PJ8sem:
(σ ⊨ ba(f ⟶ g) ⟶ (f Δ h) ⟶ (g Δ h))
using PJ8semhelp by (simp add: projection-d-def ba-defs powerinterval-def) blast

```

8.3.9 PJ9

lemma *PJ9sem*:

$(\sigma \models f \nabla (g \longrightarrow h) \longrightarrow f \triangle g \longrightarrow f \triangle h)$

by (*simp add: uprojection-d-def projection-d-def, metis*)

8.4 Axioms

lemma *BpGen*:

assumes $\vdash f$

shows $\vdash bp\ f$

using *assms*

by (*simp add: bp-d-def uprojection-d-def projection-d-def Valid-def*)

lemma *PJ1*:

$\vdash f \triangle (g \vee h) \longrightarrow f \triangle g \vee f \triangle h$

using *PJ1sem Valid-def* **by** *blast*

lemma *PJ2*:

$\vdash f \triangle empty = empty$

using *PJ2sem Valid-def* **by** *blast*

lemma *PJ3*:

$\vdash f \triangle skip = (f \wedge more)$

using *PJ3sem Valid-def* **by** *blast*

lemma *PJ4*:

$\vdash f \triangle (g;h) = (f \triangle g) ; (f \triangle h)$

using *PJ4sem Valid-def* **by** *blast*

lemma *PJ5*:

$\vdash f \triangle init(g) \longrightarrow init(g)$

using *PJ5sem Valid-def* **by** *blast*

lemma *PJ6*:

$\vdash skip \triangle g = g$

using *PJ6sem Valid-def* **by** *blast*

lemma *PJ7*:

$\vdash f \triangle (g \triangle h) = (f \triangle g) \triangle h$

using *PJ7sem Valid-def* **by** *blast*

lemma *PJ8*:

$\vdash ba(f \longrightarrow g) \longrightarrow (f \triangle h) \longrightarrow (g \triangle h)$

using *PJ8sem Valid-def* **by** *blast*

lemma *PJ9*:

$\vdash f \nabla (g \longrightarrow h) \longrightarrow f \triangle g \longrightarrow f \triangle h$

using *PJ9sem Valid-def* **by** *blast*

8.5 Time Reversal

lemma *filt-iapp*:

filt w ($l \ominus \langle x \rangle$) = (*filt* w l) \ominus (*inth* w x)

proof

(*induct* l)

case (*INil* x)

then show ?*case* **by** *simp*

next

case (*ICons* $x1a$ l)

then show ?*case*

by *simp*

qed

lemma *filt-irev*:

assumes $\forall i \leq \text{ilen } l. (\text{inth } l \ i) \leq \text{ilen } w$

shows (*filt* (*irev* w) l) = (*irev* (*filt* w (*imap* ($\lambda x. \text{ilen } w - x$) (*irev* l))))

using *assms*

proof

(*induct* l)

case (*INil* x)

then show ?*case*

proof –

have 01: *filt* (*irev* w) $\langle x \rangle$ = $\langle \text{inth } (\text{irev } w) \ x \rangle$

by *simp*

have 02: $\langle \text{inth } (\text{irev } w) \ x \rangle$ = $\langle \text{inth } w \ (\text{ilen } w - x) \rangle$

using *INil.premis irev-inth* **by** *auto*

have 03: *irev* (*filt* w (*imap* (($-$) (*ilen* w)) $\langle x \rangle$)) = $\langle \text{inth } w \ (\text{ilen } w - x) \rangle$

by *simp*

from 01 02 03 **show** ?*thesis* **by** *auto*

qed

next

case (*ICons* $x1a$ l)

then show ?*case*

proof –

have 1: *filt* (*irev* w) ($x1a \odot l$) =

(*inth* (*irev* w) $x1a$) \odot (*filt* (*irev* w) l)

by *simp*

have 2: (*inth* (*irev* w) $x1a$) = (*inth* w (*ilen* $w - x1a$))

using *ICons.premis irev-inth* **by** *fastforce*

have 3: *irev* (*filt* w ($x1a \odot l$)) = *irev* ((*inth* w $x1a$) \odot (*filt* w l))

by *simp*

have 4: *irev* (*filt* w (*imap* (($-$) (*ilen* w)) (*irev* ($x1a \odot l$)))) =
irev (*filt* w (*imap* (($-$) (*ilen* w)) ((*irev* l) \ominus $\langle x1a \rangle$))))

by *simp*

have 5: *irev* (*filt* w (*imap* (($-$) (*ilen* w)) ((*irev* l) \ominus $\langle x1a \rangle$)))) =
irev ((*filt* w (*imap* (($-$) (*ilen* w)) ((*irev* l)))) \ominus $\langle \text{inth } w \ (\text{ilen } w - x1a) \rangle$)

by (*simp* *add: filt-iapp*)

have 6: $\text{irev } ((\text{filt } w \ (\text{imap } ((-) \ (\text{ilen } w)) \ (\text{irev } l))) \ominus \langle \text{inth } w \ (\text{ilen } w - x1a) \ \rangle =$
 $(\text{inth } w \ (\text{ilen } w - x1a)) \odot \text{irev} ((\text{filt } w \ (\text{imap } ((-) \ (\text{ilen } w)) \ (\text{irev } l))))$
by *auto*
have 7: $\forall i \leq \text{ilen } l. (\text{inth } l \ i) \leq \text{ilen } w$
using *ICons(2)* **by** *auto*
have 8: $(\text{filt } (\text{irev } w) \ l) = \text{irev}(\text{filt } w \ (\text{imap } ((-) \ (\text{ilen } w)) \ (\text{irev } l)))$
using 7 *ICons.hyps* **by** *blast*
show *?thesis*
using 2 5 8 **by** *auto*
qed
qed

lemma *ProjectionRevsema:*

assumes $(\sigma \models (f \triangle g)^r)$

shows $(\sigma \models (f^r) \triangle (g^r))$

proof –

have 1: $(\sigma \models (f \triangle g)^r)$

using *assms* **by** *auto*

have 2: $\exists l. \text{index-sequence } 0 \ l \wedge \text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $\text{powerinterval } f \ (\text{irev } \sigma) \ l \wedge g \ (\text{filt } (\text{irev } \sigma) \ l)$

using 1 **by** (*simp add: projection-d-def reverse-d-def*)

obtain *l* **where** 3: $\text{index-sequence } 0 \ l \wedge \text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $\text{powerinterval } f \ (\text{irev } \sigma) \ l \wedge g \ (\text{filt } (\text{irev } \sigma) \ l)$

using 2 **by** *auto*

have 4: $\text{index-sequence } 0 \ l$

using 3 **by** *auto*

have 5: $\text{inth } l \ (\text{ilen } l) = \text{ilen } \sigma$

using 3 **by** *auto*

have 6: $\text{powerinterval } f \ (\text{irev } \sigma) \ l$

using 3 **by** *auto*

have 7: $g \ (\text{filt } (\text{irev } \sigma) \ l)$

using 3 **by** *auto*

have 8: $\forall i \leq \text{ilen } l. (\text{inth } l \ i) \leq \text{ilen } \sigma$

using 4 5 *idx-less-last-1 le-eq-less-or-eq* **by** *fastforce*

have 9: $g \ (\text{irev}(\text{filt } \sigma \ (\text{imap } (\lambda x. \text{ilen } \sigma - x) \ (\text{irev } l)) \))$

using 7 8 *filt-irev* **by** *fastforce*

have 10: $\text{inth } (\text{imap } ((-) \ (\text{ilen } \sigma)) \ (\text{irev } l)) \ 0 = 0$

by (*metis 5 diff-self-eq-0 ifirst-irev inth-imap*)

have 11: $(\forall n < \text{ilen } l.$

$\text{inth } (\text{imap } ((-) \ (\text{ilen } \sigma)) \ (\text{irev } l)) \ n$

$< \text{inth } (\text{imap } ((-) \ (\text{ilen } \sigma)) \ (\text{irev } l)) \ (\text{Suc } n))$

by (*simp add: inth-imap irev-inth*)

(*metis (no-types, lifting) 4 5 Suc-diff-Suc Suc-less-eq diff-less-Suc diff-less-mono2*
index-sequence-def idx-less-last-1)

have 12: $\text{index-sequence } 0 \ (\text{imap } (\lambda x. \text{ilen } \sigma - x) \ (\text{irev } l))$

by (*simp add: 10 11 index-sequence-def*)

have 13: $\text{ilen } (\text{imap } (\lambda x. \text{ilen } \sigma - x) \ (\text{irev } l)) = \text{ilen } l$

by *auto*

have 14: $\text{inth } (\text{imap } ((-) \ (\text{ilen } \sigma)) \ (\text{irev } l)) \ (\text{ilen } (\text{imap } ((-) \ (\text{ilen } \sigma)) \ (\text{irev } l))) =$
 $\text{ilen } \sigma$

by (metis 4 diff-zero index-sequence-def ilast-irev ilen-imap
 inth-imap)
 have 15: $\forall i < \text{ilen } l.$

$$(\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i) \leq$$

$$(\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i))$$

 by (simp add: 11 less-imp-le-nat)
 have 16: $\forall i < \text{ilen } l.$

$$(\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i)) \leq \text{ilen } \sigma$$

 by (simp add: inth-imap)
 have 17: $\forall i < \text{ilen } l.$
 irev

$$(\text{sub } (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i)$$

$$(\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i)) \sigma) =$$

$$\text{sub } (\text{ilen } \sigma - (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i)))$$

$$(\text{ilen } \sigma - (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i))$$

$$(\text{irev } \sigma)$$

 using irev-sub
 using 15 16 by blast
 have 18: $\forall i < \text{ilen } l.$

$$\text{ilen } \sigma - (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i)) =$$

$$(\text{inth } (\text{irev } l) (\text{Suc } i))$$

 by (metis 10 12 13 14 Suc-leI diff-diff-cancel idx-less-than
 ilen-gr-zero inth-imap less-le zero-less-Suc zero-less-diff)
 have 19: $\forall i < \text{ilen } l.$

$$\text{ilen } \sigma - (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i) =$$

$$(\text{inth } (\text{irev } l) i)$$

 by (metis 10 18 5 diff-zero ifirst-irev less-SucI less-Suc-eq-0-disj)
 have 20: $\forall i < \text{ilen } l.$

$$(\text{sub } (\text{ilen } \sigma - (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i)))$$

$$(\text{ilen } \sigma - (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i))$$

$$(\text{irev } \sigma)) =$$

$$(\text{sub } (\text{inth } (\text{irev } l) (\text{Suc } i))$$

$$(\text{inth } (\text{irev } l) i))$$

$$(\text{irev } \sigma))$$

 using 18 19 by simp
 have 21: $\forall i < \text{ilen } l.$

$$(\text{sub } (\text{inth } (\text{irev } l) (\text{Suc } i))$$

$$(\text{inth } (\text{irev } l) i))$$

$$(\text{irev } \sigma)) =$$

$$(\text{sub } (\text{inth } l (\text{ilen } l - (\text{Suc } i)))$$

$$(\text{inth } l (\text{ilen } l - i))$$

$$(\text{irev } \sigma))$$

 by (simp add: irev-inth)
 have 22: $\forall i < \text{ilen } l.$

$f (\text{sub} (\text{inth } l (\text{ilen } l - (\text{Suc } i)))$
 $(\text{inth } l (\text{ilen } l - i))$
 $(\text{irev } \sigma))$

by (*metis 6 Suc-diff-Suc Suc-less-eq diff-less-Suc powerinterval-def*)
have 24: $\forall i < \text{ilen } l.$
 $f (\text{irev}$
 $(\text{sub} (\text{inth} (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i)$
 $(\text{inth} (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i)) \sigma))$

by (*simp add: 17 20 21 22*)
have 25: $\text{powerinterval } (\lambda s. f (\text{irev } s)) \sigma (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l))$
by (*simp add: 24 powerinterval-def*)
have 26: $(\exists l. \text{index-sequence } 0 l \wedge$
 $\text{inth } l (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $\text{powerinterval } (\lambda s. f (\text{irev } s)) \sigma l \wedge g (\text{irev}(\text{filt } \sigma l)))$

using 12 14 25 9 by blast
from 26 show ?thesis
by (*simp add: projection-d-def reverse-d-def*)
qed

lemma ProjectionRevsemb:

assumes $(\sigma \models (f^r) \triangle (g^r))$

shows $(\sigma \models (f \triangle g)^r)$

proof –

have 1: $(\exists l. \text{index-sequence } 0 l \wedge$
 $\text{inth } l (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $\text{powerinterval } (\lambda s. f (\text{irev } s)) \sigma l \wedge g (\text{irev}(\text{filt } \sigma l)))$

using *assms* **by** (*simp add: projection-d-def reverse-d-def*)

obtain l **where 2:** $\text{index-sequence } 0 l \wedge$

$\text{inth } l (\text{ilen } l) = \text{ilen } \sigma \wedge$
 $\text{powerinterval } (\lambda s. f (\text{irev } s)) \sigma l \wedge g (\text{irev}(\text{filt } \sigma l))$

using 1 by auto

have 3: $\text{index-sequence } 0 l$

using 2 by auto

have 4: $\text{inth } l (\text{ilen } l) = \text{ilen } \sigma$

using 2 by auto

have 5: $\text{powerinterval } (\lambda s. f (\text{irev } s)) \sigma l$

using 2 by auto

have 6: $g (\text{irev}(\text{filt } \sigma l))$

using 2 by auto

have 7: $\text{ilen } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) = \text{ilen } l$

by *simp*

have 8: $\forall i \leq \text{ilen } l. (\text{inth} (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i) \leq \text{ilen } \sigma$

by (*simp add: inth-imap*)

have 9: $g (\text{filt } (\text{irev } \sigma) (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)))$

by (*metis 2 filt-irev idx-less-equal irev-ilen irev-irev-ident*
le-refl)

have 10: $\text{inth} (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{ilen } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l))) = \text{ilen } \sigma$

by (*metis 2 diff-zero index-sequence-def ilast-irev ilen-imap*)

$inth\text{-}imap$)
have 11: $inth (imap ((-) (ilen \sigma)) (irev l)) 0 = 0$
by ($metis$ 4 $diff\text{-}self\text{-}eq\text{-}0$ $ifirst\text{-}irev$ $inth\text{-}imap$)
have 12: $(\forall n < ilen\ l.$
 $inth (imap ((-) (ilen \sigma)) (irev l)) n$
 $< inth (imap ((-) (ilen \sigma)) (irev l)) (Suc\ n))$
by ($simp$ $add: inth\text{-}imap\ irev\text{-}inth$)
 $(metis (no\text{-}types, lifting) 2\ Suc\text{-}diff\text{-}Suc\ Suc\text{-}less\text{-}eq\ diff\text{-}less\text{-}Suc\ diff\text{-}less\text{-}mono2$
 $index\text{-}sequence\text{-}def\ idx\text{-}less\text{-}last\text{-}1)$
have 13: $index\text{-}sequence\ 0 (imap (\lambda x. ilen\ \sigma - x) (irev l))$
by ($simp$ $add: 11\ 12\ index\text{-}sequence\text{-}def$)
have 14: $\forall i < ilen\ l. f (irev (sub (inth\ l\ i) (inth\ l\ (Suc\ i)) \sigma))$
using 5 **by** ($simp$ $add: powerinterval\text{-}def$)
have 15: $\forall i < ilen\ l. (inth\ l\ (Suc\ i)) \leq ilen\ \sigma$
using 2 $idx\text{-}expand$ **by** $fastforce$
have 16: $\forall i < ilen\ l. (inth\ l\ i) \leq (inth\ l\ (Suc\ i))$
using 2 $idx\text{-}expand$ **by** $fastforce$
have 17: $\forall i < ilen\ l.$
 $f (sub ((ilen\ \sigma) - (inth\ l\ (Suc\ i))) ((ilen\ \sigma) - (inth\ l\ i)) (irev\ \sigma))$

using 14
by ($simp$ $add: irev\text{-}sub\ 15\ 16$)
have 18: $\forall i < ilen\ l.$
 $(inth (imap ((-) (ilen \sigma)) (irev l)) i) =$
 $ilen\ \sigma - (inth (irev\ l) i)$

using $inth\text{-}imap$ **by** $blast$
have 19: $\forall i < ilen\ l.$
 $(inth (imap ((-) (ilen \sigma)) (irev l)) (Suc\ i)) =$
 $ilen\ \sigma - (inth (irev\ l) (Suc\ i))$

using $inth\text{-}imap$ **by** $blast$
have 20: $\forall i < ilen\ l.$
 $(inth (irev\ l) i) = (inth\ l\ (ilen\ l - i))$

by ($simp$ $add: irev\text{-}inth$)
have 21: $\forall i < ilen\ l.$
 $(inth (irev\ l) (Suc\ i)) = (inth\ l\ (ilen\ l - (Suc\ i)))$
by ($simp$ $add: irev\text{-}inth$)
have 22: $\forall i < ilen\ l.$
 $f (sub (ilen\ \sigma - (inth\ l\ (ilen\ l - i)))$
 $(ilen\ \sigma - (inth\ l\ (ilen\ l - (Suc\ i))))$
 $(irev\ \sigma))$

by ($metis$ 17 $Suc\text{-}diff\text{-}Suc\ Suc\text{-}less\text{-}eq\ diff\text{-}less\text{-}Suc$)
have 23: $\forall i < ilen\ l.$
 $f (sub (ilen\ \sigma - (inth (irev\ l) i))$
 $(ilen\ \sigma - (inth (irev\ l) (Suc\ i)))$
 $(irev\ \sigma))$


```

  by (simp add: 20 21 22)
have 24:  $\forall i < \text{ilen } l.$ 
   $f (\text{sub } (\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) i)$ 
   $(\text{inth } (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l)) (\text{Suc } i)) (\text{irev } \sigma))$ 
  by (simp add: 23 inth-imap)
have 25:  $\text{powerinterval } f (\text{irev } \sigma) (\text{imap } ((-) (\text{ilen } \sigma)) (\text{irev } l))$ 
  by (simp add: 24 powerinterval-def)
have 26:  $\exists l. \text{index-sequence } 0 l \wedge \text{inth } l (\text{ilen } l) = \text{ilen } \sigma \wedge$ 
   $\text{powerinterval } f (\text{irev } \sigma) l \wedge g (\text{filt } (\text{irev } \sigma) l)$ 
  using 10 13 25 9 by blast
from 26 show ?thesis by (simp add: projection-d-def reverse-d-def)
qed

```

lemma *ProjectionRev*:
 $\vdash (f \triangle g)^r = f^r \triangle g^r$
using *ProjectionRevsema ProjectionRevsemb unl-lift2* **by** *blast*

8.6 Theorems

8.6.1 Projection

lemma *PowerProjLen*:
 $\vdash f \triangle \text{len } n = \text{power } (f \wedge \text{more}) n$
proof
 (*induct* n)
case 0
then show ?case **by** (*metis PJ2 len-d-def pow-0*)
next
case (*Suc* n)
then show ?case
by (*metis PJ3 PJ4 inteq-reflection len-d-def pow-Suc*)
qed

lemma *ProjLenExist*:
 $\vdash f \triangle (\exists n. \text{len } n) = (\exists n. f \triangle \text{len } n)$
by (*simp add: Valid-def projection-d-def, blast*)

lemma *PowerProjLenExist*:
 $\vdash (\exists n. f \triangle \text{len } n) = (\exists n. \text{power } (f \wedge \text{more}) n)$
using *PowerProjLen* **by** (*simp add: Valid-def PowerProjLen, blast*)

lemma *RightProjImpProj*:
assumes $\vdash g1 \longrightarrow g2$
shows $\vdash f \triangle g1 \longrightarrow f \triangle g2$
using *assms*
by (*simp add: Valid-def projection-d-def, blast*)

lemma *LeftProjImpProj*:

assumes $\vdash f1 \longrightarrow f2$
shows $\vdash f1 \triangle g \longrightarrow f2 \triangle g$
using *assms*
by (*simp add: Valid-def projection-d-def powerinterval-def, blast*)

lemma *RightProjEqvProj*:
assumes $\vdash g1 = g2$
shows $\vdash f \triangle g1 = f \triangle g2$
using *assms*
by (*metis (mono-tags, lifting) Valid-def inteq-reflection unl-lift2*)

lemma *LeftProjEqvProj*:
assumes $\vdash f1 = f2$
shows $\vdash f1 \triangle g = f2 \triangle g$
using *assms*
by (*metis (mono-tags, lifting) Valid-def inteq-reflection unl-lift2*)

lemma *ProjTrueEqvChopstar*:
 $\vdash f \triangle \#True = f^*$
proof –
have 1: $\vdash \#True = (\exists n. \text{len } n)$
by (*simp add: Valid-def len-defs*)
have 2: $\vdash f \triangle \#True = f \triangle (\exists n. \text{len } n)$
using 1 *RightProjEqvProj* **by** *blast*
have 3: $\vdash f \triangle (\exists n. \text{len } n) = (\exists n. \text{power } (f \wedge \text{more}) n)$
using *PowerProjLenExist ProjLenExist* **by** *fastforce*
have 4: $\vdash (\exists n. \text{power } (f \wedge \text{more}) n) = f^*$
by (*simp add: chopstar-d-def powerstar-d-def*)
show ?thesis **using** 2 3 4 **by** *fastforce*
qed

lemma *ProjChopstarEqvChopstarProj*:
 $\vdash f \triangle (g^*) = (f \triangle g)^*$
proof –
have 1: $\vdash f \triangle (g^*) = f \triangle (g \triangle \#True)$
by (*metis ProjTrueEqvChopstar RightProjEqvProj inteq-reflection*)
have 2: $\vdash f \triangle (g \triangle \#True) = (f \triangle g) \triangle \#True$
by (*simp add: PJ7*)
have 3: $\vdash (f \triangle g) \triangle \#True = (f \triangle g)^*$
by (*simp add: ProjTrueEqvChopstar*)
show ?thesis **using** 1 2 3 **by** *fastforce*
qed

lemma *ProjAndImp*:
 $\vdash f \triangle (g1 \wedge g2) \longrightarrow f \triangle g1 \wedge f \triangle g2$
by (*meson Prop12 RightProjImpProj int-iffD1 lift-and-com*)

lemma *ProjOrDist*:

$\vdash \#True \triangle (f \vee g) = (\#True \triangle f \vee \#True \triangle g)$
using *PJ1sema* **by** *blast*

lemma *StateImportProj*:

$\vdash ((init\ w) \wedge f \triangle g) = f \triangle ((init\ w) \wedge g)$
by (*auto simp add: Valid-def init-defs projection-d-def filt-inth index-sequence-def*)

lemma *ProjStateAndNextEqvStateAndMoreChopProj*:

$\vdash f \triangle ((init\ w) \wedge \bigcirc g) = ((init\ w) \wedge (f \wedge more));(f \triangle g)$
proof –
have 2: $\vdash (f \wedge more);(f \triangle g) = f \triangle \bigcirc g$
by (*metis PJ3 PJ4 inteq-reflection next-d-def*)
have 3: $\vdash f \triangle ((init\ w)) \longrightarrow init\ w$
by (*simp add: PJ5*)
have 4: $\vdash (init\ w \wedge f \triangle \bigcirc g) = f \triangle ((init\ w) \wedge \bigcirc g)$
by (*simp add: StateImportProj*)
have 5: $\vdash f \triangle ((init\ w) \wedge \bigcirc g) \longrightarrow ((init\ w) \wedge (f \wedge more));(f \triangle g)$
using 2 3 *ProjAndImp* **by** *fastforce*
from 5 4 **show** *?thesis* **using** 2 **by** *fastforce*
qed

lemma *ProjNext*:

$\vdash f \triangle \bigcirc g = (f \wedge more);(f \triangle g)$
by (*metis PJ3 PJ4 inteq-reflection next-d-def*)

lemma *ProjWnext*:

$\vdash f \triangle (wnext\ g) = (empty \vee (f \wedge more));(f \triangle g)$
proof –
have 1: $\vdash f \triangle (wnext\ g) = f \triangle (empty \vee \bigcirc g)$
by (*simp add: RightProjEqvProj WnextEqvEmptyOrNext*)
have 2: $\vdash f \triangle (empty \vee \bigcirc g) = (empty \vee f \triangle (\bigcirc g))$
using *PJ1sema PJ2* **by** *fastforce*
have 3: $\vdash f \triangle (\bigcirc g) = (f \wedge more);(f \triangle g)$
by (*metis PJ3 PJ4 inteq-reflection next-d-def*)
show *?thesis*
using 1 2 3 **by** *fastforce*
qed

lemma *ProjIntro*:

assumes $\vdash f \wedge more \longrightarrow (g \wedge more); f$
shows $\vdash f \longrightarrow g \triangle \#True$
using *assms CSIntro ProjTrueEqvChopstar* **by** *force*

lemma *RightBoxStateImportProj*:

$\vdash \Box((init\ w) \wedge f \triangle g) \longrightarrow f \triangle (\Box((init\ w) \wedge g))$
by (*simp add: Valid-def always-defs init-defs projection-d-def*)
(metis filt-ilen filt-inth idx-bound-1 suffix-zero)

lemma *LeftBoxStateImportProjhelp*:

$(\forall n \leq \text{ilen } wa. w \langle \text{inth } wa \ n \rangle) \wedge$
 $(\exists l. \text{inth } l \ 0 = 0 \wedge$
 $(\forall n < \text{ilen } l. \text{inth } l \ n < \text{inth } l \ (\text{Suc } n)) \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } wa \wedge$
 $(\forall i < \text{ilen } l. f (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa)) \wedge$
 $g (\text{filt } wa \ l)) \longrightarrow$
 $(\exists l. \text{inth } l \ 0 = 0 \wedge$
 $(\forall n < \text{ilen } l. \text{inth } l \ n < \text{inth } l \ (\text{Suc } n)) \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } wa \wedge$
 $(\forall i < \text{ilen } l.$
 $f (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa) \wedge$
 $(\forall n \leq \text{ilen } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa).$
 $w \langle \text{inth } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa) \ n \rangle)) \wedge$
 $g (\text{filt } wa \ l))$

proof

assume 0 : $(\forall n \leq \text{ilen } wa. w \langle \text{inth } wa \ n \rangle) \wedge$
 $(\exists l. \text{inth } l \ 0 = 0 \wedge$
 $(\forall n < \text{ilen } l. \text{inth } l \ n < \text{inth } l \ (\text{Suc } n)) \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } wa \wedge$
 $(\forall i < \text{ilen } l. f (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa)) \wedge$
 $g (\text{filt } wa \ l))$

show $\exists l. \text{inth } l \ 0 = 0 \wedge$
 $(\forall n < \text{ilen } l. \text{inth } l \ n < \text{inth } l \ (\text{Suc } n)) \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } wa \wedge$
 $(\forall i < \text{ilen } l.$
 $f (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa) \wedge$
 $(\forall n \leq \text{ilen } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa).$
 $w \langle \text{inth } (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa) \ n \rangle)) \wedge$
 $g (\text{filt } wa \ l)$

proof –

have 1 : $(\forall n \leq \text{ilen } wa. w \langle \text{inth } wa \ n \rangle)$

using 0 **by** *auto*

have 2 : $(\exists l. \text{inth } l \ 0 = 0 \wedge$
 $(\forall n < \text{ilen } l. \text{inth } l \ n < \text{inth } l \ (\text{Suc } n)) \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } wa \wedge$
 $(\forall i < \text{ilen } l. f (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa)) \wedge$
 $g (\text{filt } wa \ l))$

using 0 **by** *auto*

obtain l **where** 3 : $\text{inth } l \ 0 = 0 \wedge$
 $(\forall n < \text{ilen } l. \text{inth } l \ n < \text{inth } l \ (\text{Suc } n)) \wedge$
 $\text{inth } l \ (\text{ilen } l) = \text{ilen } wa \wedge$
 $(\forall i < \text{ilen } l. f (\text{sub } (\text{inth } l \ i) (\text{inth } l \ (\text{Suc } i)) \ wa)) \wedge$
 $g (\text{filt } wa \ l)$

using 2 **by** *auto*

have 4 : $\text{inth } l \ 0 = 0$

using 3 **by** *auto*

have 5 : $(\forall n < \text{ilen } l. \text{inth } l \ n < \text{inth } l \ (\text{Suc } n))$

using 3 **by** *auto*

```

have 6:  $\text{inth } l (\text{ilen } l) = \text{ilen } wa$ 
  using 3 by auto
have 7:  $(\forall i < \text{ilen } l. f (\text{sub } ( \text{inth } l i) ( \text{inth } l (\text{Suc } i)) wa))$ 
  using 3 by auto
have 8:  $g (\text{filt } wa l)$ 
  using 3 by auto
have 9:  $(\forall i < \text{ilen } l.$ 
   $f (\text{sub } ( \text{inth } l i) ( \text{inth } l (\text{Suc } i)) wa) \wedge$ 
   $(\forall n \leq (\text{inth } l (\text{Suc } i)) - (\text{inth } l i).$ 
   $w \langle \text{inth } wa ((\text{inth } l i) + n) \rangle))$ 
by (metis 1 7 inth-last-stutter nat-le-iff-add nat-le-linear)
have 10:  $(\forall i < \text{ilen } l.$ 
   $\text{ilen } (\text{sub } ( \text{inth } l i) ( \text{inth } l (\text{Suc } i)) wa) =$ 
   $(\text{inth } l (\text{Suc } i)) - (\text{inth } l i) )$ 
  by (simp add: 3 PJ6help1 index-sequence-def)
have 11:  $(\forall i < \text{ilen } l.$ 
   $(\forall n \leq (\text{inth } l (\text{Suc } i)) - (\text{inth } l i).$ 
   $\text{inth } (\text{sub } ( \text{inth } l i) ( \text{inth } l (\text{Suc } i)) wa) n =$ 
   $\text{inth } wa ((\text{inth } l i) + n) )$ 
  using 3 index-sequence-def idx-expand by fastforce
have 12:  $(\forall i < \text{ilen } l.$ 
   $f (\text{sub } ( \text{inth } l i) ( \text{inth } l (\text{Suc } i)) wa) \wedge$ 
   $(\forall n \leq \text{ilen } (\text{sub } ( \text{inth } l i) ( \text{inth } l (\text{Suc } i)) wa).$ 
   $w \langle \text{inth } (\text{sub } ( \text{inth } l i) ( \text{inth } l (\text{Suc } i)) wa) n \rangle))$ 
  using 9 10 11 by simp
show ?thesis
using 12 3 by blast
qed
qed

```

lemma *LeftBoxStateImportProj:*

$\vdash \square(\text{init } w) \wedge f \triangle g \longrightarrow (f \wedge \square(\text{init } w)) \triangle g$

using *LeftBoxStateImportProjhelp*

by (simp add: index-sequence-def Valid-def always-defs init-defs projection-d-def powerinterval-def)
blast

8.6.2 dp and bp

lemma *NotDpEqvBpNot:*

$\vdash (\neg(dp f)) = bp (\neg f)$

by (simp add: bp-d-def dp-d-def uprojection-d-def)

lemma *NotBpEqvDpNot:*

$\vdash (\neg(bp f)) = dp (\neg f)$

by (simp add: bp-d-def dp-d-def uprojection-d-def)

lemma *NowImpDp:*

$\vdash f \longrightarrow dp f$

proof –
have 1: $\vdash (\text{skip} \longrightarrow \# \text{True})$
by *simp*
have 2: $\vdash \text{ba}(\text{skip} \longrightarrow \# \text{True})$
using 1 **by** (*simp add: BaGen*)
have 3: $\vdash \text{ba}(\text{skip} \longrightarrow \# \text{True}) \longrightarrow (\text{skip} \triangle f \longrightarrow \# \text{True} \triangle f)$
using PJ8 **by** *blast*
have 4: $\vdash (\text{skip} \triangle f \longrightarrow \# \text{True} \triangle f)$
using 2 3 MP **by** *blast*
show ?thesis
by (*metis 4 PJ6 dp-d-def inteq-reflection*)
qed

lemma *BpElim*:
 $\vdash \text{bp } f \longrightarrow f$
proof –
have 1: $\vdash \neg f \longrightarrow \text{dp } (\neg f)$
by (*simp add: NowImpDp*)
hence 2: $\vdash \neg(\text{dp } (\neg f)) \longrightarrow f$
by *auto*
from 2 **show** ?thesis
by (*simp add: bp-d-def dp-d-def uprojection-d-def*)
qed

lemma *BpImpDpImpDp*:
 $\vdash \text{bp } (f \longrightarrow g) \longrightarrow \text{dp } f \longrightarrow \text{dp } g$
proof –
have 1: $\vdash \text{bp } (f \longrightarrow g) \longrightarrow (\# \text{True} \triangle f) \longrightarrow (\# \text{True} \triangle g)$
by (*simp add: PJ9 bp-d-def*)
from 1 **show** ?thesis **by** (*simp add: dp-d-def*)
qed

lemma *BpContraPosImpDist*:
 $\vdash \text{bp } (\neg g \longrightarrow \neg f) \longrightarrow (\text{bp } f) \longrightarrow (\text{bp } g)$
proof –
have 1: $\vdash \text{bp } (\neg g \longrightarrow \neg f) \longrightarrow (\text{dp } (\neg g)) \longrightarrow (\text{dp } (\neg f))$
by (*rule BpImpDpImpDp*)
hence 2: $\vdash \text{bp } (\neg g \longrightarrow \neg f) \longrightarrow (\neg (\text{dp } (\neg f))) \longrightarrow (\neg (\text{dp } (\neg g)))$ **by** *auto*
from 2 **show** ?thesis
by (*simp add: bp-d-def dp-d-def uprojection-d-def*)
qed

lemma *BpImpDist*:
 $\vdash \text{bp } (f \longrightarrow g) \longrightarrow (\text{bp } f) \longrightarrow (\text{bp } g)$
proof –
have 1: $\vdash (f \longrightarrow g) \longrightarrow (\neg g \longrightarrow \neg f)$ **by** *auto*
hence 2: $\vdash \neg (\neg g \longrightarrow \neg f) \longrightarrow \neg (f \longrightarrow g)$ **by** *auto*

hence 3: $\vdash bp (\neg (\neg g \longrightarrow \neg f) \longrightarrow \neg (f \longrightarrow g))$ **by** (rule BpGen)
have 4: $\vdash bp (\neg (\neg g \longrightarrow \neg f) \longrightarrow \neg (f \longrightarrow g))$
 \longrightarrow
 $bp (f \longrightarrow g) \longrightarrow bp (\neg g \longrightarrow \neg f)$ **by** (rule BpContraPosImpDist)
have 5: $\vdash bp (f \longrightarrow g) \longrightarrow bp (\neg g \longrightarrow \neg f)$ **using** 3 4 MP **by** blast
have 6: $\vdash bp (\neg g \longrightarrow \neg f) \longrightarrow (bp f) \longrightarrow (bp g)$ **by** (rule BpContraPosImpDist)
from 5 6 **show** ?thesis **using** lift-imp-trans **by** blast
qed

lemma DpImpDpRule:

assumes $\vdash f \longrightarrow g$
shows $\vdash dp f \longrightarrow dp g$
proof –
have 1: $\vdash f \longrightarrow g$ **using** assms **by** auto
hence 2: $\vdash \#True \triangle f \longrightarrow \#True \triangle g$
by (metis BpGen MP PJ9 bp-d-def)
from 2 **show** ?thesis **by** (simp add: dp-d-def)
qed

lemma BpImpBpRule:

assumes $\vdash f \longrightarrow g$
shows $\vdash bp f \longrightarrow bp g$
proof –
have 1: $\vdash f \longrightarrow g$ **using** assms **by** auto
hence 2: $\vdash \neg g \longrightarrow \neg f$ **by** auto
hence 3: $\vdash dp (\neg g) \longrightarrow dp (\neg f)$ **by** (rule DpImpDpRule)
hence 4: $\vdash \neg (dp (\neg f)) \longrightarrow \neg (dp (\neg g))$ **by** auto
from 4 **show** ?thesis
by (meson BpGen BpImpDist MP assms)
qed

lemma DpEqvDpRule:

assumes $\vdash f = g$
shows $\vdash dp f = dp g$
proof –
have 1: $\vdash f = g$ **using** assms **by** auto
hence 2: $\vdash \#True \triangle f = \#True \triangle g$
using RightProjEqvProj **by** blast
from 2 **show** ?thesis **by** (simp add: dp-d-def)
qed

lemma BpEqvBpRule:

assumes $\vdash f = g$
shows $\vdash bp f = bp g$
proof –
have 1: $\vdash f = g$ **using** assms **by** auto
hence 2: $\vdash (\neg f) = (\neg g)$ **by** auto
hence 3: $\vdash dp (\neg f) = dp (\neg g)$ **by** (rule DpEqvDpRule)
hence 4: $\vdash \neg (dp (\neg f)) = \neg (dp (\neg g))$ **by** auto
from 4 **show** ?thesis

by (metis BpImpBpRule assms int-iffD1 int-iffI inteq-reflection)
qed

lemma DpState:

$\vdash dp \ (init \ w) = (init \ w)$

by (metis NowImpDp PJ5 dp-d-def int-iffI)

lemma StateEqvBp:

$\vdash (init \ w) = bp \ (init \ w)$

proof –

have 1: $\vdash (init \ w) \longrightarrow bp \ (init \ w)$

by (metis (no-types, lifting) DiState DpState Initprop(2) StateEqvBi bi-d-def bp-d-def dp-d-def int-iffD1 inteq-reflection uprojection-d-def)

have 2: $\vdash bp \ (init \ w) \longrightarrow (init \ w)$ by (rule BpElim)

from 1 2 show ?thesis by fastforce

qed

lemma DpDpEqvDp:

$\vdash dp \ (dp \ f) = dp \ f$

proof –

have 2: $\vdash \#True \triangle (\#True \triangle f) = (\#True \triangle \#True) \triangle f$

by (simp add: PJ7)

have 3: $\vdash (\#True \triangle \#True) = \#True$

by (metis DpState Initprop(4) dp-d-def int-eq-true inteq-reflection)

show ?thesis by (metis 2 3 dp-d-def inteq-reflection)

qed

lemma BpBpEqvBp:

$\vdash bp \ (bp \ f) = bp \ f$

proof –

have 1: $\vdash dp \ (dp \ (\neg f)) = dp \ (\neg f)$

using DpDpEqvDp by blast

have 2: $\vdash (\neg (dp \ (dp \ (\neg f)))) = (\neg (dp \ (\neg f)))$

using 1 by auto

have 3: $\vdash (\neg (dp \ (\neg f))) = bp \ f$

by (simp add: bp-d-def dp-d-def uprojection-d-def)

have 4: $\vdash (\neg (dp \ (dp \ (\neg f)))) = bp \ (bp \ f)$

by (simp add: bp-d-def dp-d-def uprojection-d-def)

from 2 3 4 show ?thesis

by fastforce

qed

lemma DpOrEqv:

$\vdash dp \ (f \vee g) = (dp \ f \vee dp \ g)$

proof –

have 1: $\vdash \#True \triangle (f \vee g) = (\#True \triangle f \vee \#True \triangle g)$

using *ProjOrDist* **by** *auto*
from 1 **show** *?thesis* **by** (*simp add: dp-d-def*)
qed

lemma *BpAndEqv*:

$\vdash bp(f \wedge g) = (bp\ f \wedge bp\ g)$

proof –

have 1: $\vdash dp\ ((\neg f) \vee (\neg g)) = (dp\ (\neg f) \vee dp\ (\neg g))$

using *DpOrEqv* **by** *auto*

hence 2: $\vdash (\neg (dp\ ((\neg f) \vee (\neg g)))) = (\neg (dp\ (\neg f) \vee dp\ (\neg g)))$

by *auto*

have 3: $\vdash (\neg (dp\ ((\neg f) \vee (\neg g)))) = bp\ (\neg((\neg f) \vee (\neg g)))$

using *NotDpEqvBpNot* **by** *blast*

have 4: $\vdash (\neg((\neg f) \vee (\neg g))) = (f \wedge g)$

by *auto*

hence 5: $\vdash bp(\neg((\neg f) \vee (\neg g))) = bp(f \wedge g)$

by (*simp add: BpEqvBpRule*)

have 6: $\vdash (\neg(dp\ (\neg f) \vee dp\ (\neg g))) = ((\neg(dp\ (\neg f))) \wedge (\neg(dp\ (\neg g))))$

by *auto*

have 7: $\vdash ((\neg(dp\ (\neg f))) \wedge (\neg(dp\ (\neg g)))) = (bp\ f \wedge bp\ g)$

by (*simp add: bp-d-def dp-d-def uprojection-d-def*)

show *?thesis*

by (*metis 2 3 4 6 7 inteq-reflection*)

qed

lemma *DpAndA*:

$\vdash dp\ (f \wedge g) \longrightarrow dp\ f$

proof –

have 1: $\vdash \#True \triangle (f \wedge g) \longrightarrow \#True \triangle f$

by (*meson Prop12 RightProjImpProj int-iffD1 lift-and-com*)

from 1 **show** *?thesis* **by** (*simp add: dp-d-def*)

qed

lemma *BpOrA*:

$\vdash bp\ f \longrightarrow bp(f \vee g)$

by (*simp add: BpImpBpRule intI*)

lemma *BpOrB*:

$\vdash bp\ g \longrightarrow bp(f \vee g)$

by (*simp add: BpImpBpRule intI*)

lemma *BpOrImpOr*:

$\vdash bp\ f \vee bp\ g \longrightarrow bp(f \vee g)$

using *BpOrA BpOrB* **by** *fastforce*

lemma *DpAndB*:

$\vdash dp\ (f \wedge g) \longrightarrow dp\ g$

proof –

have 1: $\vdash \#True \triangle (f \wedge g) \longrightarrow \#True \triangle g$

by (meson Prop12 RightProjImpProj int-iffD2 lift-and-com)
 from 1 show ?thesis by (simp add: dp-d-def)
 qed

lemma *DpAndImpAnd*:

$\vdash dp (f \wedge g) \longrightarrow dp f \wedge dp g$

proof –

have 1: $\vdash dp (f \wedge g) \longrightarrow dp f$ by (rule DpAndA)

have 2: $\vdash dp (f \wedge g) \longrightarrow dp g$ by (rule DpAndB)

from 1 2 show ?thesis by fastforce

qed

lemma *DpSkipEqvMore*:

$\vdash dp skip = more$

proof –

have 1: $\vdash dp skip = \#True \triangle skip$

by (simp add: dp-d-def)

have 2: $\vdash \#True \triangle skip = (\#True \wedge more)$

using PJ3 by blast

have 3: $\vdash (\#True \wedge more) = more$

by auto

from 1 2 3 show ?thesis by fastforce

qed

lemma *DpMoreEqvMore*:

$\vdash dp more = more$

by (metis DpDpEqvDp DpSkipEqvMore inteq-reflection)

lemma *BpEmptyEqvEmpty*:

$\vdash bp empty = empty$

by (metis DpMoreEqvMore NotDpEqvBpNot empty-d-def inteq-reflection)

lemma *DpEmptyEqvEmpty*:

$\vdash dp empty = empty$

proof –

have 1: $\vdash dp empty = \#True \triangle empty$

by (simp add: dp-d-def)

have 2: $\vdash \#True \triangle empty = empty$

by (simp add: PJ2)

from 1 2 show ?thesis by fastforce

qed

lemma *BpMoreEqvMore*:

$\vdash bp more = more$

by (metis DpEmptyEqvEmpty NotDpEqvBpNot NotEmptyEqvMore inteq-reflection)

lemma *NextDpImpDpNext*:

```

  ⊢ ○ (dp f) → dp (○ f)
proof -
  have 1: ⊢ dp(○ f) = #True △ (skip;f)
    by (simp add: dp-d-def next-d-def)
  have 2: ⊢ #True △ (skip;f) = (#True △ skip);(#True △ f)
    by (simp add: PJ4)
  have 3: ⊢ (#True △ skip) = (#True ∧ more)
    using PJ3 by blast
  have 4: ⊢ (#True ∧ more) = more
    by auto
  have 5: ⊢ skip;(#True △ f) → more;(#True △ f)
    by (metis DpSkipEqvMore LeftChopImpChop NowImpDp inteq-reflection)
  show ?thesis
    by (metis 2 3 4 5 dp-d-def inteq-reflection next-d-def)
qed

lemma BoxStateImportBp:
  ⊢ □(init w) → bp(□(init w))
by (simp add: Valid-def always-defs init-defs projection-d-def bp-d-def uprojection-d-def
    powerinterval-def)
  (metis (mono-tags, lifting) filt-expand idx-bound-1 suffix-zero)

lemma BoxStateEqvBpBoxState:
  ⊢ □(init w) = bp(□(init w))
proof -
  have 1: ⊢ bp(□(init w)) → □(init w)
    by (simp add: BpElim)
  have 2: ⊢ bp(□(init w)) = (¬(#True △ (¬ □(init w))))
    by (simp add: bp-d-def uprojection-d-def)
  have 2: ⊢ □(init w) → dp(□(init w))
    by (metis NowImpDp)
  have 2: ⊢ □(init w) → bp(□(init w))
    using BoxStateImportBp by auto
  from 1 2 show ?thesis by fastforce
qed

end

```

9 The First Occurrence Operator in finite ITL

```

theory First
imports
  Theorems TimeReversal
begin

```

Runtime verification (RV) has gained significant interest in recent years. The behaviour of a program can be verified in real time by analysing its evolving trace. This approach has two significant benefits over static verification techniques such as model checking. Firstly, it is only necessary to verify actual

execution paths rather than all possible paths. Secondly, it is possible to react at runtime should the program diverge from its specified behaviour. RV does not replace traditional verification techniques but it does provide an extra layer of security.

Linear Temporal Logic (LTL) is a popular formalism for writing specifications from which RV monitors can be derived automatically. By contrast, Interval Temporal Logic (ITL) has not been as widely represented in this field despite being more expressive and compositional. The principal issue is efficiency. ITL uses non-deterministic operators to IConstruct sequential and iterative specifications (chop and chop-star, respectively) and these introduce combinatorial complexity. Approaches to mitigate this include using a deterministic subset of ITL or adapting the semantics to include a deterministic chop operator. This work proposes an alternative approach, wholly within existing ITL, and based upon a new, derived operator called “first occurrence”.

A theory of first occurrence is developed and used to derive an algebra of RV monitors.

9.1 Definitions

9.1.1 Definitions Strict Initial and Final

definition $bs-d :: ('a::world) formula \Rightarrow 'a formula$
where
 $bs-d f \equiv LIFT(empty \vee ((bi f) ; skip))$

definition $bt-d :: ('a::world) formula \Rightarrow 'a formula$
where
 $bt-d f \equiv LIFT(empty \vee (skip;(\Box f)))$

syntax
 $-bs-d :: lift \Rightarrow lift ((bs -) [88] 87)$
 $-bt-d :: lift \Rightarrow lift ((bt -) [88] 87)$

syntax (*ASCII*)
 $-bs-d :: lift \Rightarrow lift ((bs -) [88] 87)$
 $-bt-d :: lift \Rightarrow lift ((bt -) [88] 87)$

translations
 $-bs-d \rightleftharpoons CONST bs-d$
 $-bt-d \rightleftharpoons CONST bt-d$

definition $ds-d :: ('a::world) formula \Rightarrow 'a formula$
where
 $ds-d f \equiv LIFT(\neg (bs (\neg f)))$

definition $dt-d :: ('a::world) formula \Rightarrow 'a formula$
where
 $dt-d f \equiv LIFT(\neg (bt (\neg f)))$

syntax
 $-ds-d :: lift \Rightarrow lift ((ds -) [88] 87)$
 $-dt-d :: lift \Rightarrow lift ((dt -) [88] 87)$

syntax (*ASCII*)

$-ds-d :: lift \Rightarrow lift ((ds \ -) [88] \ 87)$

$-dt-d :: lift \Rightarrow lift ((dt \ -) [88] \ 87)$

translations

$-ds-d \Rightarrow CONST \ ds-d$

$-dt-d \Rightarrow CONST \ dt-d$

9.1.2 Definition First and Last Operators

definition $first-d :: ('a::world) \ formula \Rightarrow 'a \ formula$

where

$first-d \ f \equiv LIFT \ (f \wedge \ (bs \ (\neg \ f)))$

definition $last-d :: ('a::world) \ formula \Rightarrow 'a \ formula$

where

$last-d \ f \equiv LIFT \ (f \wedge \ (bt \ (\neg \ f)))$

syntax

$-first-d :: lift \Rightarrow lift ((\triangleright \ -) [88] \ 87)$

$-last-d :: lift \Rightarrow lift ((\triangleleft \ -) [88] \ 87)$

syntax (*ASCII*)

$-first-d :: lift \Rightarrow lift ((first \ -) [88] \ 87)$

$-last-d :: lift \Rightarrow lift ((last \ -) [88] \ 87)$

translations

$-first-d \Rightarrow CONST \ first-d$

$-last-d \Rightarrow CONST \ last-d$

9.2 First and Time Reversal

lemma *BsEqvRule*:

assumes $\vdash f = g$

shows $\vdash bs \ f = bs \ g$

proof –

have $1: \vdash f = g$ **using** *assms* **by** *auto*

hence $2: \vdash bi(f) = bi(g)$ **by** (*simp add: BiEqvBi*)

hence $3: \vdash bi(f);skip = bi(g);skip$ **by** (*simp add: LeftChopEqvChop*)

hence $4: \vdash (empty \vee bi(f);skip) = (empty \vee bi(g);skip)$ **by** *auto*

hence $5: \vdash bs(f) = bs(g)$ **by** (*simp add: bs-d-def*)

from $1 \ 2 \ 3 \ 4 \ 5$ **show** *?thesis* **by** *auto*

qed

lemma *BtEqvRule*:

assumes $\vdash f = g$

shows $\vdash bt \ f = bt \ g$

proof –

have $1: \vdash f = g$ **using** *assms* **by** *auto*

hence $2: \vdash \Box(f) = \Box(g)$ **by** (*simp add: BoxEqvBox*)

hence 3: $\vdash \text{skip};\Box(f) = \text{skip};\Box(g)$ **using** *RightChopEqvChop* **by** *blast*
 hence 4: $\vdash (\text{empty} \vee \text{skip};\Box(f)) = (\text{empty} \vee \text{skip};\Box(g))$ **by** *auto*
 hence 5: $\vdash \text{bt}(f) = \text{bt}(g)$ **by** (*simp add: bt-d-def*)
 from 1 2 3 4 5 **show** *?thesis* **by** *auto*
qed

lemma *FstEqvRule*:

assumes $\vdash f = g$
 shows $\vdash \triangleright f = \triangleright g$

proof –

have 1: $\vdash f = g$ **using** *assms* **by** *auto*
 hence 2: $\vdash (\neg f) = (\neg g)$ **by** *auto*
 hence 3: $\vdash \text{bs}(\neg f) = \text{bs}(\neg g)$ **by** (*simp add: BsEqvRule*)
 hence 4: $\vdash (f \wedge \text{bs}(\neg f)) = (g \wedge \text{bs}(\neg g))$ **using** 1 **by** *fastforce*
 from 4 **show** *?thesis* **by** (*simp add: first-d-def*)

qed

lemma *LstEqvRule*:

assumes $\vdash f = g$
 shows $\vdash \triangleleft f = \triangleleft g$

proof –

have 1: $\vdash f = g$ **using** *assms* **by** *auto*
 hence 2: $\vdash (\neg f) = (\neg g)$ **by** *auto*
 hence 3: $\vdash \text{bt}(\neg f) = \text{bt}(\neg g)$ **by** (*simp add: BtEqvRule*)
 hence 4: $\vdash (f \wedge \text{bt}(\neg f)) = (g \wedge \text{bt}(\neg g))$ **using** 1 **by** *fastforce*
 from 4 **show** *?thesis* **by** (*simp add: last-d-def*)

qed

lemma *RBsEqvBt*:

$\vdash (\text{bs } f)^r = (\text{bt } (f^r))$

proof –

have 1: $\vdash (\text{bs } f)^r = (\text{empty} \vee ((\text{bi } f) ; \text{skip}))^r$
 by (*simp add: bs-d-def*)
 have 2: $\vdash (\text{empty} \vee ((\text{bi } f) ; \text{skip}))^r = (\text{empty}^r \vee ((\text{bi } f) ; \text{skip})^r)$
 using *ROr* **by** *blast*
 have 3: $\vdash (\text{empty}^r \vee ((\text{bi } f) ; \text{skip})^r) = (\text{empty} \vee (\text{skip}^r; (\text{bi } f)^r))$
 using *REmptyEqvEmpty RevChop* **by** *fastforce*
 have 4: $\vdash (\text{empty} \vee (\text{skip}^r; (\text{bi } f)^r)) = (\text{empty} \vee (\text{skip}; \Box(f^r)))$
 by (*metis 3 RBiEqvBox RevSkip int-eq*)
 have 5: $\vdash (\text{empty} \vee (\text{skip}; \Box(f^r))) = (\text{bt } (f^r))$
 by (*simp add: bt-d-def*)
 from 1 2 3 4 5 **show** *?thesis* **by** *fastforce*

qed

lemma *RRBsEqvBt*:

$\vdash (\text{bs } (f^r))^r = (\text{bt } (f))$

proof –

have 1: $\vdash (\text{bs } (f^r))^r = \text{bt } ((f^r)^r)$ **using** *RBsEqvBt* **by** *blast*
 have 2: $\vdash \text{bt } ((f^r)^r) = \text{bt } f$ **using** *EqvReverseReverse* **using** *BtEqvRule* **by** *blast*
 from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RBtEqvBs*:

$\vdash (bt\ f)^r = (bs\ (f^r))$

proof –

have 1: $\vdash (bt\ f)^r = (empty \vee (skip; \square\ f))^r$

by (*simp add: bt-d-def*)

have 2: $\vdash (empty \vee (skip; \square\ f))^r = (empty^r \vee (skip; \square\ f)^r)$

using *ROr* **by** *blast*

have 3: $\vdash (empty^r \vee (skip; \square\ f)^r) = (empty \vee (\square\ f)^r; skip^r)$

using *REmptyEqvEmpty RevChop* **by** *fastforce*

have 4: $\vdash (empty \vee (\square\ f)^r; skip^r) = (empty \vee (bi\ (f^r)); skip)$

by (*metis 3 RBoxEqvBi RevSkip int-eq*)

show *?thesis*

by (*metis 3 4 all-rev-eq(3) bs-d-def bt-d-def inteq-reflection*)

qed

lemma *RRBtEqvBs*:

$\vdash (bt\ (f^r))^r = (bs\ (f))$

proof –

have 1: $\vdash (bt\ (f^r))^r = bs\ ((f^r)^r)$ **using** *RBtEqvBs* **by** *blast*

have 2: $\vdash bs\ ((f^r)^r) = bs\ f$ **using** *EqvReverseReverse* **using** *BsEqvRule* **by** *blast*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RFirstEqvLast*:

$\vdash (\triangleright\ f)^r = (\triangleleft\ (f^r))$

proof –

have 1: $\vdash (\triangleright\ f)^r = (f \wedge bs(\neg f))^r$ **by** (*simp add: first-d-def*)

have 2: $\vdash (f \wedge bs(\neg f))^r = (f^r \wedge (bs\ (\neg f))^r)$ **using** *RAnd* **by** *blast*

have 3: $\vdash (f^r \wedge (bs\ (\neg f))^r) = (f^r \wedge bt\ ((\neg f)^r))$ **using** *RBsEqvBt* **by** *fastforce*

have 4: $\vdash (f^r \wedge bt\ ((\neg f)^r)) = (f^r \wedge bt\ (\neg(f^r)))$ **using** *RNot int-eq* **by** *fastforce*

have 5: $\vdash (f^r \wedge bt\ (\neg(f^r))) = (\triangleleft\ (f^r))$ **by** (*simp add: last-d-def*)

from 1 2 3 4 5 **show** *?thesis* **by** *fastforce*

qed

lemma *RRFirstEqvLast*:

$\vdash (\triangleright\ (f^r))^r = (\triangleleft\ (f))$

proof –

have 1: $\vdash (\triangleright\ (f^r))^r = \triangleleft\ ((f^r)^r)$ **using** *RFirstEqvLast* **by** *blast*

have 2: $\vdash \triangleleft\ ((f^r)^r) = \triangleleft\ f$ **using** *EqvReverseReverse* **using** *LstEqvRule* **by** *blast*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *RLastEqvFirst*:

$\vdash (\triangleleft\ f)^r = (\triangleright\ (f^r))$

proof –

have 1: $\vdash (\triangleleft\ f)^r = (f \wedge bt(\neg f))^r$ **by** (*simp add: last-d-def*)

have 2: $\vdash (f \wedge bt(\neg f))^r = (f^r \wedge (bt\ (\neg f))^r)$ **using** *RAnd* **by** *blast*

have 3: $\vdash (f^r \wedge (bt\ (\neg f))^r) = (f^r \wedge bs\ ((\neg f)^r))$ **using** *RBtEqvBs* **by** *fastforce*

have 4: $\vdash (f^r \wedge bs(\neg f)^r) = (f^r \wedge bs(\neg(f^r)))$ **using** *RNot int-eq* **by** *fastforce*
have 5: $\vdash (f^r \wedge bs(\neg(f^r))) = (\triangleright(f^r))$ **by** (*simp add: first-d-def*)
from 1 2 3 4 5 **show** *?thesis* **by** *fastforce*
qed

lemma *RRLastEqvFirst*:

$\vdash (\triangleleft(f^r))^r = (\triangleright(f))$

proof –

have 1: $\vdash (\triangleleft(f^r))^r = \triangleright((f^r)^r)$ **using** *RLastEqvFirst* **by** *blast*

have 2: $\vdash \triangleright((f^r)^r) = \triangleright f$ **using** *EqvReverseReverse* **using** *FstEqvRule* **by** *blast*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

9.3 Semantic Theorems

9.3.1 Semantics First and Last Operators

lemma *FstAndBisem*:

$(ilen\ \sigma > 0 \wedge (\sigma \models f) \wedge (\sigma \models bi(\neg f); skip)) =$
 $(ilen\ \sigma > 0 \wedge (\sigma \models f) \wedge (\forall ia < ilen\ (\sigma). (prefix\ ia\ \sigma \models \neg f)))$

proof –

have $(ilen\ \sigma > 0 \wedge (\sigma \models f) \wedge (\sigma \models bi(\neg f); skip)) =$
 $(0 < ilen\ \sigma \wedge (\sigma \models f) \wedge$
 $(\exists i. (i \leq ilen\ \sigma \longrightarrow (\forall ia \leq i. \neg (prefix\ ia\ (prefix\ i\ \sigma) \models f))) \wedge$
 $ilen\ \sigma - i = Suc\ 0) \wedge i \leq ilen\ \sigma)$
 $)$

using *le-trans* **by** (*auto simp add: chop-defs bi-defs skip-defs, blast*)

also have ... =

$(0 < ilen\ \sigma \wedge (\sigma \models f) \wedge$
 $(\exists i. (i \leq ilen\ \sigma \longrightarrow (\forall ia \leq i. \neg (prefix\ ia\ (prefix\ i\ \sigma) \models f))) \wedge$
 $i = ilen\ \sigma - Suc\ 0) \wedge i \leq ilen\ \sigma)$
 $)$

by *auto*

also have ... =

$(0 < ilen\ \sigma \wedge (\sigma \models f) \wedge$
 $(\forall ia \leq (ilen\ \sigma - Suc\ 0). \neg (prefix\ ia\ (prefix\ (ilen\ \sigma - Suc\ 0)\ \sigma) \models f))$
 $)$

using *diff-le-self* **by** *blast*

also have ... =

$(ilen\ \sigma > 0 \wedge (\sigma \models f) \wedge$
 $(\forall ia < ilen\ (\sigma). \neg (prefix\ ia\ (prefix\ (ilen\ \sigma - Suc\ 0)\ \sigma) \models f))$
 $)$ **by** (*metis Suc-pred less-Suc-eq-le*)

also have ... =

$(ilen\ \sigma > 0 \wedge (\sigma \models f) \wedge$
 $(\forall ia < ilen\ (\sigma). (prefix\ ia\ (prefix\ (ilen\ \sigma - Suc\ 0)\ \sigma) \models \neg f))$
 $)$

by *auto*

also have ... =

$(ilen\ \sigma > 0 \wedge (\sigma \models f) \wedge (\forall ia < ilen\ (\sigma). (prefix\ ia\ \sigma \models \neg f)))$
by (*simp add: pref-pref-help*)

finally show $(ilen\ \sigma > 0 \wedge (\sigma \models f) \wedge (\sigma \models bi(\neg f); skip)) =$

$$(\text{ilen } \sigma > 0 \wedge (\sigma \models f) \wedge (\forall ia < \text{ilen } (\sigma). (\text{prefix } ia \ \sigma \models \neg f))) \ .$$

qed

lemma *Fstsem-0*:

$$\begin{aligned} (\sigma \models \triangleright f) = & \\ (& \\ & (\sigma \models f \wedge \text{empty}) \vee (\text{ilen } \sigma > 0 \wedge (\sigma \models f) \wedge (\sigma \models \text{bi } (\neg f); \text{skip})) \\ &) \end{aligned}$$

using *empty-defs* **by** (*auto simp add: first-d-def bs-d-def*)

lemma *Emptysem*:

$$(\sigma \models f \wedge \text{empty}) = ((\sigma \models f) \wedge \text{ilen } \sigma = 0)$$

using *empty-defs* **by** *auto*

lemma *Fstsem*:

$$\begin{aligned} (\sigma \models \triangleright f) = & \\ (& \\ & ((\sigma \models f) \wedge \text{ilen } \sigma = 0) \vee \\ & (\text{ilen } \sigma > 0 \wedge (\sigma \models f) \wedge (\forall ia < \text{ilen } (\sigma). (\text{prefix } ia \ \sigma \models \neg f))) \\ &) \end{aligned}$$

using *Fstsem-0 Emptysem FstAndBisem* **by** *metis*

lemma *Lstsem*:

$$\begin{aligned} (\sigma \models \triangleleft f) = & \\ (& \\ & ((\sigma \models f) \wedge \text{ilen } \sigma = 0) \vee \\ & (\text{ilen } \sigma > 0 \wedge (\sigma \models f) \wedge (\forall ia < \text{ilen } \sigma. (\text{suffix } ((\text{ilen } \sigma) - ia) \ \sigma \models \neg f))) \\ &) \end{aligned}$$

proof –

$$\text{have } (\sigma \models \triangleleft f) = (\sigma \models (\triangleright (f^r))^r)$$

using *RRFirstEqvLast* **by** *fastforce*

$$\text{also have } \dots = (\text{irev } \sigma \models \triangleright (f^r))$$

by (*metis reverse-d-def*)

$$\text{also have } \dots =$$

$$\begin{aligned} (& \\ & ((\text{irev } \sigma \models f^r) \wedge \text{ilen } (\text{irev } \sigma) = 0) \vee \\ & (\text{ilen } (\text{irev } \sigma) > 0 \wedge (\text{irev } \sigma \models f^r) \wedge \\ & (\forall ia < \text{ilen } (\text{irev } \sigma). (\text{prefix } ia \ (\text{irev } \sigma) \models \neg(f^r)))) \\ &) \end{aligned}$$

using *Fstsem* **by** *blast*

$$\text{also have } \dots =$$

$$\begin{aligned} (& \\ & ((\sigma \models f) \wedge \text{ilen } (\sigma) = 0) \vee \\ & (\text{ilen } (\sigma) > 0 \wedge (\sigma \models f) \wedge \\ & (\forall ia < \text{ilen } (\text{irev } \sigma). (\text{prefix } ia \ (\text{irev } \sigma) \models (\neg(f))^r))) \\ &) \end{aligned}$$

by (*simp add: reverse-d-def*)

$$\text{also have } \dots =$$

$$\begin{aligned} (& \\ & ((\sigma \models f) \wedge \text{ilen } (\sigma) = 0) \vee \end{aligned}$$

```

  ( ilen ( $\sigma$ ) > 0  $\wedge$  ( $\sigma \models f$ )  $\wedge$ 
    ( $\forall ia < i$ len (irev  $\sigma$ ). (irev (prefix ia (irev  $\sigma$ ))  $\models (\neg(f))$ )))
)
by (simp add: reverse-d-def)
also have ... =
  (
    ( ( $\sigma \models f$ )  $\wedge$  ilen ( $\sigma$ ) = 0)  $\vee$ 
    ( ilen ( $\sigma$ ) > 0  $\wedge$  ( $\sigma \models f$ )  $\wedge$ 
      ( $\forall ia < i$ len ( $\sigma$ ). ( suffix ((ilen  $\sigma$ ) - ia) ( $\sigma$ )  $\models (\neg(f))$ )))
  )
  by (simp add: irev-prefix)
finally show
  ( $\sigma \models \triangleleft f$ ) =
  ( ( ( $\sigma \models f$ )  $\wedge$  ilen  $\sigma$  = 0)  $\vee$ 
    ( ilen  $\sigma$  > 0  $\wedge$  ( $\sigma \models f$ )  $\wedge$ 
      ( $\forall ia < i$ len  $\sigma$ . (suffix ((ilen  $\sigma$ ) - ia)  $\sigma \models \neg f$ )) )
  ) .
qed

```

9.3.2 Various Semantic Lemmas

lemma *DiLensem*:

```

( $\sigma \models di$  ( $f \wedge len(i)$ )) =
  ( (prefix i  $\sigma \models f$ )  $\wedge$  i ≤ ilen  $\sigma$ )

```

using *prefix-ilen-good* **by** (*auto simp add: itl-defs*)

lemma *PrefixFstsem*:

```

( (prefix i  $\sigma \models \triangleright f$ )  $\wedge$  i ≤ ilen  $\sigma$ ) =
  ( i ≤ ilen  $\sigma \wedge$ 
    (
      ( (prefix i  $\sigma \models f$ )  $\wedge$  i = 0)  $\vee$ 
      ( i > 0  $\wedge$  (prefix i  $\sigma \models f$ )  $\wedge$  ( $\forall ia < i$ . (prefix ia  $\sigma \models \neg f$ )))
    )
  )

```

proof –

```

have 1: ( (prefix i  $\sigma \models \triangleright f$ ) ) =
  (
    ( ((prefix i  $\sigma \models f$ )  $\wedge$  ilen (prefix i  $\sigma$ ) = 0)  $\vee$ 
    ( ilen (prefix i  $\sigma$ ) > 0  $\wedge$  ((prefix i  $\sigma \models f$ )  $\wedge$ 
      ( $\forall ia < i$ len (prefix i  $\sigma$ ). (prefix ia (prefix i  $\sigma$ )  $\models \neg f$ )) )
    )
  )

```

using *Fstsem* **by** *blast*

```

hence 2: ( (prefix i  $\sigma \models \triangleright f$ )  $\wedge$  i ≤ ilen  $\sigma$ ) =
  ( i ≤ ilen  $\sigma \wedge$  (
    ( ((prefix i  $\sigma \models f$ )  $\wedge$  ilen (prefix i  $\sigma$ ) = 0)  $\vee$ 
    ( ilen (prefix i  $\sigma$ ) > 0  $\wedge$  ((prefix i  $\sigma \models f$ )  $\wedge$ 
      ( $\forall ia < i$ len (prefix i  $\sigma$ ). (prefix ia (prefix i  $\sigma$ )  $\models \neg f$ )) )
    )
  )

```

by *auto*

hence 3: $((\text{prefix } i \ \sigma) \models \triangleright f) \wedge i \leq \text{ilen } \sigma =$
 $(i \leq \text{ilen } \sigma \wedge$
 $((\text{prefix } i \ \sigma) \models f) \wedge i = 0) \vee$
 $(i > 0 \wedge ((\text{prefix } i \ \sigma) \models f) \wedge (\forall ia < i. (\text{prefix } ia \ (\text{prefix } i \ \sigma) \models \neg f)))$
 $)$
 $)$

by *auto*

hence 4: $((\text{prefix } i \ \sigma) \models \triangleright f) \wedge i \leq \text{ilen } \sigma =$
 $(i \leq \text{ilen } \sigma \wedge$
 $((\text{prefix } i \ \sigma) \models f) \wedge i = 0) \vee$
 $(i > 0 \wedge ((\text{prefix } i \ \sigma) \models f) \wedge (\forall ia < i. (\text{prefix } ia \ \sigma \models \neg f)))$
 $)$
 $)$
using *pref-pref-3* **using** *less-imp-add-positive* **by** *fastforce*
from 4 **show** *?thesis* **by** *auto*
qed

lemma *PrefixFstAndsem:*

$((\text{prefix } i \ \sigma \models \triangleright f \wedge g) \wedge i \leq \text{ilen } \sigma) =$
 $(i \leq \text{ilen } \sigma \wedge$
 $($
 $((\text{prefix } i \ \sigma \models f \wedge g) \wedge i = 0) \vee$
 $(i > 0 \wedge (\text{prefix } i \ \sigma \models f \wedge g) \wedge (\forall ia < i. (\text{prefix } ia \ \sigma \models \neg f)))$
 $)$
 $)$

using *PrefixFstsem[of f i σ]* **by** *(metis unl-lift2)*

lemma *DiLenFstsem:*

$(\sigma \models di (\triangleright f \wedge \text{len}(i))) =$
 $(i \leq \text{ilen } \sigma \wedge$
 $($
 $((\text{prefix } i \ \sigma \models f) \wedge i = 0) \vee$
 $(i > 0 \wedge (\text{prefix } i \ \sigma \models f) \wedge (\forall ia < i. (\text{prefix } ia \ \sigma \models \neg f)))$
 $)$
 $)$

by *(simp add: DiLensem PrefixFstsem)*

lemma *DiLenFstAndsem:*

$(\sigma \models di ((\triangleright f \wedge g) \wedge \text{len}(i))) =$
 $(i \leq \text{ilen } \sigma \wedge$
 $($
 $((\text{prefix } i \ \sigma \models f \wedge g) \wedge i = 0) \vee$
 $(i > 0 \wedge (\text{prefix } i \ \sigma \models f \wedge g) \wedge (\forall ia < i. (\text{prefix } ia \ \sigma \models \neg f)))$
 $)$
 $)$

using *DiLensem PrefixFstAndsem* **by** *metis*

lemma *FstLenSamesem:*

$((i \leq \text{ilen } \sigma \wedge$
 $($

$$\begin{aligned}
& ((prefix\ i\ \sigma \models f) \wedge i = 0) \vee \\
& (i > 0 \wedge (prefix\ i\ \sigma \models f) \wedge (\forall ia < i. (prefix\ ia\ \sigma \models \neg f))) \\
&) \\
&) \wedge \\
& (j \leq len\ \sigma \wedge \\
& (\\
& ((prefix\ j\ \sigma \models f) \wedge j = 0) \vee \\
& (j > 0 \wedge (prefix\ j\ \sigma \models f) \wedge (\forall ia < j. (prefix\ ia\ \sigma \models \neg f))) \\
&) \\
&) \\
&) \longrightarrow (i=j)
\end{aligned}$$

by (*metis not-less-iff-gr-or-eq unl-lift*)

9.4 Theorems

9.4.1 Fixed length intervals

lemma *LenZeroEqvEmpty*:

$\vdash len(0) = empty$

by (*simp add: len-d-def*)

lemma *LenOneEqvSkip*:

$\vdash len(1) = skip$

by (*simp add: len-d-def ChopEmpty*)

lemma *LenNPlusOneA*:

$\vdash len(n+1) = skip;len(n)$

by (*simp add: len-d-def*)

lemma *LenEqvLenChopLen*:

$\vdash len(i+j) = len(i);len(j)$

proof

(*induct i*)

case 0

then show ?case

by (*metis EmptyChop LenZeroEqvEmpty add.left-neutral inteq-reflection*)

next

case (*Suc i*)

then show ?case

by (*metis ChopAssoc add-Suc int-eq len-d-def pow-Suc*)

qed

lemma *LenNPlusOneB*:

$\vdash len(n+1) = len(n);skip$

proof –

have 1: $\vdash len(n+1) = len(n);len(1)$ **by** (*rule LenEqvLenChopLen*)

have 2: $\vdash len(1) = skip$ **by** (*rule LenOneEqvSkip*)

hence 3: $\vdash len(n);len(1) = len(n);skip$ **using** *RightChopEqvChop* **by** *blast*

from 1 3 **show** ?thesis **by** *fastforce*

qed

lemma *LenCommute*:

$\vdash (\text{skip};(\text{len } n)) = (\text{len } n);\text{skip}$

proof

(*induct n*)

case 0

then show ?*case*

by (*metis LenEqvLenChopLen LenNPlusOneA LenOneEqvSkip inteq-reflection*)

next

case (*Suc n*)

then show ?*case*

by (*metis LenEqvLenChopLen LenNPlusOneA LenOneEqvSkip inteq-reflection*)

qed

lemma *SkipTrueEqvTrueSkip*:

$\vdash \text{skip};\# \text{True} = \# \text{True};\text{skip}$

using *TrueChopSkipEqvSkipChopTrue* **by** *fastforce*

lemma *PowerCommute*:

$\vdash (f;(\text{power } f \ n)) = ((\text{power } f \ n);f)$

proof

(*induct n*)

case 0

then show ?*case* **using** *EmptyChop ChopEmpty pow-0* **by** (*metis int-eq*)

next

case (*Suc n*)

then show ?*case* **using** *ChopAssoc pow-Suc* **by** (*metis inteq-reflection*)

qed

lemma *PowerRev*:

$\vdash (\text{power } \text{skip } n)^r = (\text{power } \text{skip } n)$

proof

(*induct n*)

case 0

then show ?*case* **using** *REmptyEqvEmpty* **by** *auto*

next

case (*Suc n*)

then show ?*case* **using** *PowerCommute RevChop pow-Suc* **by** (*metis RevSkip int-eq*)

qed

lemma *RLenEqvLen*:

$\vdash (\text{len } k)^r = (\text{len } k)$

proof

(*induct k*)

case 0

then show ?*case*

by (*metis PowerRev len-d-def*)

next

case (*Suc k*)

then show *?case*
by (*metis PowerRev len-d-def*)
qed

lemma *PowerSkipEqLen*:
 $\vdash (\text{power skip } n) = (\text{len } n)$
by (*simp add: len-d-def*)

lemma *ExistsLen*:
 $\vdash \exists k. \text{len}(k)$
by (*simp add: len-defs Valid-def*)

lemma *AndExistsLen*:
 $\vdash f = (f \wedge (\exists k. \text{len}(k)))$
using *ExistsLen* **by** *fastforce*

lemma *AndExistsLenChop*:
 $\vdash (f;g) = (\exists k. (f \wedge \text{len}(k));g)$
by (*simp add: Valid-def len-defs chop-defs*)

lemma *AndExistsLenChopR*:
 $\vdash (f;g) = (\exists k. f;(g \wedge \text{len}(k)))$
by (*simp add: Valid-def len-defs chop-defs*)

lemma *LFixedAndDistr*:
 $\vdash ((f0 \wedge \text{len}(k));g0 \wedge (f1 \wedge \text{len}(k));g1) = ((f0 \wedge f1) \wedge \text{len}(k));(g0 \wedge g1)$
by (*auto simp add: Valid-def len-defs chop-defs*)

lemma *RFixedAndDistr*:
 $\vdash (f0;(g0 \wedge \text{len}(k)) \wedge f1;(g1 \wedge \text{len}(k))) = (f0 \wedge f1);((g0 \wedge g1) \wedge \text{len}(k))$
by (*simp add: Valid-def len-defs chop-defs*) (*metis diff-diff-cancel*)

lemma *LFixedAndDistrA*:
 $\vdash ((f0 \wedge \text{len}(k));g0 \wedge (f1 \wedge \text{len}(k));g0) = ((f0 \wedge f1) \wedge \text{len}(k));g0$
proof –
have 1: $\vdash ((f0 \wedge \text{len}(k));g0 \wedge (f1 \wedge \text{len}(k));g0) = ((f0 \wedge f1) \wedge \text{len}(k));(g0 \wedge g0)$
by (*rule LFixedAndDistr*)
have 2: $\vdash ((f0 \wedge f1) \wedge \text{len}(k));(g0 \wedge g0) = ((f0 \wedge f1) \wedge \text{len}(k));g0$
by *auto*
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *LFixedAndDistrB*:
 $\vdash ((f0 \wedge \text{len}(k));g0 \wedge (f0 \wedge \text{len}(k));g1) = (f0 \wedge \text{len}(k));(g0 \wedge g1)$
proof –
have 1: $\vdash ((f0 \wedge \text{len}(k));g0 \wedge (f0 \wedge \text{len}(k));g1) = ((f0 \wedge f0) \wedge \text{len}(k));(g0 \wedge g1)$
by (*rule LFixedAndDistr*)
have 2: $\vdash ((f0 \wedge f0) \wedge \text{len}(k));(g0 \wedge g1) = (f0 \wedge \text{len}(k));(g0 \wedge g1)$
by *auto*

from 1 2 show ?thesis by fastforce
qed

lemma *LFixedAndDistrB1*:

$\vdash (\text{len}(k);f \wedge \text{len}(k);g) = \text{len}(k);(f \wedge g)$

proof –

have 1: $\vdash \text{len}(k);f = (\# \text{True} \wedge \text{len}(k));f$

by auto

have 2: $\vdash \text{len}(k);g = (\# \text{True} \wedge \text{len}(k));g$

by auto

have 3: $\vdash (\text{len}(k);f \wedge \text{len}(k);g) = ((\# \text{True} \wedge \text{len}(k));f \wedge (\# \text{True} \wedge \text{len}(k));g)$

using 1 2 by auto

have 4: $\vdash ((\# \text{True} \wedge \text{len}(k));f \wedge (\# \text{True} \wedge \text{len}(k));g) = (\# \text{True} \wedge \text{len}(k));(f \wedge g)$

using *LFixedAndDistrB* by blast

have 5: $\vdash (\# \text{True} \wedge \text{len}(k));(f \wedge g) = (\text{len}(k));(f \wedge g)$

by auto

from 1 2 3 4 5 show ?thesis by auto

qed

lemma *RFixedAndDistrA*:

$\vdash (f0;(g0 \wedge \text{len}(k)) \wedge f0;(g1 \wedge \text{len}(k))) = f0;((g0 \wedge g1) \wedge \text{len}(k))$

proof –

have 1: $\vdash (f0;(g0 \wedge \text{len}(k)) \wedge f0;(g1 \wedge \text{len}(k))) = (f0 \wedge f0);((g0 \wedge g1) \wedge \text{len}(k))$

by (rule *RFixedAndDistr*)

have 2: $\vdash (f0 \wedge f0);((g0 \wedge g1) \wedge \text{len}(k)) = f0;((g0 \wedge g1) \wedge \text{len}(k))$

by auto

from 1 2 show ?thesis by fastforce

qed

lemma *RFixedAndDistrB*:

$\vdash (f0;(g0 \wedge \text{len}(k)) \wedge f1;(g0 \wedge \text{len}(k))) = (f0 \wedge f1);(g0 \wedge \text{len}(k))$

proof –

have 1: $\vdash (f0;(g0 \wedge \text{len}(k)) \wedge f1;(g0 \wedge \text{len}(k))) = (f0 \wedge f1);((g0 \wedge g0) \wedge \text{len}(k))$

by (rule *RFixedAndDistr*)

have 2: $\vdash (f0 \wedge f1);((g0 \wedge g0) \wedge \text{len}(k)) = (f0 \wedge f1);(g0 \wedge \text{len}(k))$

by auto

from 1 2 show ?thesis by fastforce

qed

lemma *ChopSkipAndChopSkip*:

$\vdash (f0;\text{skip} \wedge f1;\text{skip}) = (f0 \wedge f1);\text{skip}$

proof –

have 1: $\vdash (f0;(\# \text{True} \wedge \text{len}(1)) \wedge f1;(\# \text{True} \wedge \text{len}(1))) = (f0 \wedge f1);(\# \text{True} \wedge \text{len}(1))$

by (rule *RFixedAndDistrB*)

have 2: $\vdash (\# \text{True} \wedge \text{len}(1)) = \text{skip}$

using *LenOneEqvSkip* by fastforce

hence 3: $\vdash f0;(\# \text{True} \wedge \text{len}(1)) = f0;\text{skip}$

using *RightChopEqvChop* by blast

have 4: $\vdash f1;(\# \text{True} \wedge \text{len}(1)) = f1;\text{skip}$

using 2 *RightChopEqvChop* by blast

have 5: $\vdash (f0;(\#True \wedge len(1)) \wedge f1;(\#True \wedge len(1))) = (f0;skip \wedge f1;skip)$
using 3 4 **by** *fastforce*
have 6: $\vdash (f0 \wedge f1);(\#True \wedge len(1)) = (f0 \wedge f1);skip$
using 2 *RightChopEqvChop* **by** *blast*
from 1 5 6 **show** *?thesis* **by** *fastforce*
qed

lemma *BiAndChopSkipEqv*:
 $\vdash (bi (f \wedge g));skip = ((bi f);skip \wedge (bi g);skip)$
proof –
have 1: $\vdash bi (f \wedge g) = ((bi f) \wedge (bi g))$
by (*auto simp add: bi-defs Valid-def*)
hence 2: $\vdash (bi (f \wedge g));skip = (bi f \wedge bi g);skip$
by (*rule LeftChopEqvChop*)
have 3: $\vdash (bi f \wedge bi g);skip = ((bi f);skip \wedge (bi g);skip)$
using *ChopSkipAndChopSkip* **by** *fastforce*
from 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *DiAndChopSkipEqv*:
 $\vdash (di (f \wedge g));skip \longrightarrow (di f);skip \wedge (di g);skip$
proof –
have 1: $\vdash di (f \wedge g) \longrightarrow (di f) \wedge (di g)$
by (*simp add: DiAndImpAnd*)
hence 2: $\vdash (di (f \wedge g));skip \longrightarrow (di f \wedge di g);skip$
by (*rule LeftChopImpChop*)
have 3: $\vdash (di f \wedge di g);skip = ((di f);skip \wedge (di g);skip)$
using *ChopSkipAndChopSkip* **by** *fastforce*
from 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *ChopEmptyAndEmpty*:
 $\vdash (f;g \wedge empty) = (f \wedge g \wedge empty)$
by (*simp add: Valid-def itl-defs*)
(metis prefix-ilen suffix-zero le-zero-eq)

lemma *ChopSkipImpMore*:
 $\vdash f;skip \longrightarrow more$
using *ChopImpDiamond MoreEqvSkipChopTrue SkipTrueEqvTrueSkip TrueChopEqvDiamond* **by** *fastforce*

lemma *MoreEqvMoreChopTrue*:
 $\vdash more = more;\#True$
proof –
have 1: $\vdash more = skip;\#True$
using *MoreEqvSkipChopTrue* **by** *blast*
have 2: $\vdash \#True = \#True;\#True$
by (*auto simp add: Valid-def chop-defs*)
hence 3: $\vdash skip;\#True = skip;(\#True;\#True)$
using *RightChopEqvChop* **by** *blast*
have 4: $\vdash skip;(\#True;\#True) = (skip;\#True);\#True$

using *ChopAssoc* **by** *blast*
have 5: $\vdash (skip; \# True); \# True = more; \# True$
using *MoreEqvSkipChopTrue* **by** (*simp add: more-d-def next-d-def*)
from 1 3 4 5 **show** ?thesis **by** *fastforce*
qed

lemma *NotNotChopSkip*:
 $\vdash (\neg((\neg f); skip)) = (empty \vee (f; skip))$
by (*metis WprevEqvEmptyOrPrev prev-d-def wprev-d-def*)

lemma *NotChopFixed*:
 $\vdash (\neg(f; (g \wedge len(k)))) = (\neg(\Diamond(g \wedge len(k))) \vee ((\neg f); (g \wedge len(k))))$
by (*auto simp add: itl-defs Valid-def*)
(metis diff-diff-cancel)

lemma *NotFixedChop*:
 $\vdash (\neg((g \wedge len(k)); f)) = (\neg(di(g \wedge len(k))) \vee ((g \wedge len(k)); (\neg f)))$
by (*auto simp add: itl-defs Valid-def*)

lemma *NotChopNotSkip*:
 $\vdash (\neg(f; skip)) = (empty \vee ((\neg f); skip))$
proof –
have 1: $\vdash (\neg((\neg(\neg f)); skip)) = (empty \vee ((\neg f); skip))$ **using** *NotNotChopSkip* **by** *blast*
have 2: $\vdash (\neg((\neg(\neg f)); skip)) = (\neg(f; skip))$ **by** *auto*
from 1 2 **show** ?thesis **by** *auto*
qed

9.4.2 Additional ITL theorems

lemma *BiOrBiImpBiOr*:
 $\vdash bi\ f \vee bi\ g \longrightarrow bi(f \vee g)$
proof –
have 1: $\vdash f \longrightarrow f \vee g$ **by** *auto*
hence 2: $\vdash bi\ f \longrightarrow bi(f \vee g)$ **by** (*rule BiImpBiRule*)
have 3: $\vdash g \longrightarrow f \vee g$ **by** *auto*
hence 4: $\vdash bi\ g \longrightarrow bi(f \vee g)$ **by** (*rule BiImpBiRule*)
from 2 4 **show** ?thesis **by** *fastforce*
qed

lemma *MoreAndBiImpBiChopSkip*:
 $\vdash more \wedge bi\ f \longrightarrow (bi\ f); skip$
proof –
have 1: $\vdash (bi\ f); skip = ((\neg(di(\neg f))); skip)$ **by** (*simp add: bi-d-def*)
have 2: $\vdash (\neg(\neg(di(\neg f))); skip) = (empty \vee (di(\neg f)); skip)$ **by** (*rule NotNotChopSkip*)
have 3: $\vdash empty \longrightarrow empty \vee di(\neg f)$ **by** *auto*
have 4: $\vdash (di(\neg f)); skip \longrightarrow di(\neg f)$ **using** *ChopImpDi DiEqvDiDi* **by** *fastforce*
hence 5: $\vdash (di(\neg f)); skip \longrightarrow empty \vee di(\neg f)$ **by** (*rule Prop05*)
have 6: $\vdash \neg(\neg(di(\neg f)); skip) \longrightarrow empty \vee di(\neg f)$ **using** 2 3 5 **by** *fastforce*
hence 7: $\vdash \neg(empty \vee di(\neg f)) \longrightarrow \neg(\neg(\neg(di(\neg f)); skip))$ **by** *fastforce*

have 8: $\vdash (\neg(\neg(\neg(di(\neg f));skip))) = ((\neg(di(\neg f));skip))$ **by** *auto*
have 9: $\vdash (\neg(empty \vee di(\neg f))) = (more \wedge \neg(di(\neg f)))$
using *NotEmptyEqvMore* **by** *auto*
have 10: $\vdash (more \wedge \neg(di(\neg f))) = (more \wedge bi\ f)$ **by** (*simp add: bi-d-def*)
from 1 6 7 8 9 10 **show** *?thesis* **by** (*metis int-eq*)
qed

lemma *DiChopImpDiB*:
 $\vdash di(f;g) \longrightarrow di\ f$
proof –
have 1: $\vdash f ; (g;\#True) \longrightarrow di\ f$ **by** (*rule ChopImpDi*)
have 2: $\vdash f ; (g;\#True) = (f;g);\#True$ **by** (*rule ChopAssoc*)
from 1 2 **show** *?thesis* **by** (*metis di-d-def int-eq*)
qed

lemma *BiBiOrImpBi*:
 $\vdash bi\ (bi\ f \vee bi\ g) \longrightarrow bi\ f \vee bi\ g$
using *BiElim* **by** *auto*

lemma *BiImpBiBiOr*:
 $\vdash bi\ f \longrightarrow bi\ (bi\ f \vee bi\ g)$
proof –
have 1: $\vdash bi\ f \longrightarrow bi\ f \vee bi\ g$ **by** *auto*
hence 2: $\vdash bi\ (bi\ f) \longrightarrow bi\ (bi\ f \vee bi\ g)$ **using** *BiImpBiRule* **by** *blast*
have 3: $\vdash bi\ (bi\ f) = bi\ f$ **using** *BiEqvBiBi* **by** *fastforce*
from 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *BiImpBiBiOrB*:
 $\vdash bi\ g \longrightarrow bi\ (bi\ f \vee bi\ g)$
proof –
have 1: $\vdash bi\ g \longrightarrow bi\ f \vee bi\ g$ **by** *auto*
hence 2: $\vdash bi\ (bi\ g) \longrightarrow bi\ (bi\ f \vee bi\ g)$ **using** *BiImpBiRule* **by** *blast*
have 3: $\vdash bi\ (bi\ g) = bi\ g$ **using** *BiEqvBiBi* **by** *fastforce*
from 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *BiBiOrEqvBi*:
 $\vdash bi\ (bi\ f \vee bi\ g) = bi\ f \vee bi\ g$
proof –
have 1: $\vdash bi\ (bi\ f \vee bi\ g) \longrightarrow bi\ f \vee bi\ g$ **by** (*rule BiBiOrImpBi*)
have 2: $\vdash bi\ f \longrightarrow bi\ (bi\ f \vee bi\ g)$ **by** (*rule BiImpBiBiOr*)
have 3: $\vdash bi\ g \longrightarrow bi\ (bi\ f \vee bi\ g)$ **by** (*rule BiImpBiBiOrB*)
have 4: $\vdash bi\ f \vee bi\ g \longrightarrow bi\ (bi\ f \vee bi\ g)$ **using** 2 3 **by** *fastforce*
from 1 4 **show** *?thesis* **by** *fastforce*
qed

lemma *DiEqvOrDiChopSkipA*:
 $\vdash di\ f = (f \vee di(f;skip))$
proof –

have 1: $\vdash di\ f = f ; \#True$ **by** (*simp add: di-d-def*)
hence 2: $\vdash di\ f = f ; (empty \vee more)$ **by** (*simp add: empty-d-def*)
hence 3: $\vdash f ; (empty \vee more) = (f ; empty \vee f ; more)$ **using** *ChopOrEqv* **by** *blast*
have 4: $\vdash f ; empty = f$ **by** (*rule ChopEmpty*)
have 5: $\vdash more = skip ; \#True$ **using** *MoreEqvSkipChopTrue* **by** *blast*
hence 6: $\vdash f ; more = f ; (skip ; \#True)$ **using** *RightChopEqvChop* **by** *blast*
have 7: $\vdash f ; (skip ; \#True) = (f ; skip) ; \#True$ **by** (*rule ChopAssoc*)
from 2 3 4 6 7 **show** *?thesis* **by** (*metis di-d-def int-eq*)
qed

lemma *DiEqvOrDiChopSkipB*:

$\vdash di\ f = (f \vee (di\ f) ; skip)$

proof –

have 1: $\vdash (di\ f) = (f \vee di(f ; skip))$ **by** (*rule DiEqvOrDiChopSkipA*)
have 2: $\vdash di(f ; skip) = (f ; skip) ; \#True$ **by** (*simp add: di-d-def*)
have 3: $\vdash (f ; skip) ; \#True = f ; (skip ; \#True)$ **by** (*rule ChopAssocB*)
have 4: $\vdash di(f ; skip) = f ; (skip ; \#True)$ **using** 2 3 **by** *fastforce*
have 5: $\vdash skip ; \#True = \#True ; skip$ **by** (*rule SkipTrueEqvTrueSkip*)
hence 6: $\vdash f ; (skip ; \#True) = f ; (\#True ; skip)$ **using** *RightChopEqvChop* **by** *blast*
have 7: $\vdash di(f ; skip) = f ; (\#True ; skip)$ **using** 4 6 **by** *fastforce*
have 8: $\vdash f ; (\#True ; skip) = (f ; \#True) ; skip$ **by** (*rule ChopAssoc*)
have 9: $\vdash (f ; \#True) ; skip = (di\ f) ; skip$ **by** (*simp add: di-d-def*)
have 10: $\vdash di(f ; skip) = (di\ f) ; skip$ **using** 7 8 9 **by** *fastforce*
hence 11: $\vdash (f \vee di(f ; skip)) = (f \vee (di\ f) ; skip)$ **by** *auto*
from 1 11 **show** *?thesis* **by** *fastforce*

qed

lemma *BiEqvAndEmptyOrBiChopSkip*:

$\vdash bi\ f = (f \wedge (empty \vee (bi\ f) ; skip))$

proof –

have 1: $\vdash di\ (\neg f) = (\neg f \vee (di\ (\neg f) ; skip))$ **by** (*rule DiEqvOrDiChopSkipB*)
have 2: $\vdash di\ (\neg f) = (\neg (bi\ f))$ **by** (*rule DiNotEqvNotBi*)
have 3: $\vdash (\neg (bi\ f)) = (\neg f \vee (di\ (\neg f) ; skip))$ **using** 1 2 **by** *fastforce*
hence 4: $\vdash bi\ f = (\neg (\neg f \vee (di\ (\neg f) ; skip)))$ **by** *auto*
have 5: $\vdash (\neg (\neg f \vee (di\ (\neg f) ; skip))) = (f \wedge \neg (di\ (\neg f) ; skip))$ **by** *auto*
have 6: $\vdash di\ (\neg f) ; skip = ((\neg (bi\ f)) ; skip)$ **by** (*simp add: 2 LeftChopEqvChop*)
hence 7: $\vdash (\neg (di\ (\neg f) ; skip)) = (\neg ((\neg (bi\ f)) ; skip))$ **by** *auto*
have 8: $\vdash (\neg ((\neg (bi\ f)) ; skip)) = (empty \vee (bi\ f) ; skip)$ **using** *NotNotChopSkip* **by** *blast*
hence 9: $\vdash (f \wedge \neg (di\ (\neg f) ; skip)) = (f \wedge (empty \vee (bi\ f) ; skip))$ **using** 7 8 **by** *fastforce*
from 4 5 9 **show** *?thesis* **by** *fastforce*

qed

lemma *DiDiAndEqvDi*:

$\vdash di\ (di\ f \wedge di\ g) = (di\ f \wedge di\ g)$

proof –

have 1: $\vdash bi\ (bi\ (\neg f) \vee bi\ (\neg g)) = (bi\ (\neg f) \vee bi\ (\neg g))$
by (*meson BiBiOrImpBi BiImpBiBiOr BiImpBiBiOrB Prop02 int-iffI*)
have 2: $\vdash bi\ (\neg f) = (\neg (di\ f))$
by (*simp add: bi-d-def*)
have 3: $\vdash bi\ (\neg g) = (\neg (di\ g))$

by (simp add: bi-d-def)
 have 4: $\vdash (bi (\neg f) \vee bi (\neg g)) = (\neg (di f) \vee \neg (di g))$
 using 2 3 by fastforce
 have 5: $\vdash (\neg (di f) \vee \neg (di g)) = (\neg (di f \wedge di g))$
 by auto
 have 6: $\vdash bi (bi (\neg f) \vee bi (\neg g)) = (\neg (di f \wedge di g))$
 using 1 5 4 by fastforce
 hence 7: $\vdash (\neg (bi (bi (\neg f) \vee bi (\neg g)))) = (di f \wedge di g)$
 by auto
 have 8: $\vdash (\neg (bi (bi (\neg f) \vee bi (\neg g)))) = di (\neg (bi (\neg f) \vee bi (\neg g)))$
 using DiNotEqvNotBi by fastforce
 have 9: $\vdash (\neg (bi (\neg f) \vee bi (\neg g))) = (di f \wedge di g)$
 using 1 7 by fastforce
 hence 10: $\vdash di (\neg (bi (\neg f) \vee bi (\neg g))) = di (di f \wedge di g)$
 using DiEqvDi by blast
 from 7 8 10 show ?thesis by fastforce
 qed

lemma BiInduct:

$\vdash bi(f \longrightarrow wprev f) \wedge f \longrightarrow bi f$

proof –

have 1: $\vdash \Box((f^r) \longrightarrow wnext(f^r)) \wedge f^r \longrightarrow \Box(f^r)$ using BoxInduct by blast
 hence 2: $\vdash (\Box((f^r) \longrightarrow wnext(f^r)) \wedge f^r \longrightarrow \Box(f^r))^r$ using ReverseEqv by blast
 have 3: $\vdash ((f^r)^r) = f$ by (simp add: EqvReverseReverse)
 have 4: $\vdash (\Box(f^r))^r = bi(f)$ using RBoxEqvBi by blast
 have 5: $\vdash ((f^r) \longrightarrow wnext(f^r))^r = (f^r \longrightarrow (wnext(f^r))^r)$ by (simp add: rev-fun2)
 have 6: $\vdash (wnext(f^r))^r = wprev(f)$ using RRWNextEqvWPrev by blast
 have 7: $\vdash (f^r \longrightarrow (wnext(f^r))^r) = (f \longrightarrow wprev(f))$ using 6 3 by fastforce
 have 8: $\vdash bi(f^r \longrightarrow (wnext(f^r))^r) = bi(f \longrightarrow wprev(f))$ using 7 3 BiEqvBi by blast
 have 9: $\vdash (\Box((f^r) \longrightarrow wnext(f^r)))^r = bi((f^r) \longrightarrow wnext(f^r))^r$ using RBoxEqvBi by blast
 have 10: $\vdash (\Box((f^r) \longrightarrow wnext(f^r)))^r = bi(f \longrightarrow wprev(f))$ using 8 9 5 int-eq by fastforce
 have 11: $\vdash (\Box((f^r) \longrightarrow wnext(f^r)) \wedge f^r \longrightarrow \Box(f^r))^r =$
 $((\Box((f^r) \longrightarrow wnext(f^r)))^r \wedge (f^r)^r \longrightarrow (\Box(f^r))^r)$ by (metis int-eq rev-fun2)
 have 12: $\vdash ((\Box((f^r) \longrightarrow wnext(f^r)))^r \wedge (f^r)^r \longrightarrow (\Box(f^r))^r) =$
 $(bi(f \longrightarrow wprev(f)) \wedge f \longrightarrow bi f)$ using 8 3 4 10 by fastforce
 from 2 11 12 show ?thesis using MP by fastforce
 qed

lemma PrevLoop:

assumes $\vdash f \longrightarrow prev f$

shows $\vdash \neg f$

proof –

have 1: $\vdash f \longrightarrow prev f$ using assms by auto
 hence 2: $\vdash f \longrightarrow (more \wedge wprev f)$
 by (metis ChopSkipImpMore Prop05 Prop12 WprevEqvEmptyOrPrev inteq-reflection lift-imp-trans prev-d-def)
 hence 3: $\vdash f \longrightarrow wprev f$ by auto
 hence 4: $\vdash bi(f \longrightarrow wprev f)$ by (rule BiGen)
 have 5: $\vdash bi(f \longrightarrow wprev f) \wedge f \longrightarrow bi f$ by (rule BiInduct)
 hence 6: $\vdash bi(f \longrightarrow wprev f) \longrightarrow (f \longrightarrow bi f)$ by fastforce

have 7: $\vdash (f \longrightarrow bi\ f)$ **using** 4 6 *MP* **by** *blast*
have 8: $\vdash bi\ f \longrightarrow f$ **by** (rule *BiElim*)
have 9: $\vdash f = bi\ f$ **using** 7 8 **by** *fastforce*
have 10: $\vdash f \longrightarrow more$ **using** 2 **by** *auto*
hence 11: $\vdash bi\ f \longrightarrow bi\ more$ **using** *BiImpBiRule* **by** *blast*
have 12: $\vdash \neg(bi\ more)$ **using** *DiEmpty* *bi-d-def* *empty-d-def* **by** (*simp* *add*: *bi-d-def* *empty-d-def*)
from 7 9 11 12 **show** *?thesis* **using** *MP* **by** *fastforce*
qed

lemma *PrevImpNotPrevNot*:

$\vdash prev\ f \longrightarrow \neg(prev\ (\neg\ f))$
by (*metis* (*no-types*, *lifting*) *NextImpNotNextNot* *RPrevEqvNext* *ReverseEqv* *inteq-reflection* *rev-fun1* *rev-fun2*)

lemma *BiEqvAndWprevBi*:

$\vdash bi\ f = (f \wedge wprev(bi\ f))$
by (*metis* *BiEqvAndEmptyOrBiChopSkip* *WprevEqvEmptyOrPrev* *inteq-reflection* *prev-d-def*)

lemma *DiIntroLoop*:

assumes $\vdash (f \wedge \neg\ g) \longrightarrow prev\ f$
shows $\vdash f \longrightarrow di\ g$
using *assms* *DiamondIntro*[of *LIFT*(f^r) *LIFT*(g^r)]
by (*metis* (*no-types*, *lifting*) *RDiEqvDiamond* *RPrevEqvNext* *ReverseEqv* *inteq-reflection* *rev-fun2* *rev-fun1*)

lemma *DiEqvOrChopMore*:

$\vdash di\ f = (f \vee f;more)$
proof –
have 1: $\vdash di\ f = f; \# True$ **by** (*simp* *add*: *di-d-def*)
hence 2: $\vdash di\ f = f; (empty \vee more)$ **by** (*simp* *add*: *empty-d-def*)
have 3: $\vdash f; (empty \vee more) = (f;empty \vee f;more)$ **by** (*simp* *add*: *ChopOrEqv*)
have 4: $\vdash f;empty = f$ **by** (rule *ChopEmpty*)
from 2 3 4 **show** *?thesis* **by** *fastforce*
qed

lemma *DiAndDiEqvDiAndDiOrDiAndDi*:

$\vdash (di\ f \wedge di\ g) = (di(f \wedge di\ g) \vee di(g \wedge di\ f))$
proof –
have 1: $\vdash di\ f = (f \vee f;more)$
using *DiEqvOrChopMore* **by** *blast*
have 2: $\vdash di\ g = (g \vee g;more)$
using *DiEqvOrChopMore* **by** *blast*
have 3: $\vdash (di\ f \wedge di\ g) = ((f \vee f;more) \wedge (g \vee g;more))$
using 1 2 **by** *fastforce*
have 4: $\vdash ((f \vee f;more) \wedge (g \vee g;more)) =$
 $((f \wedge g) \vee (f \wedge g;more) \vee (g \wedge f;more) \vee (f;more \wedge g;more))$
by *auto*

have 5: $\vdash \text{more} = \# \text{True}; \text{skip}$
using *MoreEqvSkipChopTrue SkipTrueEqvTrueSkip* **by** *fastforce*
hence 6: $\vdash f; \text{more} = f; (\# \text{True}; \text{skip})$
using *RightChopEqvChop* **by** *blast*
have 7: $\vdash f; (\# \text{True}; \text{skip}) = (f; \# \text{True}); \text{skip}$
by (*rule ChopAssoc*)
have 8: $\vdash f; \text{more} = \text{prev} (di f)$
using 6 7 **by** (*metis di-d-def int-eq prev-d-def*)
have 9: $\vdash g; \text{more} = g; (\# \text{True}; \text{skip})$
using 5 *RightChopEqvChop* **by** *blast*
have 10: $\vdash g; (\# \text{True}; \text{skip}) = (g; \# \text{True}); \text{skip}$
by (*rule ChopAssoc*)
have 11: $\vdash g; \text{more} = \text{prev} (di g)$
using 9 10 **by** (*metis di-d-def int-eq prev-d-def*)
have 12: $\vdash (f; \text{more} \wedge g; \text{more}) = (\text{prev} (di f) \wedge \text{prev} (di g))$
using 8 11 **by** *fastforce*
hence 13: $\vdash (f; \text{more} \wedge g; \text{more}) = \text{prev} (di f \wedge di g)$
by (*metis ChopSkipAndChopSkip int-eq prev-d-def*)
have 14: $\vdash (di f \wedge di g) =$
 $((f \wedge g) \vee (f \wedge g; \text{more}) \vee (g \wedge f; \text{more})) \vee (f; \text{more} \wedge g; \text{more})$
using 3 4 **by** *auto*
have 15: $\vdash (di f \wedge di g) =$
 $((f \wedge g) \vee (f \wedge g; \text{more}) \vee (g \wedge f; \text{more})) \vee \text{prev} (di f \wedge di g)$
by (*metis 10 14 6 7 9 ChopSkipAndChopSkip di-d-def int-eq prev-d-def*)
hence 16: $\vdash (di f \wedge di g) \longrightarrow$
 $((f \wedge g) \vee (f \wedge g; \text{more}) \vee (g \wedge f; \text{more})) \vee \text{prev} (di f \wedge di g)$
by *fastforce*
hence 17: $\vdash (di f \wedge di g) \wedge \neg((f \wedge g) \vee (f \wedge g; \text{more}) \vee (g \wedge f; \text{more})) \longrightarrow$
 $\text{prev} (di f \wedge di g)$
by *fastforce*
hence 18: $\vdash (di f \wedge di g) \longrightarrow di((f \wedge g) \vee (f \wedge g; \text{more}) \vee (g \wedge f; \text{more}))$
using *DiIntroLoop* **by** *blast*
have 19: $\vdash di((f \wedge g) \vee (f \wedge g; \text{more}) \vee (g \wedge f; \text{more})) =$
 $(di(f \wedge g) \vee di(f \wedge g; \text{more}) \vee di(g \wedge f; \text{more}))$
by (*meson DiOrEqv Prop06*)
have 20: $\vdash f \longrightarrow di f$
using *DiIntro* **by** *blast*
hence 21: $\vdash f \wedge g \longrightarrow g \wedge di f$
by *auto*
hence 22: $\vdash di(f \wedge g) \longrightarrow di(g \wedge di f)$
using *DiImpDi* **by** *blast*
hence 23: $\vdash di(f \wedge g) \longrightarrow di(g \wedge di f) \vee di(f \wedge di g)$
by *auto*
have 24: $\vdash g; \text{more} \longrightarrow di g$
by (*simp add: ChopImpDi*)
hence 25: $\vdash f \wedge g; \text{more} \longrightarrow f \wedge di g$
by *auto*
hence 26: $\vdash di(f \wedge g; \text{more}) \longrightarrow di(f \wedge di g)$
using *DiImpDi* **by** *blast*
hence 27: $\vdash di(f \wedge g; \text{more}) \longrightarrow di(f \wedge di g) \vee di(g \wedge di f)$

by *auto*
 have 28: $\vdash f; \text{more} \longrightarrow di\ f$
 by (*simp add: ChopImpDi*)
 hence 29: $\vdash g \wedge f; \text{more} \longrightarrow g \wedge di\ f$
 by *auto*
 hence 30: $\vdash di(g \wedge f; \text{more}) \longrightarrow di(g \wedge di\ f)$
 using *DiImpDi* by *blast*
 hence 31: $\vdash di(g \wedge f; \text{more}) \longrightarrow di(f \wedge di\ g) \vee di(g \wedge di\ f)$
 by *auto*
 have 32: $\vdash di(f \wedge g) \vee di(f \wedge g; \text{more}) \vee di(g \wedge f; \text{more}) \longrightarrow$
 $di(f \wedge di\ g) \vee di(g \wedge di\ f)$
 using 23 27 31 by *fastforce*
 have 33: $\vdash di((f \wedge g) \vee (f \wedge g; \text{more}) \vee (g \wedge f; \text{more})) \longrightarrow$
 $di(f \wedge di\ g) \vee di(g \wedge di\ f)$
 using 19 32 by *fastforce*
 have 34: $\vdash (di\ f \wedge di\ g) \longrightarrow di(f \wedge di\ g) \vee di(g \wedge di\ f)$
 using 18 33 by *fastforce*
 have 35: $\vdash f \longrightarrow di\ f$
 using *DiIntro* by *blast*
 hence 36: $\vdash f \wedge di\ g \longrightarrow di\ f \wedge di\ g$
 by *auto*
 hence 37: $\vdash di(f \wedge di\ g) \longrightarrow di(di\ f \wedge di\ g)$
 using *DiImpDi* by *blast*
 have 38: $\vdash di(di\ f \wedge di\ g) = (di\ f \wedge di\ g)$
 using *DiDiAndEqvDi* by *blast*
 have 39: $\vdash di(f \wedge di\ g) \longrightarrow di\ f \wedge di\ g$
 using 37 38 by *fastforce*
 have 40: $\vdash g \longrightarrow di\ g$
 using *DiIntro* by *blast*
 hence 41: $\vdash g \wedge di\ f \longrightarrow di\ f \wedge di\ g$
 by *auto*
 hence 42: $\vdash di(g \wedge di\ f) \longrightarrow di(di\ f \wedge di\ g)$
 using *DiImpDi* by *blast*
 have 43: $\vdash di(di\ f \wedge di\ g) = (di\ f \wedge di\ g)$
 using *DiDiAndEqvDi* by *fastforce*
 have 44: $\vdash di(g \wedge di\ f) \longrightarrow di\ f \wedge di\ g$
 using 42 43 by *fastforce*
 have 45: $\vdash di(f \wedge di\ g) \vee di(g \wedge di\ f) \longrightarrow di\ f \wedge di\ g$
 using 39 44 by *fastforce*
 from 34 45 show ?thesis by *fastforce*
 qed

lemma *BoxStateEqvBiFinState*:

$\vdash \square (init\ w) = bi\ (fin\ (init\ w))$

proof –

have 1: $\vdash \Diamond (\neg (init\ w)) = \#True ; (\neg (init\ w))$

by (*simp add: sometimes-d-def*)

have 2: $\vdash \Diamond (init(\neg w)) = \#True ; init(\neg w)$

by (*simp add: sometimes-d-def*)

have 3: $\vdash di(\#True \wedge fin\ (init(\neg w))) = \#True ; init(\neg w)$

```

    using DiAndFinEqvChopState by blast
  have 4:  $\vdash \Diamond (init(\neg w)) = di (\#True \wedge fin (init (\neg w)))$ 
    using 1 2 3 by fastforce
  have 5:  $\vdash (\neg (\Diamond (init(\neg w)))) = (\neg (di (\#True \wedge fin (init (\neg w)))))$ 
    using 4 by fastforce
  have 6:  $\vdash \Box (init w) = (\neg (di (\#True \wedge fin (init (\neg w)))))$ 
    using 5 always-d-def Initprop(2) by (metis int-eq)
  have 7:  $\vdash \Box (init w) = bi (\neg (fin (init (\neg w))))$ 
    using 6 by (simp add: bi-d-def)
  have 8:  $\vdash init (\neg w) = (\neg (init w))$ 
    using Initprop(2) by fastforce
  have 9:  $\vdash fin (init (\neg w)) = fin (\neg (init w))$ 
    using 8 FinEqvFin by blast
  have 10:  $\vdash fin (init (\neg w)) = (\neg (fin (init w)))$ 
    using 8 FinNotStateEqvNotFinState FinEqvFin by blast
  have 11:  $\vdash (\neg (fin (init (\neg w)))) = (fin (init w))$ 
    using 10 by fastforce
  have 12:  $\vdash bi (\neg (fin (init (\neg w)))) = bi (fin (init w))$ 
    using 11 by (simp add: BiEqvBi)
  have 13:  $\vdash \Box (init w) = bi (fin (init w))$ 
    using 7 12 by fastforce
  from 13 show ?thesis by simp
qed

```

lemma *DiamondStateEqvDiFinState*:

$\vdash \Diamond (init w) = di (fin (init w))$

proof –

```

  have 1:  $\vdash \Box (init (\neg w)) = bi (fin (init (\neg w)))$ 
    using BoxStateEqvBiFinState by blast
  have 2:  $\vdash (\neg (\Box (init (\neg w)))) = (\neg (bi (fin (init (\neg w)))))$ 
    using 1 by auto
  have 3:  $\vdash \Diamond (\neg (init (\neg w))) = di (\neg (fin (init (\neg w))))$ 
    using 2 by (simp add: always-d-def bi-d-def)
  have 4:  $\vdash \Diamond (init w) = di (\neg (fin (init (\neg w))))$ 
    by (metis 3 DiEqvNotBiNot DiState Initprop(2) StateEqvBi int-eq)
  have 5:  $\vdash \Diamond (init w) = di (fin (init w))$  using 4 FinNotStateEqvNotFinState
    by (metis DiEqvNotBiNot DiNotEqvNotBi inteq-reflection)
  from 1 2 3 4 5 show ?thesis by simp
qed

```

lemma *OrDiEqvDi*:

$\vdash (f \vee di f) = di f$

proof –

```

  have 1:  $\vdash f \longrightarrow di f$  using DiIntro by blast
  from 1 show ?thesis by auto
qed

```

lemma *AndDiEqv*:

$\vdash (f \wedge di f) = f$

proof –

have 1: $\vdash f \longrightarrow di\ f$ **using** *DiIntro* **by** *blast*
from 1 **show** *?thesis* **by** *auto*
qed

lemma *BiEmptyEqvEmpty*:

$\vdash bi\ empty = empty$

proof –

have 1: $\vdash bi\ empty = (\neg (di\ (\neg empty)))$ **by** (*simp add: bi-d-def*)
have 2: $\vdash (\neg (di\ (\neg empty))) = (\neg ((\neg empty); \# True))$ **by** (*simp add: di-d-def*)
have 3: $\vdash (\neg ((\neg empty); \# True)) = (\neg (more; \# True))$ **by** (*simp add: empty-d-def*)
have 4: $\vdash more; \# True = more$ **using** *MoreEqvMoreChopTrue* **by** *auto*
hence 5: $\vdash (\neg (more; \# True)) = (\neg more)$ **by** *fastforce*
from 1 2 3 5 **show** *?thesis* **using** *NotEmptyEqvMore* **by** *fastforce*
qed

lemma *EmptyChopSkipInduct*:

assumes $\vdash empty \longrightarrow f$

$\vdash prev\ f \longrightarrow f$

shows $\vdash f$

proof –

have 1: $\vdash empty \longrightarrow f$ **using** *assms(1)* **by** *auto*
have 2: $\vdash prev\ f \longrightarrow f$ **using** *assms(2)* **by** *blast*
have 3: $\vdash (empty \vee prev\ f) \longrightarrow f$ **using** 1 2 **by** *fastforce*
have 4: $\vdash wprev\ f = (empty \vee prev\ f)$ **by** (*simp add: WprevEqvEmptyOrPrev*)
hence 5: $\vdash wprev\ f \longrightarrow f$ **using** 3 **by** *fastforce*
hence 6: $\vdash \neg f \longrightarrow \neg (wprev\ f)$ **by** *fastforce*
hence 7: $\vdash \neg f \longrightarrow prev\ (\neg f)$ **by** (*simp add: wprev-d-def*)
hence 8: $\vdash \neg \neg f$ **by** (*rule PrevLoop*)
from 8 **show** *?thesis* **by** *auto*
qed

lemma *MoreImpImpChopSkipEqv*:

$\vdash more \longrightarrow ((f \longrightarrow g); skip = ((f; skip) \longrightarrow (g; skip)))$

proof –

have 01: $\vdash (f \longrightarrow g) = (\neg f \vee g)$ **by** *auto*
hence 02: $\vdash (f \longrightarrow g); skip = (\neg f \vee g); skip$ **by** (*simp add: LeftChopEqvChop*)
hence 1: $\vdash (more \wedge (f \longrightarrow g); skip) = (more \wedge (\neg f \vee g); skip)$ **by** *fastforce*
have 2: $\vdash (\neg f \vee g); skip = ((\neg f); skip \vee g; skip)$
using *OrChopEqv* **by** *auto*
hence 3: $\vdash (more \wedge (\neg f \vee g); skip) = (more \wedge ((\neg f); skip \vee g; skip))$
by *auto*
have 4: $\vdash (\neg ((\neg f); skip)) = (empty \vee (f; skip))$
using *NotNotChopSkip* **by** *blast*
hence 5: $\vdash ((\neg f); skip) = (\neg (empty \vee (f; skip)))$
by *fastforce*
have 6: $\vdash \neg (empty \vee (f; skip)) = (more \wedge \neg (f; skip))$
using 5 *NotChopSkipEqvMoreAndNotChopSkip* **by** *fastforce*
have 7: $\vdash ((\neg f); skip \vee g; skip) = ((more \wedge \neg (f; skip)) \vee g; skip)$
using 5 6 **by** *fastforce*
hence 8: $\vdash (more \wedge (\neg f; skip \vee g; skip)) = (more \wedge ((more \wedge \neg (f; skip)) \vee g; skip))$

by auto
 have 9: $\vdash (more \wedge ((more \wedge \neg(f;skip)) \vee g;skip)) = (more \wedge (\neg(f;skip) \vee g;skip))$
 by auto
 have 10: $\vdash (more \wedge (\neg(f;skip) \vee g;skip)) = (more \wedge ((f;skip) \longrightarrow (g;skip)))$
 by auto
 have 11: $\vdash (more \wedge (f \longrightarrow g);skip) = (more \wedge ((f;skip) \longrightarrow (g;skip)))$
 using 1 2 3 8 9 10 7 by fastforce
 from 11 show ?thesis using MP by fastforce
 qed

lemma MoreImpImpPrevEqv:
 $\vdash more \longrightarrow (prev(f \longrightarrow g) = (prev f \longrightarrow prev g))$
 by (simp add: MoreImpImpChopSkipEqv prev-d-def)

lemma BiBoxNotEqvNotTrueChopChopTrue:
 $\vdash bi(\Box (\neg f)) = (\neg((\# True;f); \# True))$
 by (simp add: bi-d-def always-d-def di-d-def sometimes-d-def)

lemma DiAndEmptyEqvAndEmpty:
 $\vdash (di f \wedge empty) = (f \wedge empty)$
 proof –
 have 1: $\vdash di f = (f \vee di f;skip)$
 using DiEqvOrDiChopSkipB by blast
 hence 2: $\vdash (di f \wedge empty) = ((f \vee di f;skip) \wedge empty)$
 by fastforce
 have 3: $\vdash ((f \vee di f;skip) \wedge empty) = ((f \wedge empty) \vee (di f;skip \wedge empty))$
 by auto
 have 4: $\vdash \neg(di f;skip \wedge empty)$
 by (metis AndDiEqv DiSkipEqvMore NotChopAndMoreAndEmpty inteq-reflection)
 hence 5: $\vdash ((f \wedge empty) \vee (di f;skip \wedge empty)) = (f \wedge empty)$
 by auto
 from 2 3 5 show ?thesis by fastforce
 qed

9.4.3 Strict initial intervals

lemma DsMoreDi:
 $\vdash ds f = (more \wedge (di f);skip)$
 proof –
 have 1: $\vdash ds f = (\neg(bs (\neg f)))$
 by (simp add: ds-d-def)
 have 2: $\vdash (\neg(bs (\neg f))) = (\neg(empty \vee (bi (\neg f));skip))$
 by (simp add: bs-d-def)
 have 3: $\vdash (\neg(empty \vee (bi (\neg f));skip)) = (\neg empty \wedge \neg((bi (\neg f));skip))$
 by auto
 have 4: $\vdash (\neg empty \wedge \neg((bi (\neg f));skip)) = (more \wedge \neg((bi (\neg f));skip))$
 using NotEmptyEqvMore by auto
 have 5: $\vdash (more \wedge \neg((bi (\neg f));skip)) = (more \wedge \neg(\neg(di f);skip))$
 by (metis DiEqvNotBiNot DiIntro DiSkipEqvMore NotChopSkipEqvMoreAndNotChopSkip)

Prop10 RightChopImpMoreRule int-simps(4) inteq-reflection lift-and-com)

have 6: $\vdash (\text{more} \wedge \neg((\neg(\text{di } f));\text{skip})) = (\text{more} \wedge (\text{empty} \vee (\text{di } f);\text{skip}))$
using *NotNotChopSkip* **by** *fastforce*
have 7: $\vdash (\text{more} \wedge (\text{empty} \vee (\text{di } f);\text{skip})) = (\text{more} \wedge (\text{di } f);\text{skip})$
using *NotEmptyEqvMore* **by** *auto*
from 1 2 3 4 5 6 7 **show** *?thesis* **by** *fastforce*
qed

lemma *DsDi*:
 $\vdash \text{ds } f = (\text{di } f);\text{skip}$
proof –
have 1: $\vdash \text{ds } f = (\text{more} \wedge (\text{di } f);\text{skip})$ **by** (*rule DsMoreDi*)
have 2: $\vdash (\text{di } f);\text{skip} \longrightarrow \text{more}$ **by** (*metis DiIntro DiSkipEqvMore RightChopImpMoreRule int-eq*)
hence 3: $\vdash (\text{more} \wedge (\text{di } f);\text{skip}) = (\text{di } f);\text{skip}$ **by** *auto*
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *BsEqvNotDsNot*:
 $\vdash \text{bs } f = (\neg(\text{ds } (\neg f)))$
proof –
have 1: $\vdash \text{ds } (\neg f) = (\text{more} \wedge (\text{di } (\neg f));\text{skip})$
by (*rule DsMoreDi*)
hence 2: $\vdash (\neg(\text{ds } (\neg f))) = (\neg(\text{more} \wedge (\text{di } (\neg f));\text{skip}))$
by *auto*
have 3: $\vdash (\neg(\text{more} \wedge (\text{di } (\neg f));\text{skip})) = (\text{empty} \vee \neg((\text{di } (\neg f));\text{skip}))$
using *NotEmptyEqvMore* **by** *auto*
have 4: $\vdash (\text{empty} \vee \neg((\text{di } (\neg f));\text{skip})) = (\text{empty} \vee \neg((\neg(\text{bi } f));\text{skip}))$
using *DiNotEqvNotBi* **by** (*metis 3 inteq-reflection*)
have 5: $\vdash (\neg((\neg(\text{bi } f));\text{skip})) = (\text{empty} \vee (\text{bi } f);\text{skip})$
by (*rule NotNotChopSkip*)
hence 6: $\vdash (\text{empty} \vee \neg((\neg(\text{bi } f));\text{skip})) = (\text{empty} \vee (\text{bi } f);\text{skip})$
by *auto*
from 2 3 4 6 **show** *?thesis* **by** (*metis bs-d-def inteq-reflection*)
qed

lemma *NotBsEqvDsNot*:
 $\vdash (\neg(\text{bs } f)) = \text{ds } (\neg f)$
proof –
have 1: $\vdash \text{bs } f = (\neg(\text{ds } (\neg f)))$ **by** (*rule BsEqvNotDsNot*)
hence 2: $\vdash (\neg(\text{bs } f)) = (\neg\neg(\text{ds } (\neg f)))$ **by** *auto*
from 2 **show** *?thesis* **by** *auto*
qed

lemma *NotDsEqvBsNot*:
 $\vdash (\neg(\text{ds } f)) = \text{bs } (\neg f)$
proof –
have 1: $\vdash (\neg(\text{ds } f)) = (\neg\neg(\text{bs } (\neg f)))$ **by** (*simp add: ds-d-def*)
from 1 **show** *?thesis* **by** *auto*
qed

lemma *NotDsAndEmpty*:

$\vdash \neg(ds\ f \wedge empty)$

proof –

have 1: $\vdash ds\ f = (more \wedge (di\ f);skip)$ **by** (*rule DsMoreDi*)

have 2: $\vdash more \wedge (di\ f);skip \wedge empty \longrightarrow \#False$ **using** *NotEmptyEqvMore* **by** *auto*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *BsMoreEqvEmpty*:

$\vdash bs\ more = empty$

proof –

have 1: $\vdash bs\ more = (empty \vee (bi\ more);skip)$ **by** (*simp add: bs-d-def*)

have 2: $\vdash bi\ more \longrightarrow \#False$ **using** *DiEmpty NotEmptyEqvMore* **by** (*simp add: bi-d-def empty-d-def*)

hence 3: $\vdash (bi\ more);skip \longrightarrow \#False;skip$ **using** *LeftChopImpChop* **by** *blast*

have 31: $\vdash \#False;skip \longrightarrow \#False$ **by** (*simp add: Valid-def skip-defs chop-defs*)

have 32: $\vdash (bi\ more);skip \longrightarrow \#False$ **using** 3 31 **by** *fastforce*

hence 4: $\vdash (empty \vee ((bi\ more);skip)) = empty$ **by** *fastforce*

from 1 4 **show** *?thesis* **by** *fastforce*

qed

lemma *BsAndEqv*:

$\vdash (bs\ f \wedge bs\ g) = bs(f \wedge g)$

proof –

have 1: $\vdash bs\ f = (empty \vee (bi\ f);skip)$

by (*simp add: bs-d-def*)

have 2: $\vdash bs\ g = (empty \vee (bi\ g);skip)$

by (*simp add: bs-d-def*)

have 3: $\vdash (bs\ f \wedge bs\ g) = ((empty \vee (bi\ f);skip) \wedge (empty \vee (bi\ g);skip))$

using 1 2 **by** *fastforce*

have 4: $\vdash ((empty \vee (bi\ f);skip) \wedge (empty \vee (bi\ g);skip)) =$

$(empty \vee ((bi\ f);skip \wedge (bi\ g);skip))$

by *auto*

have 5: $\vdash (((bi\ f);skip \wedge (bi\ g);skip)) = bi(f \wedge g);skip$

using *BiAndChopSkipEqv* **by** *fastforce*

hence 6: $\vdash (empty \vee ((bi\ f);skip \wedge (bi\ g);skip)) = (empty \vee bi(f \wedge g);skip)$

by *auto*

from 3 4 6 **show** *?thesis* **by** (*metis bs-d-def inteq-reflection*)

qed

lemma *DsEqvRule*:

assumes $\vdash f = g$

shows $\vdash ds\ f = ds\ g$

using *assms* **using** *int-eq* **by** *force*

lemma *DsOrEqv*:

$\vdash (ds\ f \vee ds\ g) = ds\ (f \vee g)$

proof –

have 1: $\vdash ds\ f = (\neg(bs\ (\neg f)))$ **by** (*simp add: ds-d-def*)

have 2: $\vdash ds\ g = (\neg(bs\ (\neg g)))$ **by** (*simp add: ds-d-def*)

have 3: $\vdash (ds\ f \vee ds\ g) = (\neg(bs\ (\neg f)) \vee \neg(bs\ (\neg g)))$ **using** 1 2 **by** *fastforce*

have 4: $\vdash (\neg(bs(\neg f)) \vee \neg(bs(\neg g))) = (\neg(bs(\neg f) \wedge bs(\neg g)))$ **by** *auto*
have 5: $\vdash (bs(\neg f) \wedge bs(\neg g)) = bs(\neg f \wedge \neg g)$ **by** (*rule BsAndEqv*)
hence 6: $\vdash (\neg(bs(\neg f) \wedge bs(\neg g))) = (\neg(bs(\neg f \wedge \neg g)))$ **by** *auto*
have 7: $\vdash (\neg(bs(\neg f \wedge \neg g))) = ds(\neg(\neg f \wedge \neg g))$ **by** (*rule NotBsEqvDsNot*)
have 8: $\vdash (\neg(\neg f \wedge \neg g)) = (f \vee g)$ **by** *auto*
hence 9: $\vdash ds(\neg(\neg f \wedge \neg g)) = ds(f \vee g)$ **by** (*rule DsEqvRule*)
from 3 4 6 7 9 **show** *?thesis* **by** *fastforce*
qed

lemma *BsOrImp*:

$\vdash bs f \vee bs g \longrightarrow bs(f \vee g)$

proof –

have 1: $\vdash bi f \vee bi g \longrightarrow bi(f \vee g)$

by (*rule BiOrBiImpBiOr*)

hence 2: $\vdash (bi f \vee bi g);skip \longrightarrow (bi(f \vee g));skip$

by (*rule LeftChopImpChop*)

have 3: $\vdash (bi f);skip \vee (bi g);skip \longrightarrow (bi(f \vee g));skip$

using 1 *OrChopEqv* 2 **by** *fastforce*

hence 4: $\vdash empty \vee (bi f);skip \vee (bi g);skip \longrightarrow empty \vee (bi(f \vee g));skip$

by *auto*

hence 5: $\vdash (empty \vee (bi f);skip) \vee (empty \vee (bi g);skip) \longrightarrow empty \vee (bi(f \vee g));skip$

by *auto*

from 5 **show** *?thesis* **by** (*simp add: bs-d-def*)

qed

lemma *DsAndImp*:

$\vdash ds(f \wedge g) \longrightarrow ds f \wedge ds g$

proof –

have 1: $\vdash bs(\neg f) \vee bs(\neg g) \longrightarrow bs(\neg f \vee \neg g)$ **by** (*rule BsOrImp*)

have 2: $\vdash (\neg f \vee \neg g) = (\neg(f \wedge g))$ **by** *auto*

hence 3: $\vdash bs(\neg f \vee \neg g) = bs(\neg(f \wedge g))$ **by** (*rule BsEqvRule*)

have 4: $\vdash bs(\neg f) \vee bs(\neg g) \longrightarrow bs(\neg(f \wedge g))$ **using** 1 3 **by** *fastforce*

have 5: $\vdash bs(\neg f) = (\neg(ds f))$ **using** *NotDsEqvBsNot* **by** *fastforce*

have 6: $\vdash bs(\neg g) = (\neg(ds g))$ **using** *NotDsEqvBsNot* **by** *fastforce*

have 7: $\vdash bs(\neg(f \wedge g)) = (\neg(ds(f \wedge g)))$ **using** *NotDsEqvBsNot* **by** *fastforce*

have 8: $\vdash \neg(ds f) \vee \neg(ds g) \longrightarrow \neg(ds(f \wedge g))$ **using** 4 5 6 7 **by** *fastforce*

hence 9: $\vdash \neg(ds f \wedge ds g) \longrightarrow \neg(ds(f \wedge g))$ **by** *auto*

from 9 **show** *?thesis* **by** *auto*

qed

lemma *DsAndImpElimL*:

$\vdash ds(f \wedge g) \longrightarrow ds f$

using *DsAndImp* **by** *fastforce*

lemma *DsAndImpElimR*:

$\vdash ds(f \wedge g) \longrightarrow ds g$

using *DsAndImp* **by** *fastforce*

lemma *BiImpBs*:

$\vdash bi f \longrightarrow bs f$

proof –
have 1: $\vdash \text{empty} \longrightarrow \text{empty} \vee (\text{bi } f); \text{skip}$ **by** *auto*
hence 2: $\vdash \text{empty} \wedge \text{bi } f \longrightarrow \text{empty} \vee (\text{bi } f); \text{skip}$ **by** *auto*
have 2: $\vdash \text{more} \wedge \text{bi } f \longrightarrow (\text{bi } f); \text{skip}$ **by** (*rule MoreAndBiImpBiChopSkip*)
hence 3: $\vdash \text{more} \wedge \text{bi } f \longrightarrow \text{empty} \vee (\text{bi } f); \text{skip}$ **by** *auto*
have 4: $\vdash \text{bi } f = ((\text{bi } f \wedge \text{empty}) \vee (\text{bi } f \wedge \text{more}))$ **by** (*auto simp add: empty-d-def*)
have 5: $\vdash (\text{empty} \vee (\text{bi } f); \text{skip}) = \text{bs } f$ **by** (*simp add: bs-d-def*)
from 2 3 4 5 **show** *?thesis* **by** *fastforce*
qed

lemma *BsImpBsBs*:

$\vdash \text{bs } f \longrightarrow \text{bs } (\text{bs } f)$

proof –

have 1: $\vdash \text{bi } f \longrightarrow \text{bs } f$ **by** (*rule BiImpBs*)
hence 2: $\vdash \text{bi } (\text{bi } f) \longrightarrow \text{bi } (\text{bs } f)$ **by** (*rule BiImpBiRule*)
hence 3: $\vdash (\text{bi } f) \longrightarrow \text{bi } (\text{bs } f)$ **using** *BiEqvBiBi* **by** *fastforce*
hence 4: $\vdash (\text{bi } f); \text{skip} \longrightarrow (\text{bi } (\text{bs } f)); \text{skip}$ **by** (*rule LeftChopImpChop*)
hence 5: $\vdash \text{empty} \vee (\text{bi } f); \text{skip} \longrightarrow \text{empty} \vee (\text{bi } (\text{bs } f)); \text{skip}$ **by** *auto*
from 5 **show** *?thesis* **by** (*simp add: bs-d-def*)

qed

lemma *DsImpDi*:

$\vdash \text{ds } f \longrightarrow \text{di } f$

proof –

have 1: $\vdash \text{bi } (\neg f) \longrightarrow \text{bs } (\neg f)$ **by** (*rule BiImpBs*)
hence 2: $\vdash \neg (\text{bs } (\neg f)) \longrightarrow \neg (\text{bi } (\neg f))$ **by** *fastforce*
from 2 **show** *?thesis* **using** *NotBsEqvDsNot DiEqvNotBiNot* **by** *fastforce*

qed

lemma *BsImpBsRule*:

assumes $\vdash f \longrightarrow g$

shows $\vdash \text{bs } f \longrightarrow \text{bs } g$

proof –

have 1: $\vdash f \longrightarrow g$ **using** *assms* **by** *auto*
hence 2: $\vdash \text{bi } f \longrightarrow \text{bi } g$ **by** (*rule BiImpBiRule*)
hence 3: $\vdash (\text{bi } f); \text{skip} \longrightarrow (\text{bi } g); \text{skip}$ **by** (*rule LeftChopImpChop*)
hence 4: $\vdash \text{empty} \vee (\text{bi } f); \text{skip} \longrightarrow \text{empty} \vee (\text{bi } g); \text{skip}$ **by** *auto*
from 4 **show** *?thesis* **by** (*simp add: bs-d-def*)

qed

lemma *DsChopImpDsB*:

$\vdash \text{ds } (f;g) \longrightarrow \text{ds } f$

proof –

have 1: $\vdash \text{di } (f;g) \longrightarrow \text{di } f$ **by** (*rule DiChopImpDiB*)
hence 2: $\vdash (\text{di } (f;g)); \text{skip} \longrightarrow (\text{di } f); \text{skip}$ **by** (*rule LeftChopImpChop*)
from 2 **show** *?thesis* **using** *DsDi* **by** *fastforce*

qed

lemma *NotBsImpBsNotChop*:

$\vdash \text{bs } (\neg f) \longrightarrow \text{bs } (\neg (f;g))$

proof –
have 1: $\vdash ds(f;g) \longrightarrow ds f$ **by** (rule *DsChopImpDsB*)
hence 2: $\vdash \neg(ds f) \longrightarrow \neg(ds(f;g))$ **by** *fastforce*
from 2 **show** *?thesis* **using** *NotDsEqvBsNot* **by** *fastforce*
qed

lemma *BsOrBsEqvBsBiOrBi*:

$\vdash (bs f \vee bs g) = bs(bi f \vee bi g)$

proof –

have 1: $\vdash (bs f \vee bs g) = ((empty \vee (bi f);skip) \vee (empty \vee (bi g);skip))$
by (*simp add: bs-d-def*)

have 2: $\vdash ((empty \vee (bi f);skip) \vee (empty \vee (bi g);skip)) = (empty \vee (bi f);skip \vee (bi g);skip)$
by *auto*

have 3: $\vdash ((bi f);skip \vee (bi g);skip) = (bi f \vee bi g);skip$
using *OrChopEqv* **by** *fastforce*

hence 4: $\vdash (empty \vee (bi f);skip \vee (bi g);skip) = (empty \vee (bi f \vee bi g);skip)$
by *auto*

have 5: $\vdash (bi f \vee bi g) = bi (bi f \vee bi g)$
by (*meson BiBiOrImpBi BiImpBiBiOr BiImpBiBiOrB Prop02 int-iffI*)

hence 6: $\vdash (bi f \vee bi g);skip = bi (bi f \vee bi g);skip$
by (*simp add: LeftChopEqvChop*)

hence 7: $\vdash (empty \vee bi (bi f \vee bi g);skip) = (empty \vee (bi f \vee bi g);skip)$
by *auto*

have 8: $\vdash (empty \vee (bi f \vee bi g);skip) = bs(bi f \vee bi g)$ **using** *bs-d-def*
by (*metis 4 5 inteq-reflection*)

from 1 2 4 8 **show** *?thesis* **by** (*metis inteq-reflection*)

qed

lemma *DiOrDsEqvDi*:

$\vdash di f \vee ds f = di f$

proof –

have 1: $\vdash di f \longrightarrow di f \vee ds f$ **by** *auto*

have 2: $\vdash di f \longrightarrow di f$ **by** *auto*

have 3: $\vdash ds f \longrightarrow di f$ **by** (rule *DsImpDi*)

have 4: $\vdash di f \vee ds f \longrightarrow di f$ **using** 2 3 **by** *auto*

from 1 4 **show** *?thesis* **by** *auto*

qed

lemma *DiAndDsEqvDs*:

$\vdash (di f \wedge ds f) = ds f$

proof –

have 1: $\vdash di f \wedge ds f \longrightarrow ds f$ **by** *auto*

have 2: $\vdash ds f \longrightarrow ds f$ **by** *auto*

have 3: $\vdash ds f \longrightarrow di f$ **by** (rule *DsImpDi*)

have 4: $\vdash ds f \longrightarrow di f \wedge ds f$ **using** 2 3 **by** *auto*

from 1 4 **show** *?thesis* **by** *auto*

qed

lemma *OrDsEqvDi*:

$\vdash (f \vee ds\ f) = di\ f$
proof –
have 1: $\vdash ds\ f = (di\ f);skip$ **by** (rule *DsDi*)
hence 2: $\vdash (f \vee ds\ f) = (f \vee (di\ f);skip)$ **by** *auto*
from 2 **show** ?thesis **using** *DiEqvOrDiChopSkipB[of f]* **by** *fastforce*
qed

lemma *AndBsEqvBi*:
 $\vdash (f \wedge bs\ f) = bi\ f$
proof –
have 1: $\vdash (f \wedge bs\ f) = (f \wedge (empty \vee (bi\ f);skip))$ **by** (*simp add: bs-d-def*)
from 1 **show** ?thesis
by (*meson BiEqvAndEmptyOrBiChopSkip Prop04 int-simps(4)*)
qed

lemma *BsEqvBsBi*:
 $\vdash bs\ f = bs\ (bi\ f)$
proof –
have 1: $\vdash bs\ f = (empty \vee (bi\ f);skip)$ **by** (*simp add: bs-d-def*)
have 2: $\vdash bi\ f = bi\ (bi\ f)$ **by** (rule *BiEqvBiBi*)
hence 3: $\vdash (bi\ f);skip = bi\ (bi\ f);skip$ **using** *LeftChopEqvChop* **by** *blast*
hence 4: $\vdash (empty \vee (bi\ f);skip) = (empty \vee bi\ (bi\ f);skip)$ **by** *auto*
from 1 4 **show** ?thesis **by** (*simp add: bs-d-def*)
qed

lemma *StateImpBs*:
 $\vdash init\ w \longrightarrow bs\ (init\ w)$
proof –
have 1: $\vdash init\ w = bi\ (init\ w)$ **by** (rule *StateEqvBi*)
have 2: $\vdash bi\ (init\ w) \longrightarrow bs\ (init\ w)$ **by** (rule *BiImpBs*)
from 1 2 **show** ?thesis **using** *StateImpBi* **by** *fastforce*
qed

lemma *DsAndDsEqvDsAndDiOrDsAndDi*:
 $\vdash (ds\ f \wedge ds\ g) = (ds\ (f \wedge di\ g) \vee ds\ (g \wedge di\ f))$
proof –
have 1: $\vdash (di\ f \wedge di\ g) = (di\ (f \wedge di\ g) \vee di\ (g \wedge di\ f))$
by (rule *DiAndDiEqvDiAndDiOrDiAndDi*)
hence 2: $\vdash (di\ f \wedge di\ g);skip = (di\ (f \wedge di\ g) \vee di\ (g \wedge di\ f));skip$
by (rule *LeftChopEqvChop*)
have 3: $\vdash (di\ f \wedge di\ g);skip = ((di\ f);skip \wedge (di\ g);skip)$
using *ChopSkipAndChopSkip* **by** *fastforce*
have 4: $\vdash ((di\ f);skip \wedge (di\ g);skip) = (di\ (f \wedge di\ g) \vee di\ (g \wedge di\ f));skip$
using 2 3 **by** *fastforce*
have 5: $\vdash (di\ (f \wedge di\ g) \vee di\ (g \wedge di\ f));skip = (di\ (f \wedge di\ g);skip \vee di\ (g \wedge di\ f);skip)$
using *OrChopEqv* **by** *blast*
have 6: $\vdash ds\ f = (di\ f);skip$
using *DsDi* **by** *blast*
have 7: $\vdash ds\ g = (di\ g);skip$
using *DsDi* **by** *blast*

have 8: $\vdash ((di\ f);skip \wedge (di\ g);skip) = (ds\ f \wedge ds\ g)$
using 6 7 by fastforce
have 9: $\vdash ds(f \wedge di\ g) = di(f \wedge di\ g);skip$
using DsDi by blast
have 10: $\vdash ds(g \wedge di\ f) = di(g \wedge di\ f);skip$
using DsDi by blast
have 11: $\vdash (di(f \wedge di\ g);skip \vee di(g \wedge di\ f);skip) = (ds(f \wedge di\ g) \vee ds(g \wedge di\ f))$
using 9 10 by fastforce
from 4 5 8 11 show ?thesis by (meson Prop04)
qed

lemma BsEqvBiMoreImpChop:

$\vdash bs\ f = bi(more \longrightarrow f;skip)$

proof –

have 1: $\vdash bs\ f = (empty \vee (bi\ f;skip))$
by (simp add: bs-d-def)
have 2: $\vdash (empty \vee (bi\ f;skip)) = ((\neg(\neg(bi\ f)));skip)$
using NotNotChopSkip by fastforce
have 3: $\vdash \neg(\neg(bi\ f));skip = (\neg(di\ (\neg f));skip)$
by (simp add: bi-d-def)
have 4: $\vdash (\neg(di\ (\neg f));skip) = (\neg(((\neg f) ;\# True);skip))$
by (simp add: di-d-def)
have 5: $\vdash (\neg(((\neg f) ;\# True);skip)) = (\neg((\neg f) ;(\# True;skip)))$
using ChopAssocB by fastforce
have 6: $\vdash (\neg((\neg f) ;(\# True;skip))) = (\neg((\neg f) ;(skip;\# True)))$
by (metis 5 SkipTrueEqvTrueSkip int-eq)
have 7: $\vdash (\neg((\neg f) ;(skip;\# True))) = (\neg(((\neg f) ;skip);\# True))$
using ChopAssoc by fastforce
have 8: $\vdash (\neg(((\neg f) ;skip);\# True)) = (\neg(di\ ((\neg f);skip)))$
by (simp add: di-d-def)
have 9: $\vdash (\neg(di\ ((\neg f);skip))) = bi\ (\neg((\neg f) ;skip))$
using NotDiEqvBiNot by blast
have 10: $\vdash bi\ (\neg((\neg f) ;skip)) = bi\ (empty \vee (f;skip))$
using NotNotChopSkip using BiEqvBi by blast
have 11: $\vdash bi\ (empty \vee (f;skip)) = bi\ (\neg more \vee (f;skip))$
by (simp add: empty-d-def)
have 12: $\vdash (\neg more \vee (f;skip)) = (more \longrightarrow f;skip)$ **by auto**
have 13: $\vdash bi\ (\neg more \vee (f;skip)) = bi(more \longrightarrow f;skip)$ **using 12 using BiEqvBi by blast**
have 14: $\vdash bs\ f = (\neg(((\neg f);skip);\# True))$ **using 1 2 3 4 5 6 7 by fastforce**
have 15: $\vdash (\neg(((\neg f);skip);\# True)) = bi(more \longrightarrow f;skip)$ **using 8 9 10 11 13 by fastforce**
from 14 15 show ?thesis by fastforce

qed

lemma BoxMoreStateEqvBsFinState:

$\vdash \Box(more \longrightarrow \neg (init\ w)) = bs(\neg(fin(init\ w)))$

proof –

have 1: $\vdash \Box(more \longrightarrow \neg (init\ w)) = (\neg(\Diamond(\neg(more \longrightarrow \neg (init\ w)))))$
by (simp add: always-d-def)
have 01: $\vdash (\neg(more \longrightarrow \neg (init\ w))) = (init\ w \wedge more)$ **by auto**
hence 2: $\vdash \neg(\Diamond(\neg(more \longrightarrow \neg (init\ w)))) = (\neg(\# True;(init\ w \wedge more)))$

by (metis int-eq int-iffD1 int-simps(14) int-simps(6) sometimes-d-def)
have 3: $\vdash \text{more} = \# \text{True}; \text{skip}$
 using MoreEqvSkipChopTrue SkipTrueEqvTrueSkip by fastforce
have 4: $\vdash (\text{init } w \wedge \text{more}) = (\text{init } w \wedge (\# \text{True}; \text{skip}))$
 using 3 by auto
have 5: $\vdash (\text{init } w \wedge (\# \text{True}; \text{skip})) = ((\text{init } w \wedge \text{empty}); (\# \text{True}; \text{skip}))$
 using StateAndEmptyChop by fastforce
have 6: $\vdash (\text{init } w \wedge \text{more}) = ((\text{init } w \wedge \text{empty}); (\# \text{True}; \text{skip}))$
 using 4 5 by fastforce
have 7: $\vdash (\# \text{True}; (\text{init } w \wedge \text{more})) = (\# \text{True}; ((\text{init } w \wedge \text{empty}); (\# \text{True}; \text{skip})))$
 using 6 RightChopEqvChop by blast
have 8: $\vdash (\# \text{True}; ((\text{init } w \wedge \text{empty}); (\# \text{True}; \text{skip}))) =$
 $((\# \text{True}; (\text{init } w \wedge \text{empty})); (\# \text{True}; \text{skip}))$
 using ChopAssoc by blast
have 9: $\vdash (((\# \text{True}; (\text{init } w \wedge \text{empty})); (\# \text{True}; \text{skip}))) =$
 $((((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True}); \text{skip}))$
 using ChopAssoc by blast
have 10: $\vdash (\# \text{True}; (\text{init } w \wedge \text{more})) =$
 $((((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True}); \text{skip}))$
 using 7 8 9 by fastforce
hence 11: $\vdash (\neg(\# \text{True}; (\text{init } w \wedge \text{more}))) =$
 $(\neg(((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True}); \text{skip})))$
 by auto
have 12: $\vdash \neg(((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True}); \text{skip})) =$
 $\text{empty} \vee (\neg((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True}); \text{skip}))$
 using NotChopNotSkip by fastforce
have 13: $\vdash (\neg((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True})) = \text{bi}(\Box (\neg(\text{init } w \wedge \text{empty})))$
 using BiBoxNotEqvNotTrueChopChopTrue by fastforce
hence 14: $\vdash (\neg((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True})); \text{skip} =$
 $(\text{bi}(\Box (\neg(\text{init } w \wedge \text{empty})))); \text{skip}$
 using RightChopEqvChop by (simp add: LeftChopEqvChop)
hence 15: $\vdash \text{empty} \vee (\neg((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True})); \text{skip} =$
 $\text{empty} \vee (\text{bi}(\Box (\neg(\text{init } w \wedge \text{empty})))); \text{skip}$
 by auto
have 16: $\vdash (\neg(((\# \text{True}; (\text{init } w \wedge \text{empty})); \# \text{True}); \text{skip})) =$
 $(\text{empty} \vee (\text{bi}(\Box (\neg(\text{init } w \wedge \text{empty})))); \text{skip}))$
 using 12 15 using 14 NotChopNotSkip int-eq by fastforce
have 171: $\vdash (\neg(\text{init } w \wedge \text{empty})) = (\neg(\text{init } w) \vee \neg \text{empty})$
 by auto
hence 172: $\vdash \Box(\neg(\text{init } w \wedge \text{empty})) = \Box(\neg(\text{init } w) \vee \neg \text{empty})$
 by (simp add: BoxEqvBox)
hence 173: $\vdash \text{bi}(\Box(\neg(\text{init } w \wedge \text{empty}))) = \text{bi}(\Box(\neg(\text{init } w) \vee \neg \text{empty}))$
 by (simp add: BiEqvBi)
hence 174: $\vdash \text{bi}(\Box(\neg(\text{init } w \wedge \text{empty}))); \text{skip} = \text{bi}(\Box(\neg(\text{init } w) \vee \neg \text{empty})); \text{skip}$
 using LeftChopEqvChop by blast
hence 17: $\vdash (\text{empty} \vee (\text{bi}(\Box (\neg(\text{init } w \wedge \text{empty}))); \text{skip})) =$
 $(\text{empty} \vee (\text{bi}(\Box (\neg(\text{init } w) \vee \neg \text{empty})); \text{skip}))$
 by auto
have 181: $\vdash (\neg(\text{init } w) \vee \neg \text{empty}) = (\neg \text{empty} \vee \neg(\text{init } w))$
 by auto

hence 18: $\vdash \Box (\neg(\text{init } w) \vee \neg \text{empty}) = \Box (\neg \text{empty} \vee \neg(\text{init } w))$
by (*simp add: BoxEqvBox*)
have 191: $\vdash (\neg \text{empty} \vee \neg(\text{init } w)) = (\text{empty} \longrightarrow \neg(\text{init } w))$
by *auto*
hence 19: $\vdash \Box (\neg \text{empty} \vee \neg(\text{init } w)) = \Box(\text{empty} \longrightarrow \neg(\text{init } w))$
by (*simp add: BoxEqvBox*)
have 20: $\vdash \Box(\text{empty} \longrightarrow \neg(\text{init } w)) = \text{fin } (\neg(\text{init } w))$
by (*simp add: fin-d-def*)
have 21: $\vdash \text{fin } (\neg(\text{init } w)) = (\neg(\text{fin } (\text{init } w)))$
using *FinEqvFin FinNotStateEqvNotFinState Initprop(2)* **by** *fastforce*
have 22: $\vdash \text{bi}(\Box (\neg(\text{init } w) \vee \neg \text{empty})) = \text{bi } (\neg(\text{fin } (\text{init } w)))$
using 18 19 20 21 *BiEqvBi* **by** (*metis int-eq*)
hence 23: $\vdash (\text{bi}(\Box (\neg(\text{init } w) \vee \neg \text{empty}))); \text{skip} = (\text{bi } (\neg(\text{fin } (\text{init } w)))); \text{skip}$
using *RightChopEqvChop* **by** (*simp add: LeftChopEqvChop*)
hence 24: $\vdash (\text{empty} \vee (\text{bi}(\Box (\neg(\text{init } w) \vee \neg \text{empty}))); \text{skip}) =$
 $(\text{empty} \vee (\text{bi } (\neg(\text{fin } (\text{init } w)))); \text{skip})$
by *auto*
hence 25: $\vdash (\text{empty} \vee (\text{bi } (\neg(\text{fin } (\text{init } w)))); \text{skip}) = \text{bs}(\neg(\text{fin } (\text{init } w)))$
by (*simp add: bs-d-def*)
show *?thesis*
by (*metis 01 1 11 16 172 18 19 20 21 25 inteq-reflection sometimes-d-def*)
qed

lemma *BsFalseEqvEmpty:*

$\vdash \text{bs } \# \text{False} = \text{empty}$

proof –

have 1: $\vdash \text{bs } \# \text{False} = (\text{empty} \vee \text{bi } \# \text{False}; \text{skip})$

by (*simp add: bs-d-def*)

have 2: $\vdash \neg(\text{bi } \# \text{False}; \text{skip})$

by (*metis BiEqvAndWprevBi MoreEqvSkipChopTrue NotChopSkipEqvMoreAndNotChopSkip*
SkipTrueEqvTrueSkip int-eq int-iffD1 int-simps(14) int-simps(19) int-simps(2)
int-simps(21))

from 1 2 show *?thesis* **by** *fastforce*

qed

9.4.4 First occurrence

lemma *FstWithAndImp:*

$\vdash \triangleright f \wedge g \longrightarrow \triangleright (f \wedge g)$

proof –

have 1: $\vdash (\triangleright f \wedge g) = ((f \wedge (\text{bs } (\neg f)))) \wedge g$

by (*simp add: first-d-def*)

have 2: $\vdash ((f \wedge (\text{bs } (\neg f)))) \wedge g = (f \wedge \neg(ds f) \wedge g)$

using *NotDsEqvBsNot* **by** *fastforce*

have 3: $\vdash \neg(ds f) \longrightarrow \neg(ds(f \wedge g))$

using *DsAndImpElimL* **by** *fastforce*

hence 4: $\vdash f \wedge \neg(ds f) \wedge g \longrightarrow f \wedge g \wedge \neg(ds(f \wedge g))$

by *auto*

have 5: $\vdash (f \wedge g \wedge \neg(ds(f \wedge g))) = ((f \wedge g) \wedge (\text{bs } (\neg(f \wedge g))))$

using *NotDsEqvBsNot* **by** *fastforce*

have 6: $\vdash ((f \wedge g) \wedge (bs (\neg(f \wedge g)))) = \triangleright(f \wedge g)$
by (*simp add: first-d-def*)
from 1 2 4 5 6 **show** ?thesis **by** fastforce
qed

lemma *FstWithOrEqv*:

$\vdash \triangleright(f \vee g) = ((\triangleright f \wedge bs (\neg g)) \vee (\triangleright g \wedge bs (\neg f)))$

proof –

have 1: $\vdash \triangleright(f \vee g) = ((f \vee g) \wedge bs (\neg(f \vee g)))$

by (*simp add: first-d-def*)

have 2: $\vdash (\neg(f \vee g)) = (\neg f \wedge \neg g)$

by auto

hence 3: $\vdash bs (\neg(f \vee g)) = bs (\neg f \wedge \neg g)$

using BsEqvRule **by** blast

have 4: $\vdash bs (\neg f \wedge \neg g) = (bs (\neg f) \wedge bs (\neg g))$

using BsAndEqv **by** fastforce

have 5: $\vdash ((f \vee g) \wedge bs (\neg(f \vee g))) = ((f \vee g) \wedge bs (\neg f) \wedge bs (\neg g))$

using 3 4 **by** fastforce

have 6: $\vdash ((f \vee g) \wedge bs (\neg f) \wedge bs (\neg g)) =$

$((f \wedge bs (\neg f)) \wedge bs (\neg g)) \vee (g \wedge bs (\neg f) \wedge bs (\neg g))$

by auto

have 7: $\vdash ((f \wedge bs (\neg f)) \wedge bs (\neg g)) = (\triangleright f \wedge bs (\neg g))$

by (*simp add: first-d-def*)

have 8: $\vdash (g \wedge bs (\neg f) \wedge bs (\neg g)) = ((g \wedge bs (\neg g)) \wedge bs (\neg f))$

by auto

have 9: $\vdash ((g \wedge bs (\neg g)) \wedge bs (\neg f)) = (\triangleright g \wedge bs (\neg f))$

by (*simp add: first-d-def*)

have 10: $\vdash ((f \wedge bs (\neg f)) \wedge bs (\neg g)) \vee (g \wedge bs (\neg f) \wedge bs (\neg g)) =$

$(\triangleright f \wedge bs (\neg g)) \vee (\triangleright g \wedge bs (\neg f))$

using 7 8 9 **by** fastforce

from 1 5 6 10 **show** ?thesis **by** (metis 7 8 9 int-eq)

qed

lemma *FstFstAndEqvFstAnd*:

$\vdash \triangleright(\triangleright f \wedge g) = (\triangleright f \wedge g)$

proof –

have 1: $\vdash (\triangleright f \wedge g) = ((f \wedge (bs (\neg f))) \wedge g)$ **by** (*simp add: first-d-def*)

hence 2: $\vdash \triangleright f \wedge g \longrightarrow (bs (\neg f))$ **by** auto

hence 3: $\vdash \triangleright f \wedge g \longrightarrow \triangleright f \wedge g \wedge (bs (\neg f))$ **by** auto

have 4: $\vdash \neg f \longrightarrow \neg f \vee \neg(bs (\neg f)) \vee \neg g$ **by** auto

hence 5: $\vdash bs (\neg f) \longrightarrow bs(\neg f \vee \neg(bs (\neg f)) \vee \neg g)$ **using** BsImpBsRule **by** blast

have 6: $\vdash (\neg f \vee \neg(bs (\neg f)) \vee \neg g) = (\neg(f \wedge bs (\neg f) \wedge g))$ **by** auto

hence 7: $\vdash bs(\neg f \vee \neg(bs (\neg f)) \vee \neg g) = bs(\neg(f \wedge bs (\neg f) \wedge g))$ **using** BsEqvRule **by** blast

have 8: $\vdash ((f \wedge bs (\neg f)) \wedge g) = (\triangleright f \wedge g)$ **by** (*simp add: first-d-def*)

hence 9: $\vdash (\neg(f \wedge bs (\neg f) \wedge g)) = (\neg(\triangleright f \wedge g))$ **by** auto

hence 10: $\vdash bs (\neg(f \wedge bs (\neg f) \wedge g)) = bs (\neg(\triangleright f \wedge g))$ **using** BsEqvRule **by** blast

have 11: $\vdash \triangleright f \wedge g \longrightarrow (\triangleright f \wedge g) \wedge bs (\neg(\triangleright f \wedge g))$ **using** 3 5 7 10 **by** fastforce

hence 12: $\vdash \triangleright f \wedge g \longrightarrow \triangleright(\triangleright f \wedge g)$ **by** (*simp add: first-d-def*)

have 13: $\vdash \triangleright(\triangleright f \wedge g) = ((\triangleright f \wedge g) \wedge bs (\neg(\triangleright f \wedge g)))$ **by** (*simp add: first-d-def*)

hence 14: $\vdash \triangleright(\triangleright f \wedge g) \longrightarrow \triangleright f \wedge g$ **by** auto

from 12 14 show ?thesis by fastforce
qed

lemma *FstTrue*:

$\vdash \triangleright \#True = empty$

proof –

have 1: $\vdash \triangleright \#True = (\#True \wedge bs (\neg \#True))$

by (simp add: first-d-def)

have 2: $\vdash bs (\neg \#True) = (empty \vee (bi (\neg \#True));skip)$

by (simp add: bs-d-def)

have 3: $\vdash \neg(bi (\neg \#True))$

using BiElim by fastforce

have 4: $\vdash \neg((bi (\neg \#True));skip)$

by (metis AndChopA BiEqvAndEmptyOrBiChopSkip MoreEqvSkipChopTrue

NotChopSkipEqvMoreAndNotChopSkip SkipTrueEqvTrueSkip int-eq int-simps(14) int-simps(21))

have 5: $\vdash bs (\neg \#True) = empty$

using 2 4 by fastforce

from 1 5 show ?thesis by fastforce

qed

lemma *FstFalse*:

$\vdash \neg(\triangleright \#False)$

proof –

have 1: $\vdash \triangleright \#False = (\#False \wedge bs \#True)$ by (simp add: first-d-def)

from 1 show ?thesis by auto

qed

lemma *FstChopFalseEqvFalse*:

$\vdash \neg(\triangleright f ; \#False)$

by (simp add: Valid-def chop-defs)

lemma *FstEmpty*:

$\vdash \triangleright empty = empty$

proof –

have 1: $\vdash \triangleright empty = (empty \wedge bs (\neg empty))$ by (simp add: first-d-def)

have 2: $\vdash bs (\neg empty) = (empty \vee bi (\neg empty);skip)$ by (simp add: bs-d-def)

from 1 2 show ?thesis by fastforce

qed

lemma *FstAndEmptyEqvAndEmpty*:

$\vdash (\triangleright f \wedge empty) = (f \wedge empty)$

proof –

have 1: $\vdash (\triangleright f \wedge empty) = ((f \wedge bs (\neg f)) \wedge empty)$ by (simp add: first-d-def)

have 2: $\vdash bs (\neg f) = (empty \vee bi (\neg f);skip)$ by (simp add: bs-d-def)

from 1 2 show ?thesis by fastforce

qed

lemma *FstEmptyOrEqvEmpty*:

$\vdash \triangleright(empty \vee f) = empty$

proof –

have 1: $\vdash \triangleright(\text{empty} \vee f) = ((\triangleright \text{empty} \wedge \text{bs}(\neg f)) \vee (\triangleright f \wedge \text{bs}(\neg \text{empty})))$ **using** *FstWithOrEqv* **by** *blast*
have 2: $\vdash (\neg \text{empty}) = \text{more}$ **by** (*simp add: empty-d-def*)
hence 3: $\vdash \text{bs}(\neg \text{empty}) = \text{bs more}$ **using** *BsEqvRule* **by** *blast*
have 4: $\vdash \text{bs more} = \text{empty}$ **using** *BsMoreEqvEmpty* **by** *blast*
have 5: $\vdash (\triangleright f \wedge \text{bs}(\neg \text{empty})) = (\triangleright f \wedge \text{empty})$ **using** 3 4 **by** *fastforce*
have 6: $\vdash \triangleright \text{empty} = \text{empty}$ **using** *FstEmpty* **by** *blast*
hence 7: $\vdash (\triangleright \text{empty} \wedge \text{bs}(\neg f)) = (\text{empty} \wedge \text{bs}(\neg f))$ **by** *auto*
have 8: $\vdash (\text{empty} \wedge \text{bs}(\neg f)) = (\text{empty} \wedge (\text{empty} \vee \text{bi}(\neg f); \text{skip}))$ **by** (*simp add: bs-d-def*)
have 9: $\vdash (\text{empty} \wedge (\text{empty} \vee \text{bi}(\neg f); \text{skip})) = \text{empty}$ **by** *auto*
have 10: $\vdash (\text{empty} \wedge \text{bs}(\neg f)) = \text{empty}$ **using** 8 9 **by** *auto*
have 11: $\vdash ((\triangleright \text{empty} \wedge \text{bs}(\neg f)) \vee (\triangleright f \wedge \text{bs}(\neg \text{empty}))) =$
 $(\text{empty} \vee (\triangleright f \wedge \text{empty}))$ **using** 7 10 5 **by** *fastforce*
have 12: $\vdash (\text{empty} \vee (\triangleright f \wedge \text{empty})) = \text{empty}$ **by** *auto*
from 1 11 12 **show** *?thesis* **by** *fastforce*
qed

lemma *FstChopEmptyEqvFstChopFstEmpty*:

$\vdash (\triangleright f; g \wedge \text{empty}) = (\triangleright f; \triangleright g \wedge \text{empty})$

proof –

have 1: $\vdash (\triangleright f; g \wedge \text{empty}) = (\triangleright f \wedge g \wedge \text{empty})$ **using** *ChopEmptyAndEmpty* **by** *blast*
have 2: $\vdash (\triangleright g \wedge \text{empty}) = (g \wedge \text{empty})$ **using** *FstAndEmptyEqvAndEmpty* **by** *blast*
hence 3: $\vdash (\triangleright f \wedge g \wedge \text{empty}) = (\triangleright f \wedge \triangleright g \wedge \text{empty})$ **by** *auto*
have 4: $\vdash (\triangleright f; \triangleright g \wedge \text{empty}) = (\triangleright f \wedge \triangleright g \wedge \text{empty})$ **using** *ChopEmptyAndEmpty* **by** *blast*
from 1 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *FstMoreEqvSkip*:

$\vdash \triangleright \text{more} = \text{skip}$

proof –

have 1: $\vdash \triangleright \text{more} = (\text{more} \wedge \text{bs}(\neg \text{more}))$ **by** (*simp add: first-d-def*)
have 2: $\vdash (\text{more} \wedge \text{bs}(\neg \text{more})) = (\text{more} \wedge (\text{empty} \vee \text{bi}(\neg \text{more}); \text{skip}))$ **by** (*simp add: bs-d-def*)
have 3: $\vdash (\text{more} \wedge (\text{empty} \vee \text{bi}(\neg \text{more}); \text{skip})) = (\text{more} \wedge \text{bi}(\neg \text{more}); \text{skip})$ **using** *empty-d-def*
using *MoreAndEmptyOrEqvMoreAnd* **by** *fastforce*
have 4: $\vdash (\text{more} \wedge ((\text{bi}(\neg \text{more}); \text{skip}))) = ((\text{bi}(\neg \text{more}); \text{skip}))$ **using** *ChopSkipImpMore* **by** *fastforce*
have 5: $\vdash ((\text{bi}(\neg \text{more}); \text{skip})) = \text{bi empty}; \text{skip}$ **by** (*simp add: empty-d-def*)
have 6: $\vdash \text{bi empty} = \text{empty}$ **using** *BiEmptyEqvEmpty* **by** *auto*
hence 7: $\vdash \text{bi empty}; \text{skip} = \text{empty}; \text{skip}$ **using** *LeftChopEqvChop* **by** *blast*
have 8: $\vdash \text{empty}; \text{skip} = \text{skip}$ **using** *EmptyChop* **by** *blast*
from 1 2 3 4 5 7 8 **show** *?thesis* **by** (*metis int-eq*)

qed

lemma *FstEqvBsNotAndDi*:

$\vdash \triangleright f = (\text{bs}(\neg f) \wedge \text{di } f)$

proof –

have 1: $\vdash \text{bs}(\neg f) = (\neg(\text{ds } f))$ **by** (*simp add: ds-d-def*)
hence 2: $\vdash (\text{bs}(\neg f) \wedge \text{di } f) = (\neg(\text{ds } f) \wedge \text{di } f)$ **by** *auto*
have 3: $\vdash \text{di } f = (\text{ds } f \vee f)$ **using** *OrDsEqvDi* **by** *fastforce*
hence 4: $\vdash (\neg(\text{ds } f) \wedge \text{di } f) = (\neg(\text{ds } f) \wedge (\text{ds } f \vee f))$ **by** *auto*
have 5: $\vdash (\neg(\text{ds } f) \wedge (\text{ds } f \vee f)) = (\neg(\text{ds } f) \wedge f)$ **by** *auto*
have 6: $\vdash (\neg(\text{ds } f) \wedge f) = (f \wedge \text{bs}(\neg f))$ **using** 1 **by** *auto*

from 2 4 5 6 show ?thesis by (metis first-d-def int-eq)
qed

lemma FstOrDiEqvDi:

$\vdash (\triangleright f \vee di\ f) = di\ f$

proof –

have 1: $\vdash (\triangleright f \vee di\ f) = ((f \wedge bs\ (\neg f)) \vee di\ f)$ by (simp add: first-d-def)

have 2: $\vdash ((f \wedge bs\ (\neg f)) \vee di\ f) = ((f \vee di\ f) \wedge (bs\ (\neg f) \vee di\ f))$ by auto

have 3: $\vdash (f \vee di\ f) = di\ f$

by (metis 2 DiIntro RRDiamondEqvDi int-eq Prop02 Prop03 Prop11 Prop12)

hence 4: $\vdash ((f \vee di\ f) \wedge (bs\ (\neg f) \vee di\ f)) = (di\ f \wedge (bs\ (\neg f) \vee di\ f))$ by auto

have 5: $\vdash (di\ f \wedge (bs\ (\neg f) \vee di\ f)) = di\ f$ by auto

from 1 2 4 5 show ?thesis by fastforce

qed

lemma FstAndDiEqvFst:

$\vdash (\triangleright f \wedge di\ f) = \triangleright f$

proof –

have 1: $\vdash (\triangleright f \wedge di\ f) = ((f \wedge bs\ (\neg f)) \wedge di\ f)$ by (simp add: first-d-def)

have 2: $\vdash (f \wedge di\ f) = f$ by (meson DiIntro Prop10 Prop11)

hence 3: $\vdash (f \wedge bs\ (\neg f) \wedge di\ f) = (f \wedge bs\ (\neg f))$ by auto

from 1 3 show ?thesis by (metis first-d-def int-iffD2 int-iffI Prop12)

qed

lemma DiEqvDiFst:

$\vdash di\ f = di\ (\triangleright f)$

proof –

have 1: $\vdash di\ (\triangleright f) = di\ (f \wedge bs\ (\neg f))$

by (simp add: first-d-def)

have 2: $\vdash di\ (f \wedge bs\ (\neg f)) \longrightarrow di\ f \wedge di\ (bs\ (\neg f))$

using DiAndImpAnd by auto

hence 3: $\vdash di\ (f \wedge bs\ (\neg f)) \longrightarrow di\ f$

by auto

have 4: $\vdash di\ (\triangleright f) \longrightarrow di\ f$ using 1 3

by fastforce

have 5: $\vdash (di\ f \wedge empty) = (f \wedge empty)$

using DiAndEmptyEqvAndEmpty by blast

have 6: $\vdash (\triangleright f \wedge empty) = (f \wedge empty)$

using FstAndEmptyEqvAndEmpty by auto

have 7: $\vdash di\ f \wedge empty \longrightarrow \triangleright f$

using 5 6 by fastforce

have 8: $\vdash \triangleright f \longrightarrow di\ (\triangleright f)$

using DiIntro by auto

have 9: $\vdash di\ f \wedge empty \longrightarrow di\ (\triangleright f)$

using 7 8 using lift-imp-trans by blast

hence 10: $\vdash empty \longrightarrow (di\ f \longrightarrow di\ (\triangleright f))$

by auto

have 11: $\vdash prev\ (di\ f \longrightarrow di\ (\triangleright f)) \longrightarrow more$

by (simp add: ChopSkipImpMore prev-d-def)

have 12: $\vdash more \longrightarrow (prev\ (di\ f \longrightarrow di\ (\triangleright f)) = (prev\ (di\ f) \longrightarrow prev\ (di\ (\triangleright f))))$

using *MoreImpImpPrevEqv* **by** *auto*
have 13: $\vdash (more \wedge prev (di f \longrightarrow di (\triangleright f))) = (more \wedge (prev(di f) \longrightarrow prev(di (\triangleright f))))$
using 12 **by** *fastforce*
have 14: $\vdash prev (di f \longrightarrow di (\triangleright f)) = (more \wedge (prev(di f) \longrightarrow prev(di (\triangleright f))))$
using 11 13 **by** *fastforce*
have 15: $\vdash di f = (f \vee ds f)$
using *OrDsEqvDi* **by** *fastforce*
have 16: $\vdash di f = (di f \wedge (bs (\neg f) \vee \neg(bs (\neg f))))$
by *auto*
have 17: $\vdash (di f \wedge (bs (\neg f) \vee \neg(bs (\neg f)))) = ((di f \wedge bs (\neg f)) \vee (di f \wedge \neg(bs (\neg f))))$
by *auto*
have 18: $\vdash (di f \wedge bs (\neg f)) = ((f \vee ds f) \wedge bs (\neg f))$
using 15 **by** *auto*
have 19: $\vdash ((f \vee ds f) \wedge bs (\neg f)) = ((f \wedge bs (\neg f)) \vee (ds f \wedge bs (\neg f)))$
by *auto*
have 20: $\vdash \neg(ds f \wedge bs (\neg f))$
by (*simp add: ds-d-def*)
have 21: $\vdash ((f \wedge bs (\neg f)) \vee (ds f \wedge bs (\neg f))) = (f \wedge bs (\neg f))$
using 20 **by** *auto*
have 22: $\vdash (di f \wedge bs (\neg f)) = (f \wedge bs (\neg f))$
using 18 19 21 **by** *fastforce*
have 23: $\vdash (f \wedge bs (\neg f)) = \triangleright f$
by (*simp add: first-d-def*)
have 24: $\vdash (\triangleright f) \longrightarrow di (\triangleright f)$
using *DiIntro* **by** *auto*
have 25: $\vdash (f \wedge bs (\neg f)) \longrightarrow di (\triangleright f)$
using 23 24 **by** *fastforce*
have 26: $\vdash (di f \wedge bs (\neg f)) \longrightarrow di (\triangleright f)$
using 25 22 **by** *fastforce*
hence 27: $\vdash (di f \wedge bs (\neg f) \wedge (prev (di f \longrightarrow di (\triangleright f)))) \longrightarrow di (\triangleright f)$
by *auto*
have 28: $\vdash (di f \wedge \neg(bs (\neg f))) = (di f \wedge ds f)$
by (*simp add: ds-d-def*)
hence 29: $\vdash (di f \wedge \neg(bs (\neg f)) \wedge (prev (di f \longrightarrow di (\triangleright f)))) =$
 $(di f \wedge ds f \wedge (prev (di f \longrightarrow di (\triangleright f))))$
by *auto*
have 30: $\vdash ds f = prev(di f)$
using *DsDi* **by** (*metis prev-d-def*)
hence 31: $\vdash (di f \wedge ds f \wedge (prev (di f \longrightarrow di (\triangleright f)))) =$
 $(di f \wedge prev(di f) \wedge (prev (di f \longrightarrow di (\triangleright f))))$
by *auto*
have 32: $\vdash prev (di f \longrightarrow di (\triangleright f)) \longrightarrow (prev(di f) \longrightarrow prev(di (\triangleright f)))$
using 14 **by** *auto*
hence 33: $\vdash di f \wedge prev(di f) \wedge prev (di f \longrightarrow di (\triangleright f)) \longrightarrow$
 $di f \wedge prev(di f) \wedge (prev(di f) \longrightarrow prev(di (\triangleright f)))$
by *auto*
have 34: $\vdash di f \wedge prev(di f) \wedge (prev(di f) \longrightarrow prev(di (\triangleright f))) \longrightarrow prev(di (\triangleright f))$
by *auto*
have 35: $\vdash prev(di (\triangleright f)) = (di (\triangleright f)); skip$
by (*simp add: prev-d-def*)

have 36: $\vdash (di \ (\triangleright f)); skip \longrightarrow di(di \ (\triangleright f))$
using *ChopImpDi* **by** *auto*
have 37: $\vdash di(di \ (\triangleright f)) = di \ (\triangleright f)$
using *DiEqvDiDi* **by** *fastforce*
have 38: $\vdash di \ f \wedge prev(di \ f) \wedge (prev(di \ f) \longrightarrow prev(di \ (\triangleright f))) \longrightarrow di \ (\triangleright f)$
using 37 36 35 34 **by** *fastforce*
have 39: $\vdash di \ f \wedge \neg(bs \ (\neg f)) \wedge (prev \ (di \ f \longrightarrow di \ (\triangleright f))) \longrightarrow di \ (\triangleright f)$
using 29 31 33 38 **by** *fastforce*
hence 40: $\vdash \neg(bs \ (\neg f)) \wedge (prev \ (di \ f \longrightarrow di \ (\triangleright f))) \longrightarrow (di \ f \longrightarrow di \ (\triangleright f))$
by *fastforce*
have 41: $\vdash bs \ (\neg f) \wedge (prev \ (di \ f \longrightarrow di \ (\triangleright f))) \longrightarrow (di \ f \longrightarrow di \ (\triangleright f))$
using 27 **by** *fastforce*
have 42: $\vdash (\neg(bs \ (\neg f)) \vee bs \ (\neg f)) \wedge (prev \ (di \ f \longrightarrow di \ (\triangleright f))) \longrightarrow (di \ f \longrightarrow di \ (\triangleright f))$
using 40 41 **by** *fastforce*
have 43: $\vdash (\neg(bs \ (\neg f)) \vee bs \ (\neg f))$
by *auto*
have 44: $\vdash (prev \ (di \ f \longrightarrow di \ (\triangleright f))) \longrightarrow (di \ f \longrightarrow di \ (\triangleright f))$
using 42 43 **by** *fastforce*
have 45: $\vdash di \ f \longrightarrow di \ (\triangleright f)$
using 10 44 *EmptyChopSkipInduct* **by** *blast*
from 4 45 **show** *?thesis* **by** *fastforce*
qed

lemma *FstDiEqvFst*:

$\vdash \triangleright(di \ f) = \triangleright f$

proof –

have 1: $\vdash \triangleright(di \ f) = (di \ f \wedge bs \ (\neg (di \ f)))$ **by** (*simp add: first-d-def*)
have 2: $\vdash \neg (di \ f) = bi \ (\neg f)$ **by** (*simp add: NotDiEqvBiNot*)
hence 3: $\vdash bs \ (\neg (di \ f)) = bs \ (bi \ (\neg f))$ **using** *BsEqvRule* **by** *blast*
have 4: $\vdash bs \ (bi \ (\neg f)) = bs \ (\neg f)$ **using** *BsEqvBsBi* **by** *fastforce*
hence 5: $\vdash (di \ f \wedge bs \ (\neg (di \ f))) = (di \ f \wedge bs \ (\neg f))$ **using** 3 **by** *fastforce*
have 6: $\vdash di \ f = (f \vee ds \ f)$ **using** *OrDsEqvDi* **by** *fastforce*
hence 7: $\vdash (di \ f \wedge bs \ (\neg f)) = ((f \vee ds \ f) \wedge bs \ (\neg f))$ **by** *auto*
have 8: $\vdash ((f \vee ds \ f) \wedge bs \ (\neg f)) = ((f \wedge bs \ (\neg f)) \vee (ds \ f \wedge bs \ (\neg f)))$ **by** *auto*
have 9: $\vdash \neg(ds \ f \wedge bs \ (\neg f))$ **by** (*simp add: ds-d-def*)
have 10: $\vdash (f \wedge bs \ (\neg f)) = \triangleright f$ **by** (*simp add: first-d-def*)
have 11: $\vdash ((f \wedge bs \ (\neg f)) \vee (ds \ f \wedge bs \ (\neg f))) = \triangleright f$ **using** 9 10 **by** *fastforce*
from 1 5 7 8 11 **show** *?thesis* **by** (*metis int-eq*)

qed

lemma *DiAndFstOrEqvFstOrDiAnd*:

$\vdash (di \ f \wedge (\triangleright f \vee g)) = (\triangleright f \vee (di \ f \wedge g))$

proof –

have 1: $\vdash (di \ f \wedge (\triangleright f \vee g)) = (\triangleright f \wedge di \ f) \vee (di \ f \wedge g)$ **by** *auto*
have 2: $\vdash (\triangleright f \wedge di \ f) = \triangleright f$ **using** *FstAndDiEqvFst* **by** *blast*
from 1 2 **show** *?thesis* **by** *auto*

qed

lemma *DiOrFstAndEqvDi*:

$\vdash di \ f \vee (\triangleright f \wedge g) = di \ f$

proof –

have 1: $\vdash (di\ f \vee (\triangleright f \wedge g)) = ((\triangleright f \vee di\ f) \wedge (di\ f \vee g))$ **by** *auto*

have 2: $\vdash (\triangleright f \vee di\ f) = di\ f$ **using** *FstOrDiEqvDi* **by** *blast*

from 1 2 **show** *?thesis* **by** *auto*

qed

lemma *FstDiAndDiEqv*:

$\vdash \triangleright(di\ f \wedge di\ g) = ((\triangleright f \wedge di\ g) \vee (\triangleright g \wedge di\ f))$

proof –

have 1: $\vdash \triangleright(di\ f \wedge di\ g) = ((di\ f \wedge di\ g) \wedge bs(\neg(di\ f \wedge di\ g)))$ **by** (*simp add: first-d-def*)

have 2: $\vdash (\neg(di\ f \wedge di\ g)) = (bi(\neg f) \vee bi(\neg g))$ **by** (*auto simp add: bi-d-def*)

hence 3: $\vdash bs(\neg(di\ f \wedge di\ g)) = bs(bi(\neg f) \vee bi(\neg g))$ **using** *BsEqvRule* **by** *blast*

hence 4: $\vdash ((di\ f \wedge di\ g) \wedge bs(\neg(di\ f \wedge di\ g))) =$
 $(di\ f \wedge di\ g \wedge bs(bi(\neg f) \vee bi(\neg g)))$ **by** *auto*

have 5: $\vdash (bs(\neg f) \vee bs(\neg g)) = bs(bi(\neg f) \vee bi(\neg g))$ **using** *BsOrBsEqvBsBiOrBi* **by** *blast*

hence 6: $\vdash (di\ f \wedge di\ g \wedge bs(bi(\neg f) \vee bi(\neg g))) =$
 $(di\ f \wedge di\ g \wedge (bs(\neg f) \vee bs(\neg g)))$ **by** *auto*

have 7: $\vdash (di\ f \wedge di\ g \wedge (bs(\neg f) \vee bs(\neg g))) =$
 $((bs(\neg f) \wedge di\ f \wedge di\ g) \vee (di\ f \wedge bs(\neg g) \wedge di\ g))$ **by** *auto*

have 8: $\vdash \triangleright f = (bs(\neg f) \wedge di\ f)$ **using** *FstEqvBsNotAndDi* **by** *blast*

hence 9: $\vdash (bs(\neg f) \wedge di\ f \wedge di\ g) = (\triangleright f \wedge di\ g)$ **by** *auto*

have 10: $\vdash \triangleright g = (bs(\neg g) \wedge di\ g)$ **using** *FstEqvBsNotAndDi* **by** *blast*

hence 11: $\vdash (di\ f \wedge bs(\neg g) \wedge di\ g) = (di\ f \wedge \triangleright g)$ **by** *auto*

show *?thesis*

by (*metis 11 4 6 7 9 first-d-def inteq-reflection lift-and-com*)

qed

lemma *BiNotFstEqvBiNot*:

$\vdash bi(\neg(\triangleright f)) = bi(\neg f)$

proof –

have 1: $\vdash di\ f = di(\triangleright f)$ **using** *DiEqvDiFst* **by** *blast*

hence 2: $\vdash (\neg(di\ f)) = (\neg(di(\triangleright f)))$ **by** *auto*

from 1 2 **show** *?thesis* **using** *NotDiEqvBiNot* **by** *fastforce*

qed

lemma *BsNotFstEqvBsNot*:

$\vdash bs(\neg(\triangleright f)) = bs(\neg f)$

proof –

have 1: $\vdash bs(\neg(\triangleright f)) = (empty \vee bi(\neg(\triangleright f));skip)$ **by** (*simp add: bs-d-def*)

have 2: $\vdash bi(\neg(\triangleright f)) = bi(\neg f)$ **using** *BiNotFstEqvBiNot* **by** *blast*

hence 3: $\vdash bi(\neg(\triangleright f));skip = bi(\neg f);skip$ **using** *LeftChopEqvChop* **by** *blast*

hence 4: $\vdash (empty \vee bi(\neg(\triangleright f));skip) = (empty \vee bi(\neg f);skip)$ **by** *auto*

from 1 4 **show** *?thesis* **by** (*simp add: bs-d-def*)

qed

lemma *FstState*:

$\vdash \triangleright(init\ w) = (empty \wedge init\ w)$

proof –

have 1: $\vdash \triangleright(init\ w) = (init\ w \wedge bs(\neg(init\ w)))$ **by** (*simp add: first-d-def*)

hence 2: $\vdash \triangleright(init\ w) \longrightarrow init\ w$ **by** *auto*

have 3: $\vdash \text{init } w \longrightarrow \text{bs } (\text{init } w)$ **using** *StateImpBs* **by** *auto*
have 4: $\vdash \triangleright (\text{init } w) \longrightarrow \text{bs } (\text{init } w)$ **using** 2 3 **by** *fastforce*
have 5: $\vdash \triangleright (\text{init } w) \longrightarrow \text{bs } (\neg(\text{init } w))$ **using** 1 **by** *auto*
have 6: $\vdash \triangleright (\text{init } w) \longrightarrow \text{bs } (\text{init } w) \wedge \text{bs } (\neg(\text{init } w))$ **using** 4 5 **by** *fastforce*
have 7: $\vdash (\text{bs } (\text{init } w) \wedge \text{bs } (\neg(\text{init } w))) = (\text{bs } ((\text{init } w) \wedge \neg(\text{init } w)))$ **using** *BsAndEqv* **by** *blast*
have 8: $\vdash ((\text{init } w) \wedge \neg(\text{init } w)) = \#False$ **by** *auto*
hence 9: $\vdash (\text{bs } ((\text{init } w) \wedge \neg(\text{init } w))) = \text{bs } \#False$ **using** *BsEqvRule* **by** *blast*
have 10: $\vdash \text{bs } \#False = \text{empty}$ **using** *BsFalseEqvEmpty* **by** *auto*
have 11: $\vdash \triangleright (\text{init } w) \longrightarrow \text{empty}$ **using** 10 9 7 6 **by** *fastforce*
have 12: $\vdash \triangleright (\text{init } w) \longrightarrow \text{empty} \wedge \text{init } w$ **using** 11 2 **by** *fastforce*
have 13: $\vdash \text{empty} \wedge \text{init } w \longrightarrow \text{empty}$ **by** *auto*
hence 14: $\vdash \text{empty} \wedge \text{init } w \longrightarrow \text{empty} \vee \text{bi } (\neg(\text{init } w)); \text{skip}$ **by** *auto*
hence 15: $\vdash \text{empty} \wedge \text{init } w \longrightarrow \text{bs } (\neg(\text{init } w))$ **by** (*simp add: bs-d-def*)
have 16: $\vdash \text{empty} \wedge \text{init } w \longrightarrow \text{init } w$ **by** *auto*
have 17: $\vdash \text{empty} \wedge \text{init } w \longrightarrow \text{init } w \wedge \text{bs } (\neg(\text{init } w))$ **using** 16 15 **by** *auto*
hence 18: $\vdash \text{empty} \wedge \text{init } w \longrightarrow \triangleright(\text{init } w)$ **by** (*simp add: first-d-def*)
from 12 18 **show** *?thesis* **by** *fastforce*
qed

lemma *FstStateAndBsNotEmpty*:

$\vdash (\triangleright (\text{init } w) \wedge \text{bs } (\neg \text{empty})) = \triangleright (\text{init } w)$

proof –

have 1: $\vdash (\triangleright (\text{init } w) \wedge \text{bs } (\neg \text{empty})) = (\triangleright (\text{init } w) \wedge \text{bs } \text{more})$
using *BsEqvRule NotEmptyEqvMore* **by** (*simp add: empty-d-def*)
have 2: $\vdash (\triangleright (\text{init } w) \wedge \text{bs } \text{more}) = (\triangleright (\text{init } w) \wedge \text{empty})$
using *BsMoreEqvEmpty* **by** *fastforce*
have 3: $\vdash \triangleright (\text{init } w) = (\text{empty} \wedge (\text{init } w))$
using *FstState* **by** *blast*
hence 4: $\vdash (\triangleright (\text{init } w) \wedge \text{empty}) = (\text{empty} \wedge (\text{init } w) \wedge \text{empty})$
by *auto*
have 5: $\vdash (\text{empty} \wedge (\text{init } w) \wedge \text{empty}) = (\text{empty} \wedge (\text{init } w))$
by *auto*
have 6: $\vdash (\text{empty} \wedge (\text{init } w)) = \triangleright(\text{init } w)$
using *FstState* **by** *fastforce*
from 1 2 4 5 6 **show** *?thesis* **by** *fastforce*
qed

lemma *FstStateImpFstStateOr*:

$\vdash \triangleright(\text{init } w) \longrightarrow \triangleright(\text{init } w \vee f)$

proof –

have 1: $\vdash \triangleright(\text{init } w) = (\text{empty} \wedge \text{init } w)$
using *FstState* **by** *blast*
have 2: $\vdash (\text{empty} \wedge \text{init } w) = (\text{empty} \wedge (\text{empty} \vee \text{bi } (\neg f); \text{skip}) \wedge \text{init } w)$
by *auto*
have 3: $\vdash (\text{empty} \wedge (\text{empty} \vee \text{bi } (\neg f); \text{skip}) \wedge \text{init } w) =$
 $(\text{empty} \wedge \text{bs } (\neg f) \wedge \text{init } w)$
by (*simp add: bs-d-def*)
have 4: $\vdash (\text{empty} \wedge \text{bs } (\neg f) \wedge \text{init } w) = (\text{empty} \wedge \text{init } w \wedge \text{bs } (\neg f))$
by *auto*
have 5: $\vdash (\text{empty} \wedge \text{init } w) = \triangleright (\text{init } w)$

using *FstState* **by** *fastforce*
hence 6: $\vdash (\text{empty} \wedge \text{init } w \wedge \text{bs } (\neg f)) = (\triangleright (\text{init } w) \wedge \text{bs } (\neg f))$
by *auto*
have 7: $\vdash \triangleright (\text{init } w) \wedge \text{bs } (\neg f) \longrightarrow (\triangleright (\text{init } w) \wedge \text{bs } (\neg f)) \vee (\triangleright f \wedge \text{bs } (\neg(\text{init } w)))$
by *auto*
have 8: $\vdash \triangleright (\text{init } w \vee f) = ((\triangleright (\text{init } w) \wedge \text{bs } (\neg f)) \vee (\triangleright f \wedge \text{bs } (\neg(\text{init } w))))$
using *FstWithOrEqv* **by** *blast*
show ?thesis **using** 1 3 8 **by** *fastforce*
qed

lemma *FstLenSame*:
 $(\forall \sigma. (\sigma \models \text{di } (\triangleright f \wedge \text{len}(i)) \wedge \text{di } (\triangleright f \wedge \text{len}(j))) \longrightarrow (i=j))$
by (*simp add: DiLenFstsem FstLenSamesem*)

lemma *FstLenSame-1*:
 $\vdash \text{di } (\triangleright f \wedge \text{len}(i)) \wedge \text{di } (\triangleright f \wedge \text{len}(j)) \longrightarrow (\#i=\#j)$
using *FstLenSame Valid-def* **by** *fastforce*

lemma *FstAndLenSame*:
 $(\forall \sigma. (\sigma \models \text{di } ((\triangleright f \wedge g1) \wedge \text{len}(i)) \wedge \text{di } ((\triangleright f \wedge g2) \wedge \text{len}(j))) \longrightarrow (i=j))$
using *linorder-neqE-nat* **by** (*simp add: DiLenFstAndsem*) *blast*

lemma *FstAndLenSame-1*:
 $\vdash \text{di } ((\triangleright f \wedge g1) \wedge \text{len}(i)) \wedge \text{di } ((\triangleright f \wedge g2) \wedge \text{len}(j)) \longrightarrow (\#i=\#j)$
using *FstAndLenSame Valid-def* **by** *fastforce*

lemma *FstLenSameChop*:
 $(\forall \sigma. (\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i));h1 \wedge ((\triangleright f \wedge g2) \wedge \text{len}(j));h2) \longrightarrow (i=j))$
proof
fix σ
show $(\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i));h1 \wedge ((\triangleright f \wedge g2) \wedge \text{len}(j));h2) \longrightarrow (i=j)$
proof
assume 0: $(\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i));h1 \wedge ((\triangleright f \wedge g2) \wedge \text{len}(j));h2)$
have 1: $(\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i));h1)$ **using** 0 **by** *auto*
have 2: $(\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i));h1) \longrightarrow$
 $(\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i));\# \text{True})$ **by** (*metis ChopImpDi Valid-def di-d-def unl-lift2*)
have 3: $(\sigma \models \text{di}((\triangleright f \wedge g1) \wedge \text{len}(i)))$ **using** 1 2 **by** (*simp add: di-d-def*)
have 4: $(\sigma \models ((\triangleright f \wedge g2) \wedge \text{len}(j));h2)$ **using** 0 **by** *auto*
have 5: $(\sigma \models ((\triangleright f \wedge g2) \wedge \text{len}(j));h2) \longrightarrow$
 $(\sigma \models ((\triangleright f \wedge g2) \wedge \text{len}(j));\# \text{True})$ **by** (*metis ChopImpDi Valid-def di-d-def unl-lift2*)
have 6: $(\sigma \models \text{di}((\triangleright f \wedge g2) \wedge \text{len}(j)))$ **using** 4 5 **by** (*simp add: di-d-def*)
have 7: $(\sigma \models \text{di}((\triangleright f \wedge g1) \wedge \text{len}(i)) \wedge \text{di}((\triangleright f \wedge g2) \wedge \text{len}(j)))$ **using** 3 6 **by** *auto*
thus $(i=j)$ **using** *FstAndLenSame* **by** *blast*
qed
qed

lemma *FstLenSameChop-1*:
 $\vdash ((\triangleright f \wedge g1) \wedge \text{len}(i));h1 \wedge ((\triangleright f \wedge g2) \wedge \text{len}(j));h2 \longrightarrow (\#i=\#j)$
using *FstLenSameChop Valid-def* **by** *fastforce*

lemma *DiImpExistsOneDiLenAndFst*:

$(\forall \sigma. (\sigma \models di\ f) \longrightarrow (\exists! k. (\sigma \models di(\triangleright f \wedge len(k)))))$

proof

fix σ

show $(\sigma \models di\ f) \longrightarrow (\exists! k. (\sigma \models di(\triangleright f \wedge len(k)))))$

proof

assume $0: (\sigma \models di\ f)$

have $1: (\sigma \models di(\triangleright f))$

using $0\ DiEqvDiFst\ Valid-def$ **by** *force*

have $2: (\sigma \models \triangleright f) = ((\sigma \models \triangleright f) \wedge (\exists k. (\sigma \models len(k))))$

using *AndExistsLen[of TEMP $\triangleright f$]* **by** (*simp add: Valid-def*)

have $3: ((\sigma \models \triangleright f) \wedge (\exists k. (\sigma \models len(k)))) =$
 $(\exists k. (\sigma \models \triangleright f) \wedge (\sigma \models len(k)))$

by *auto*

have $4: (\sigma \models di(\triangleright f)) = (\exists k. (\sigma \models di(\triangleright f \wedge len(k))))$

using $2\ 3$ **by** (*metis 1 DiLensem di-defs*)

have $5: (\exists k. (\sigma \models di(\triangleright f \wedge len(k))))$

using 1 **using** 4 **by** *auto*

then obtain i **where** $6: (\sigma \models di(\triangleright f \wedge len(i)))$ **by** *blast*

from 5 **obtain** j **where** $7: (\sigma \models di(\triangleright f \wedge len(j)))$ **by** *blast*

have $8: (\sigma \models di(\triangleright f \wedge len(i))) \wedge (\sigma \models di(\triangleright f \wedge len(j)))$

using $6\ 7$ **by** *auto*

hence $9: (\sigma \models di(\triangleright f \wedge len(i)) \wedge di(\triangleright f \wedge len(j)))$

by *simp*

hence $10: i=j$

using *FstLenSame* **by** *blast*

have $11: \bigwedge j. (\sigma \models di(\triangleright f \wedge len(j))) \longrightarrow (j=i)$

using $9\ 10$ **using** *FstLenSame* **by** *auto*

thus $(\exists! k. (\sigma \models di(\triangleright f \wedge len(k))))$

using $11\ 5$ **by** *blast*

qed

qed

lemma *DiImpExistsOneDiLenAndFst-1*:

$\vdash di\ f \longrightarrow (\exists! k. (di(\triangleright f \wedge len(k))))$

using *Valid-def DiImpExistsOneDiLenAndFst* **by** *fastforce*

lemma *LFstAndDist-help*:

$(\sigma \models ((\triangleright f \wedge g1) \wedge len(k)); h1 \wedge ((\triangleright f \wedge g2) \wedge len(k)); h2) =$
 $(\sigma \models (((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge len(k)); (h1 \wedge h2))$

using *LFixedAndDistr* **by** *fastforce*

lemma *LFstAndDist-help-1*:

$(\exists k. (\sigma \models ((\triangleright f \wedge g1) \wedge len(k)); h1 \wedge ((\triangleright f \wedge g2) \wedge len(k)); h2)) =$
 $(\exists k. (\sigma \models (((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge len(k)); (h1 \wedge h2)))$

proof

assume $0: \exists k. \sigma \models ((\triangleright f \wedge g1) \wedge len\ k) ; h1 \wedge ((\triangleright f \wedge g2) \wedge len\ k) ; h2$

obtain k **where** $1: \sigma \models ((\triangleright f \wedge g1) \wedge len\ k) ; h1 \wedge ((\triangleright f \wedge g2) \wedge len\ k) ; h2$

using 0 **by** *auto*

hence 2: $(\sigma \models ((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge \text{len}(k)); (h1 \wedge h2))$
 using *LFstAndDist-help* by *blast*
 show $(\exists k. (\sigma \models ((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge \text{len}(k)); (h1 \wedge h2))$
 using 2 by *auto*
 next
 assume 3: $(\exists k. (\sigma \models ((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge \text{len}(k)); (h1 \wedge h2))$
 obtain *k* where 4: $(\sigma \models ((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge \text{len}(k)); (h1 \wedge h2))$
 using 3 by *auto*
 hence 5: $(\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(k)); h1 \wedge ((\triangleright f \wedge g2) \wedge \text{len}(k)); h2)$
 using *LFstAndDist-help* by *blast*
 show $(\exists k. (\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(k)); h1 \wedge ((\triangleright f \wedge g2) \wedge \text{len}(k)); h2))$
 using 5 by *auto*
 qed

lemma *LFstAndDistrsem*:

$(\forall \sigma. (\sigma \models ((\triangleright f \wedge g1); h1 \wedge (\triangleright f \wedge g2); h2) = (\triangleright f \wedge g1 \wedge g2); (h1 \wedge h2)))$
 proof
 fix σ
 show $(\sigma \models ((\triangleright f \wedge g1); h1 \wedge (\triangleright f \wedge g2); h2) = (\triangleright f \wedge g1 \wedge g2); (h1 \wedge h2))$
 proof –
 have 1: $(\sigma \models (\triangleright f \wedge g1); h1) = (\exists i. (\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i)); h1))$
 using *AndExistsLenChop[of TEMP $\triangleright f \wedge g1$]* by *fastforce*
 have 2: $(\sigma \models (\triangleright f \wedge g2); h2) = (\exists j. (\sigma \models ((\triangleright f \wedge g2) \wedge \text{len}(j)); h2))$
 using *AndExistsLenChop[of TEMP $\triangleright f \wedge g2$]* by *fastforce*
 have 3: $(\sigma \models (\triangleright f \wedge g1); h1 \wedge (\triangleright f \wedge g2); h2) =$
 $(\exists i j. (\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i)); h1 \wedge$
 $((\triangleright f \wedge g2) \wedge \text{len}(j)); h2))$
)
 using 1 2 by *auto*
 have 4: $(\exists i j. (\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(i)); h1 \wedge$
 $((\triangleright f \wedge g2) \wedge \text{len}(j)); h2))$
)
 $=$
 $(\exists k. (\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(k)); h1 \wedge$
 $((\triangleright f \wedge g2) \wedge \text{len}(k)); h2))$
)
 using *FstLenSameChop* by *blast*
 have 5: $(\exists k. (\sigma \models ((\triangleright f \wedge g1) \wedge \text{len}(k)); h1 \wedge ((\triangleright f \wedge g2) \wedge \text{len}(k)); h2)) =$
 $(\exists k. (\sigma \models (((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge \text{len}(k)); (h1 \wedge h2)))$
 using *LFstAndDist-help-1* by *blast*
 have 6: $(\exists k. (\sigma \models (((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)) \wedge \text{len}(k)); (h1 \wedge h2))) =$
 $(\sigma \models ((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)); (h1 \wedge h2))$
 using *AndExistsLenChop[of TEMP $((\triangleright f \wedge g1) \wedge \triangleright f \wedge g2)$]* by *fastforce*
 have 7: $(\sigma \models ((\triangleright f \wedge g1) \wedge (\triangleright f \wedge g2)); (h1 \wedge h2)) =$
 $(\sigma \models (\triangleright f \wedge g1 \wedge g2); (h1 \wedge h2))$
 by *(auto simp add: chop-defs)*
 from 3 4 5 6 7 show ?thesis by *auto*
 qed
 qed

lemma *LFstAndDistr*:

$\vdash ((\triangleright f \wedge g1);h1 \wedge (\triangleright f \wedge g2);h2) = (\triangleright f \wedge g1 \wedge g2);(h1 \wedge h2)$
using *LFstAndDistrsem* **by** *fastforce*

lemma *LFstAndDistrA*:

$\vdash ((\triangleright f \wedge g1);h \wedge (\triangleright f \wedge g2);h) = (\triangleright f \wedge g1 \wedge g2);h$

proof –

have 1: $\vdash ((\triangleright f \wedge g1);h \wedge (\triangleright f \wedge g2);h) = (\triangleright f \wedge g1 \wedge g2);(h \wedge h)$ **using** *LFstAndDistr* **by** *blast*

have 2: $\vdash (\triangleright f \wedge g1 \wedge g2);(h \wedge h) = (\triangleright f \wedge g1 \wedge g2);h$ **by** *auto*

from 1 2 **show** *?thesis* **by** *auto*

qed

lemma *LFstAndDistrB*:

$\vdash ((\triangleright f \wedge g);h1 \wedge (\triangleright f \wedge g);h2) = (\triangleright f \wedge g);(h1 \wedge h2)$

proof –

have 1: $\vdash ((\triangleright f \wedge g);h1 \wedge (\triangleright f \wedge g);h2) = (\triangleright f \wedge g \wedge g);(h1 \wedge h2)$ **using** *LFstAndDistr* **by** *blast*

have 2: $\vdash (\triangleright f \wedge g \wedge g);(h1 \wedge h2) = (\triangleright f \wedge g);(h1 \wedge h2)$ **by** *auto*

from 1 2 **show** *?thesis* **by** *auto*

qed

lemma *LFstAndDistrC*:

$\vdash ((\triangleright f);h1 \wedge (\triangleright f);h2) = (\triangleright f);(h1 \wedge h2)$

proof –

have 1: $\vdash ((\triangleright f \wedge \#True);h1 \wedge (\triangleright f \wedge \#True);h2) = (\triangleright f \wedge \#True \wedge \#True);(h1 \wedge h2)$
using *LFstAndDistr* **by** *blast*

have 2: $\vdash (\triangleright f \wedge \#True);h1 = (\triangleright f);h1$
by *auto*

have 3: $\vdash (\triangleright f \wedge \#True);h2 = (\triangleright f);h2$
by *auto*

have 4: $\vdash (\triangleright f \wedge \#True \wedge \#True);(h1 \wedge h2) = (\triangleright f);(h1 \wedge h2)$
by *auto*

from 1 2 3 4 **show** *?thesis* **by** *auto*

qed

lemma *LFstAndDistrD*:

$\vdash (di(\triangleright f \wedge g1) \wedge di(\triangleright f \wedge g2)) = di(\triangleright f \wedge g1 \wedge g2)$

proof –

have 1: $\vdash ((\triangleright f \wedge g1);\#True \wedge (\triangleright f \wedge g2);\#True) = (\triangleright f \wedge g1 \wedge g2);(\#True \wedge \#True)$
using *LFstAndDistr* **by** *blast*

have 2: $\vdash (\triangleright f \wedge g1);\#True = di(\triangleright f \wedge g1)$
by (*simp add: di-d-def*)

have 3: $\vdash (\triangleright f \wedge g2);\#True = di(\triangleright f \wedge g2)$
by (*simp add: di-d-def*)

have 4: $\vdash (\triangleright f \wedge g1 \wedge g2);(\#True \wedge \#True) = di(\triangleright f \wedge g1 \wedge g2)$
by (*simp add: di-d-def*)

from 1 2 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *LstAndDistr*:

$\vdash (h1;(\triangleleft f \wedge g1) \wedge h2;(\triangleleft f \wedge g2)) = (h1 \wedge h2);(\triangleleft f \wedge g1 \wedge g2)$

proof –

have 1: $\vdash ((\triangleright(f^r) \wedge g1^r);(h1^r) \wedge (\triangleright(f^r) \wedge (g2^r));(h2^r)) =$
 $(\triangleright(f^r) \wedge (g1^r) \wedge (g2^r));((h1^r) \wedge (h2^r))$
using *LFstAndDistr* **by** *blast*
hence 2: $\vdash ((\triangleright(f^r) \wedge g1^r);(h1^r) \wedge (\triangleright(f^r) \wedge (g2^r));(h2^r))^r =$
 $((\triangleright(f^r) \wedge (g1^r) \wedge (g2^r));((h1^r) \wedge (h2^r)))^r$
using *1 REqvRule* **by** *blast*
have 3: $\vdash (((\triangleright(f^r) \wedge g1^r);(h1^r))^r \wedge ((\triangleright(f^r) \wedge (g2^r));(h2^r))^r) =$
 $((\triangleright(f^r) \wedge g1^r);(h1^r) \wedge (\triangleright(f^r) \wedge (g2^r));(h2^r))^r$

using *RAnd* **by** *fastforce*
have 4: $\vdash ((h1^r)^r;(\triangleright(f^r) \wedge g1^r)^r \wedge (h2^r)^r;(\triangleright(f^r) \wedge (g2^r))^r) =$
 $((\triangleright(f^r) \wedge g1^r);(h1^r))^r \wedge ((\triangleright(f^r) \wedge (g2^r));(h2^r))^r$

using *RevChop* **by** *fastforce*
have 5: $\vdash (h1^r)^r = h1$
using *EqvReverseReverse* **by** *blast*
have 6: $\vdash (h2^r)^r = h2$
using *EqvReverseReverse* **by** *blast*
have 7: $\vdash (g1^r)^r = g1$
using *EqvReverseReverse* **by** *blast*
have 8: $\vdash (g2^r)^r = g2$
using *EqvReverseReverse* **by** *blast*
have 9: $\vdash (f^r)^r = f$
using *EqvReverseReverse* **by** *blast*
have 10: $\vdash (\triangleright(f^r) \wedge g1^r)^r = ((\triangleright(f^r))^r \wedge (g1^r)^r)$
using *RAnd* **by** *blast*
have 11: $\vdash (\triangleright(f^r) \wedge g2^r)^r = ((\triangleright(f^r))^r \wedge (g2^r)^r)$
using *RAnd* **by** *blast*
have 12: $\vdash (\triangleright(f^r))^r = \triangleleft(f)$
using *RRFirstEqvLast* **by** *blast*
have 13: $\vdash ((\triangleright(f^r))^r \wedge (g1^r)^r) = (\triangleleft f \wedge g1)$
using *12 7* **by** *fastforce*
have 14: $\vdash ((\triangleright(f^r))^r \wedge (g2^r)^r) = (\triangleleft f \wedge g2)$
using *12 8* **by** *fastforce*
have 15: $\vdash (h1;(\triangleleft f \wedge g1) \wedge h2;(\triangleleft f \wedge g2)) =$
 $((h1^r)^r;(\triangleright(f^r) \wedge g1^r)^r \wedge (h2^r)^r;(\triangleright(f^r) \wedge (g2^r))^r)$

using *14 13 10 11 5 6* **by** (*metis 4 int-eq*)
have 16: $\vdash (((\triangleright(f^r) \wedge (g1^r) \wedge (g2^r));((h1^r) \wedge (h2^r))))^r =$
 $((h1^r) \wedge (h2^r))^r;((\triangleright(f^r) \wedge (g1^r) \wedge (g2^r)))^r$
by (*simp add: RevChop*)
have 17: $\vdash ((\triangleright(f^r)) \wedge (g1^r) \wedge (g2^r))^r = ((\triangleright(f^r))^r \wedge (g1^r)^r \wedge (g2^r)^r)$
by (*metis inteq-reflection rev-fun2*)
have 18: $\vdash ((\triangleright(f^r))^r \wedge (g1^r)^r \wedge (g2^r)^r) = (\triangleleft f \wedge g1 \wedge g2)$
using *12 7 8* **by** *fastforce*
have 19: $\vdash ((h1^r) \wedge (h2^r))^r = (h1 \wedge h2)$
using *RRAnd* **by** *auto*
have 20: $\vdash ((h1^r) \wedge (h2^r))^r;((\triangleright(f^r) \wedge (g1^r) \wedge (g2^r)))^r =$
 $(h1 \wedge h2);(\triangleleft f \wedge g1 \wedge g2)$
using *19 17 18* **using** *ChopEqvChop* **by** (*metis int-eq*)

from 15 4 3 2 16 20 show ?thesis using int-eq by metis
qed

lemma *LstAndDistrA*:

$\vdash (h;(\triangleleft f \wedge g1) \wedge h;(\triangleleft f \wedge g2)) = h;(\triangleleft f \wedge g1 \wedge g2)$

proof –

have 1: $\vdash (h;(\triangleleft f \wedge g1) \wedge h;(\triangleleft f \wedge g2)) = (h \wedge h);(\triangleleft f \wedge g1 \wedge g2)$

using *LstAndDistr* **by** *blast*

have 2: $\vdash (h \wedge h);(\triangleleft f \wedge g1 \wedge g2) = h;(\triangleleft f \wedge g1 \wedge g2)$

by *auto*

from 1 2 **show** ?thesis **by** *auto*

qed

lemma *LstAndDistrB*:

$\vdash (h1;(\triangleleft f \wedge g) \wedge h2;(\triangleleft f \wedge g)) = (h1 \wedge h2);(\triangleleft f \wedge g)$

proof –

have 1: $\vdash (h1;(\triangleleft f \wedge g) \wedge h2;(\triangleleft f \wedge g)) = (h1 \wedge h2);(\triangleleft f \wedge g \wedge g)$

using *LstAndDistr* **by** *blast*

have 2: $\vdash (h1 \wedge h2);(\triangleleft f \wedge g \wedge g) = (h1 \wedge h2);(\triangleleft f \wedge g)$

by *auto*

from 1 2 **show** ?thesis **by** *auto*

qed

lemma *LstAndDistrC*:

$\vdash (h1;(\triangleleft f) \wedge h2;(\triangleleft f)) = (h1 \wedge h2);(\triangleleft f)$

proof –

have 1: $\vdash (h1;(\triangleleft f \wedge \#True) \wedge h2;(\triangleleft f \wedge \#True)) = (h1 \wedge h2);(\triangleleft f \wedge \#True \wedge \#True)$

using *LstAndDistr* **by** *blast*

have 2: $\vdash (h1 \wedge h2);(\triangleleft f \wedge \#True \wedge \#True) = (h1 \wedge h2);(\triangleleft f)$

by *auto*

have 3: $\vdash h1;(\triangleleft f \wedge \#True) = h1;(\triangleleft f)$

by *auto*

have 4: $\vdash h2;(\triangleleft f \wedge \#True) = h2;(\triangleleft f)$

by *auto*

from 1 2 3 4 **show** ?thesis **by** *auto*

qed

lemma *LstAndDistrD*:

$\vdash (\Diamond(\triangleleft f \wedge g1) \wedge \Diamond(\triangleleft f \wedge g2)) = \Diamond(\triangleleft f \wedge g1 \wedge g2)$

proof –

have 1: $\vdash (\#True;(\triangleleft f \wedge g1) \wedge \#True;(\triangleleft f \wedge g2)) = (\#True \wedge \#True);(\triangleleft f \wedge g1 \wedge g2)$

using *LstAndDistr* **by** *blast*

have 2: $\vdash (\#True \wedge \#True);(\triangleleft f \wedge g1 \wedge g2) = \Diamond(\triangleleft f \wedge g1 \wedge g2)$

by (*simp add: sometimes-d-def*)

have 3: $\vdash \#True;(\triangleleft f \wedge g1) = \Diamond(\triangleleft f \wedge g1)$

by (*simp add: sometimes-d-def*)

have 4: $\vdash \#True;(\triangleleft f \wedge g2) = \Diamond(\triangleleft f \wedge g2)$

by (*simp add: sometimes-d-def*)

from 1 2 3 4 **show** ?thesis **by** *fastforce*

qed

lemma *NotFstChop*:

$$\vdash (\neg(\triangleright f ; g)) = (\neg(di(\triangleright f)) \vee (\triangleright f;(\neg g)))$$

proof –

have 1: $\vdash g \longrightarrow \#True$ **by** *auto*

hence 2: $\vdash \triangleright f;g \longrightarrow \triangleright f;\#True$ **using** *RightChopImpChop* **by** *blast*

hence 3: $\vdash \triangleright f;g \longrightarrow di(\triangleright f)$ **by** (*simp add: di-d-def*)

hence 4: $\vdash \neg(di(\triangleright f)) \longrightarrow \neg(\triangleright f;g)$ **by** *auto*

have 5: $\vdash (\triangleright f;(\neg g) \longrightarrow \neg(\triangleright f;g)) = ((\triangleright f;(\neg g)) \wedge (\triangleright f;g) \longrightarrow \#False)$ **by** *auto*

have 6: $\vdash ((\triangleright f;(\neg g)) \wedge (\triangleright f;g)) = \triangleright f;(\neg g \wedge g)$ **using** *LFstAndDistrC* **by** *blast*

have 7: $\vdash \neg(\triangleright f;(\neg g \wedge g))$ **by** (*simp add: FstChopFalseEqvFalse*)

have 8: $\vdash \triangleright f;(\neg g) \longrightarrow \neg(\triangleright f;g)$ **using** 5 6 7 **by** *fastforce*

have 9: $\vdash \neg(di(\triangleright f)) \vee (\triangleright f;(\neg g)) \longrightarrow \neg(\triangleright f;g)$ **using** 4 8 **by** *fastforce*

have 10: $\vdash di(\triangleright f) \vee \neg(di(\triangleright f))$ **by** *auto*

hence 11: $\vdash (\triangleright f;\#True) \vee \neg(di(\triangleright f))$ **by** (*simp add: di-d-def*)

hence 12: $\vdash (\triangleright f;(g \vee \neg g)) \vee \neg(di(\triangleright f))$ **by** *auto*

have 13: $\vdash (\triangleright f;(g \vee \neg g)) = ((\triangleright f;g) \vee (\triangleright f;(\neg g)))$ **using** *ChopOrEqv* **by** *fastforce*

have 14: $\vdash ((\triangleright f;g) \vee (\triangleright f;(\neg g))) \vee \neg(di(\triangleright f))$ **using** 12 13 **by** *fastforce*

hence 15: $\vdash \neg(\triangleright f;g) \longrightarrow \neg(di(\triangleright f)) \vee (\triangleright f;(\neg g))$ **by** *auto*

from 9 15 **show** *?thesis* **by** *fastforce*

qed

lemma *BsNotFstChop*:

$$\vdash bs(\neg(\triangleright f;g)) = (empty \vee \neg(di(\triangleright f)) \vee (\triangleright f;bs(\neg g)))$$

proof –

have 1: $\vdash bs(\neg(\triangleright f;g)) = (empty \vee bi(\neg(\triangleright f;g));skip)$

by (*simp add: bs-d-def*)

have 2: $\vdash (empty \vee bi(\neg(\triangleright f;g));skip) = (empty \vee (\neg(di(\triangleright f;g)));skip)$

by (*metis 1 NotDiEqvBiNot int-eq*)

have 3: $\vdash (empty \vee (\neg(di(\triangleright f;g)));skip) = (empty \vee (\neg((\triangleright f;g);\#True));skip)$

by (*simp add: di-d-def*)

have 4: $\vdash (\neg((\triangleright f;g);\#True));skip = (\neg(\triangleright f;(g;\#True));skip)$

by (*metis ChopAssocB LeftChopEqvChop int-simps(15) inteq-reflection*)

hence 5: $\vdash (empty \vee (\neg((\triangleright f;g);\#True));skip) = (empty \vee (\neg(\triangleright f;(g;\#True));skip))$

by *auto*

have 6: $\vdash (empty \vee (\neg(\triangleright f;(g;\#True));skip) = (empty \vee (\neg(\triangleright f;di(g));skip))$

by (*simp add: di-d-def*)

have 7: $\vdash (empty \vee (\neg(\triangleright f;di(g));skip) = (empty \vee \neg(\neg(\neg(\triangleright f;di(g));skip)))$

by *auto*

have 8: $\vdash \neg(\neg(\neg(\neg(\triangleright f;di(g));skip))) = (\neg(empty \vee (\triangleright f;di(g));skip))$

using *NotNotChopSkip* **by** *fastforce*

hence 9: $\vdash (empty \vee \neg(\neg(\neg(\triangleright f;di(g));skip))) = (empty \vee \neg(empty \vee (\triangleright f;di(g));skip))$

by *auto*

have 10: $\vdash (empty \vee \neg(empty \vee (\triangleright f;di(g));skip)) = (empty \vee (more \wedge \neg((\triangleright f;di(g));skip)))$

by (*meson 6 7 9 NotChopSkipEqvMoreAndNotChopSkip Prop04 Prop06*)

have 11: $\vdash (empty \vee (more \wedge \neg((\triangleright f;di(g));skip))) = (empty \vee \neg((\triangleright f;di(g));skip))$

by (*auto simp add: empty-d-def*)

have 12: $\vdash (empty \vee \neg((\triangleright f;di(g));skip)) = (empty \vee \neg(\triangleright f;(di(g);skip)))$

using *ChopAssocB 11* **by** *fastforce*

have 13: $\vdash (\neg(\triangleright f;(di(g);skip))) = (\neg(\triangleright f;(ds(g))))$

using *DsDi* using *RightChopEqvChop* by *fastforce*
 hence 14: $\vdash (\text{empty} \vee \neg(\triangleright f;(\text{di}(g);\text{skip}))) = (\text{empty} \vee \neg(\triangleright f;(\text{ds}(g))))$
 by *auto*
 have 15: $\vdash (\text{empty} \vee \neg(\triangleright f;(\text{ds}(g)))) = (\text{empty} \vee \neg(\text{di}(\triangleright f)) \vee (\triangleright f;(\neg(\text{ds } g))))$
 using *NotFstChop* by *fastforce*
 have 16: $\vdash (\triangleright f;(\neg(\text{ds } g))) = (\triangleright f;(bs(\neg g)))$
 using *NotDsEqvBsNot RightChopEqvChop* by *blast*
 hence 17: $\vdash ((\text{empty} \vee \neg(\text{di}(\triangleright f))) \vee (\triangleright f;(\neg(\text{ds } g)))) = ((\text{empty} \vee \neg(\text{di}(\triangleright f))) \vee (\triangleright f;(bs(\neg g))))$
 by *auto*
 show ?thesis
 by (metis 11 12 15 16 2 4 *DsDi*
NotChopSkipEqvMoreAndNotChopSkip bs-d-def di-d-def inteq-reflection)

qed

lemma *FstFstChopEqvFstChopFst*:

$\vdash \triangleright(\triangleright f;g) = \triangleright f;\triangleright g$

proof –

have 1: $\vdash \triangleright(\triangleright f;g) = ((\triangleright f;g) \wedge bs(\neg(\triangleright f;g)))$
 by (simp add: *first-d-def*)
 have 2: $\vdash bs(\neg(\triangleright f;g)) = (\text{empty} \vee \neg(\text{di}(\triangleright f)) \vee (\triangleright f;bs(\neg g)))$
 using *BsNotFstChop* by *auto*
 hence 3: $\vdash ((\triangleright f;g) \wedge bs(\neg(\triangleright f;g))) = ((\triangleright f;g) \wedge (\text{empty} \vee \neg(\text{di}(\triangleright f)) \vee (\triangleright f;bs(\neg g))))$
 by *auto*
 have 4: $\vdash ((\triangleright f;g) \wedge (\text{empty} \vee \neg(\text{di}(\triangleright f)) \vee (\triangleright f;bs(\neg g)))) =$
 $((\triangleright f;g) \wedge \text{empty}) \vee ((\triangleright f;g) \wedge \neg(\text{di}(\triangleright f))) \vee ((\triangleright f;g) \wedge (\triangleright f;bs(\neg g)))$
 by *auto*
 have 5: $\vdash \neg((\triangleright f;g) \wedge \neg(\text{di}(\triangleright f)))$
 using *ChopImpDi* by *fastforce*
 hence 6: $\vdash (((\triangleright f;g) \wedge \text{empty}) \vee ((\triangleright f;g) \wedge \neg(\text{di}(\triangleright f))) \vee ((\triangleright f;g) \wedge (\triangleright f;bs(\neg g)))) =$
 $((\triangleright f;g) \wedge \text{empty}) \vee ((\triangleright f;g) \wedge (\triangleright f;bs(\neg g)))$
 by *auto*
 have 7: $\vdash ((\triangleright f;g) \wedge (\triangleright f;bs(\neg g))) = ((\triangleright f;(g \wedge (bs(\neg g)))))$
 using *LFstAndDistrC* by *blast*
 hence 8: $\vdash (((\triangleright f;g) \wedge \text{empty}) \vee ((\triangleright f;g) \wedge (\triangleright f;bs(\neg g)))) =$
 $((\triangleright f;g) \wedge \text{empty}) \vee ((\triangleright f;(g \wedge (bs(\neg g)))))$
 by *auto*
 have 9: $\vdash (((\triangleright f;g) \wedge \text{empty}) \vee ((\triangleright f;(g \wedge (bs(\neg g))))) = (((\triangleright f;g) \wedge \text{empty}) \vee \triangleright f;\triangleright g)$
 by (simp add: *first-d-def*)
 have 10: $\vdash ((\triangleright f;g) \wedge \text{empty}) = ((\triangleright f;\triangleright g) \wedge \text{empty})$
 using *FstChopEmptyEqvFstChopFstEmpty* by *blast*
 hence 11: $\vdash (((\triangleright f;g) \wedge \text{empty}) \vee \triangleright f;\triangleright g) = (((\triangleright f;\triangleright g) \wedge \text{empty}) \vee \triangleright f;\triangleright g)$
 by *auto*
 have 12: $\vdash (((\triangleright f;\triangleright g) \wedge \text{empty}) \vee \triangleright f;\triangleright g) = \triangleright f;\triangleright g$
 by *auto*
 from 1 3 4 6 8 9 11 12 show ?thesis by (metis *inteq-reflection*)

qed

lemma *FstFixFst*:

$\vdash \triangleright(\triangleright f) = \triangleright f$

proof –

have 1: $\vdash \triangleright f = (\triangleright f); \text{empty}$ **using** *ChopEmpty* **by** (*metis int-eq*)
hence 2: $\vdash \triangleright (\triangleright f) = \triangleright ((\triangleright f); \text{empty})$ **using** *FstEqvRule* **by** *blast*
have 3: $\vdash \triangleright ((\triangleright f); \text{empty}) = \triangleright f; \triangleright \text{empty}$ **using** *FstFstChopEqvFstChopFst* **by** *auto*
have 4: $\vdash \triangleright f; \triangleright \text{empty} = \triangleright f; \text{empty}$ **using** *FstEmpty* **using** *RightChopEqvChop* **by** *blast*
have 5: $\vdash \triangleright f; \text{empty} = \triangleright f$ **using** *ChopEmpty* **by** *blast*
from 2 3 4 5 **show** *?thesis* **by** *fastforce*
qed

lemma *FstCSEqvEmpty*:

$\vdash \triangleright (f^*) = \text{empty}$

proof –

have 1: $\vdash \triangleright (f^*) = \triangleright (\text{empty} \vee ((f \wedge \text{more}); f^*))$ **using** *ChopstarEqv FstEqvRule* **by** *blast*

from 1 **show** *?thesis* **using** *FstEmptyOrEqvEmpty* **by** *fastforce*

qed

lemma *FstIterFixFst*:

$\vdash \text{power } (\triangleright f) \ n = \triangleright (\text{power } (\triangleright f) \ n)$

proof

(*induct n*)

case 0

then show *?case*

proof –

have 1: $\vdash \text{power } (\triangleright f) \ 0 = \text{empty}$ **by** *auto*

have 2: $\vdash \text{empty} = \triangleright \text{empty}$ **using** *FstEmpty* **by** *auto*

have 3: $\vdash \triangleright \text{empty} = \triangleright (\text{power } (\triangleright f) \ 0)$ **by** *auto*

from 1 2 3 **show** *?thesis* **by** *auto*

qed

next

case (*Suc n*)

then show *?case*

proof –

have 4: $\vdash (\text{power } (\triangleright f) \ (\text{Suc } n)) = (\triangleright f) ; (\text{power } (\triangleright f) \ n)$

by (*simp*)

have 5: $\vdash (\triangleright f) ; (\text{power } (\triangleright f) \ n) = (\triangleright f) ; \triangleright (\text{power } (\triangleright f) \ n)$

using *RightChopEqvChop Suc.hyps* **by** *blast*

have 6: $\vdash (\triangleright f) ; \triangleright (\text{power } (\triangleright f) \ n) = \triangleright (\triangleright f; (\text{power } (\triangleright f) \ n))$

using *FstFstChopEqvFstChopFst* **by** *fastforce*

have 7: $\vdash \triangleright (\triangleright f; (\text{power } (\triangleright f) \ n)) = \triangleright (\text{power } (\triangleright f) \ (\text{Suc } n))$

by *simp*

from 4 5 6 7 **show** *?thesis* **by** *fastforce*

qed

qed

lemma *DsImpNotFst*:

$\vdash \text{ds } f \longrightarrow (\neg(\triangleright f))$

proof –

have 1: $\vdash (\text{ds } f \wedge \triangleright f) = (\text{ds } f \wedge (f \wedge \text{bs } (\neg f)))$ **by** (*simp add: first-d-def*)

have 2: $\vdash (\text{ds } f \wedge (f \wedge \text{bs } (\neg f))) = (\text{ds } f \wedge f \wedge \neg(\text{ds } f))$ **using** *NotDsEqvBsNot* **by** *fastforce*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *FstLenAndEqvLenAnd*:

$\vdash \triangleright(\text{len}(k) \wedge f) = (\text{len}(k) \wedge f)$

proof –

have 1: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow \text{ds}(\text{len}(k))$

using *DsAndImpElimL* **by** *fastforce*

hence 2: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow (\text{di}(\text{len}(k))); \text{skip}$

using *DsDi* **by** *fastforce*

hence 3: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow ((\text{len}(k); \# \text{True}); \text{skip})$

by (*simp add: di-d-def*)

hence 4: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow (\text{len}(k); (\# \text{True}; \text{skip}))$

using *ChopAssocB* **by** *fastforce*

hence 5: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow (\text{len}(k); (\text{skip}; \# \text{True}))$

by (*metis SkipTrueEqvTrueSkip inteq-reflection*)

hence 6: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow (\text{len}(k); (\text{skip}; \# \text{True})) \wedge \text{len}(k)$

by *auto*

hence 7: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow (\text{len}(k); (\text{skip}; \# \text{True})) \wedge \text{len}(k); \text{empty}$

using *ChopEmpty* **by** (*metis int-eq*)

hence 8: $\vdash \text{len}(k) \wedge f \wedge \text{ds}(\text{len}(k) \wedge f) \longrightarrow (\text{len}(k); ((\text{skip}; \# \text{True}) \wedge \text{empty}))$

using *LFixedAndDistrB1* **by** *fastforce*

have 9: $\vdash \neg(\text{len}(k); ((\text{skip}; \# \text{True}) \wedge \text{empty}))$

by (*simp add: empty-d-def more-d-def next-d-def chop-defs Valid-def*)

have 10: $\vdash \text{len}(k) \wedge f \longrightarrow \neg(\text{ds}(\text{len}(k) \wedge f))$

using 8 9 **by** *fastforce*

hence 11: $\vdash \text{len}(k) \wedge f \longrightarrow \text{bs}(\neg(\text{len}(k) \wedge f))$

using *NotDsEqvBsNot* **by** *fastforce*

hence 12: $\vdash \text{len}(k) \wedge f \longrightarrow (\text{len}(k) \wedge f) \wedge \text{bs}(\neg(\text{len}(k) \wedge f))$

by *auto*

hence 13: $\vdash \text{len}(k) \wedge f \longrightarrow \triangleright(\text{len}(k) \wedge f)$

by (*simp add: first-d-def*)

have 14: $\vdash \triangleright(\text{len}(k) \wedge f) \longrightarrow \text{len}(k) \wedge f$

by (*auto simp add: first-d-def*)

from 13 14 **show** *?thesis* **by** *fastforce*

qed

lemma *FstAndElimL*:

$\vdash \triangleright f \longrightarrow f$

by (*auto simp add: first-d-def*)

lemma *FstImpNotDiChopSkip*:

$\vdash \triangleright f \longrightarrow \neg(\text{di } f; \text{skip})$

proof –

have 1: $\vdash \triangleright f \longrightarrow \text{bs}(\neg f)$ **by** (*auto simp add: first-d-def*)

hence 2: $\vdash \triangleright f \longrightarrow \neg(\text{ds } f)$ **using** *NotDsEqvBsNot* **by** *fastforce*

have 3: $\vdash \text{ds } f = \text{di } f ; \text{skip}$ **using** *DsDi* **by** *blast*

hence 4: $\vdash \neg(\text{ds } f) = \neg(\text{di } f; \text{skip})$ **by** *auto*

from 2 4 **show** *?thesis* **by** *fastforce*

qed

lemma *FstImpNotDiChopSkipB*:

$\vdash \triangleright f \longrightarrow \neg(di\ f; skip)$
proof –
have 1: $\vdash \triangleright f \longrightarrow bs\ (\neg f)$
 by (*auto simp add: first-d-def*)
hence 2: $\vdash \triangleright f \longrightarrow \neg(ds\ f)$
 using *NotDsEqvBsNot* **by** *fastforce*
have 3: $\vdash ds\ f = di\ f ; skip$
 using *DsDi* **by** *blast*
have 4: $\vdash di\ f ; skip = (f; \# True); skip$
 by (*simp add: di-d-def*)
have 5: $\vdash (f; \# True); skip = f; (\# True; skip)$
 using *ChopAssocB* **by** *blast*
have 6: $\vdash f; (\# True; skip) = f; (skip; \# True)$
 using *SkipTrueEqvTrueSkip* **using** *TrueChopSkipEqvSkipChopTrue* *RightChopEqvChop* **by** *blast*
have 7: $\vdash f; (skip; \# True) = (f; skip); \# True$
 using *ChopAssoc* **by** *blast*
have 8: $\vdash (f; skip); \# True = di(f; skip)$
 by (*simp add: di-d-def*)
have 9: $\vdash (\neg(ds\ f)) = (\neg(di(f; skip)))$
 using 3 4 5 6 7 8 **by** *fastforce*
from 2 9 **show** *?thesis* **by** *fastforce*
qed

lemma *FstImpDiEqv*:

$\vdash \triangleright f \longrightarrow (di\ f = f)$
proof –
have 1: $\vdash \triangleright f \longrightarrow \neg(di\ f; skip)$ **using** *FstImpNotDiChopSkip* **by** *blast*
have 2: $\vdash di\ f \longrightarrow f \vee (di\ f; skip)$ **using** *DiEqvOrDiChopSkipB* **by** *fastforce*
have 3: $\vdash \triangleright f \wedge di\ f \longrightarrow (f \vee (di\ f; skip)) \wedge \neg(di\ f; skip)$ **using** 1 2 **by** *fastforce*
have 4: $\vdash ((f \vee (di\ f; skip)) \wedge \neg(di\ f; skip)) = (f \wedge \neg(di\ f; skip))$ **by** *auto*
have 5: $\vdash \triangleright f \wedge di\ f \longrightarrow f \wedge \neg(di\ f; skip)$ **using** 3 4 **by** *fastforce*
hence 6: $\vdash \triangleright f \wedge di\ f \longrightarrow f$ **by** *fastforce*
hence 7: $\vdash \triangleright f \longrightarrow (di\ f \longrightarrow f)$ **using** *FstAndElimL* **by** *fastforce*
have 8: $\vdash f \longrightarrow di\ f$ **using** *DiIntro* **by** *auto*
hence 9: $\vdash \triangleright f \longrightarrow (f \longrightarrow (di\ f))$ **by** *auto*
from 7 9 **show** *?thesis* **by** *fastforce*
qed

lemma *FstAndDiFstAndEqvFstAnd*:

$\vdash (\triangleright f \wedge di(\triangleright f \wedge g)) = (\triangleright f \wedge g)$
proof –
have 1: $\vdash \triangleright f \wedge di(\triangleright f \wedge g) \longrightarrow \triangleright f$
 by *auto*
have 2: $\vdash \triangleright f \wedge di(\triangleright f \wedge g) \longrightarrow di(\triangleright f \wedge g)$
 by *auto*
have 3: $\vdash di(\triangleright f \wedge g) = ((\triangleright f \wedge g) \vee di((\triangleright f \wedge g); skip))$
 using *DiEqvOrDiChopSkipA* **by** *blast*
have 4: $\vdash di((\triangleright f \wedge g); skip) = ((\triangleright f \wedge g); skip); \# True$
 by (*simp add: di-d-def*)
have 5: $\vdash \triangleright f \wedge di(\triangleright f \wedge g) \longrightarrow (\triangleright f \wedge g) \vee ((\triangleright f \wedge g); skip); \# True$

using 2 3 4 by fastforce
 have 6: $\vdash \triangleright f \wedge g \longrightarrow f$
 using FstAndElimL by fastforce
 hence 7: $\vdash ((\triangleright f \wedge g); \text{skip}); \# \text{True} \longrightarrow (f; \text{skip}); \# \text{True}$
 by (simp add: LeftChopImpChop)
 hence 8: $\vdash ((\triangleright f \wedge g); \text{skip}); \# \text{True} \longrightarrow \text{di}(f; \text{skip})$
 by (simp add: di-d-def)
 have 9: $\vdash \triangleright f \longrightarrow \neg(\text{di}(f; \text{skip}))$
 using FstImpNotDiChopSkipB by blast
 have 10: $\vdash \triangleright f \wedge \text{di}(\triangleright f \wedge g) \longrightarrow ((\triangleright f \wedge g) \vee \text{di}(f; \text{skip}))$
 using 5 8 by fastforce
 have 11: $\vdash \triangleright f \wedge \text{di}(\triangleright f \wedge g) \longrightarrow \neg(\text{di}(f; \text{skip})) \wedge ((\triangleright f \wedge g) \vee \text{di}(f; \text{skip}))$
 using 9 10 1 by fastforce
 have 12: $\vdash (\neg(\text{di}(f; \text{skip})) \wedge ((\triangleright f \wedge g) \vee \text{di}(f; \text{skip}))) = (\neg(\text{di}(f; \text{skip})) \wedge ((\triangleright f \wedge g)))$
 by auto
 have 13: $\vdash \triangleright f \wedge \text{di}(\triangleright f \wedge g) \longrightarrow (\triangleright f \wedge g)$
 using 11 12 by auto
 have 14: $\vdash (\triangleright f \wedge g) \longrightarrow \triangleright f$
 by auto
 hence 15: $\vdash (\triangleright f \wedge g) \longrightarrow \text{di}(\triangleright f \wedge g)$
 using DiIntro by auto
 have 16: $\vdash (\triangleright f \wedge g) \longrightarrow \triangleright f \wedge \text{di}(\triangleright f \wedge g)$
 using 14 15 by auto
 from 13 16 show ?thesis by fastforce
 qed

lemma FstAndDiImpBsNotAndDi:

$\vdash (\triangleright f \wedge \text{di } g) \longrightarrow (\text{bs } (\neg(\text{di } f \wedge g)))$

proof –

have 1: $\vdash (\triangleright f \wedge \text{di } g) \wedge \neg(\text{bs } (\neg(\text{di } f \wedge g))) \longrightarrow \text{ds}(\text{di } f \wedge g)$
 by (auto simp add: ds-d-def)
 hence 2: $\vdash (\triangleright f \wedge \text{di } g) \wedge \neg(\text{bs } (\neg(\text{di } f \wedge g))) \longrightarrow \text{ds}(\text{di } f)$
 using DsAndImp by fastforce
 hence 3: $\vdash (\triangleright f \wedge \text{di } g) \wedge \neg(\text{bs } (\neg(\text{di } f \wedge g))) \longrightarrow \text{di}(\text{di } f); \text{skip}$
 using DsDi by fastforce
 hence 4: $\vdash (\triangleright f \wedge \text{di } g) \wedge \neg(\text{bs } (\neg(\text{di } f \wedge g))) \longrightarrow \text{di } f; \text{skip}$
 using DiEqvDiDi by (metis int-eq)
 hence 5: $\vdash (\triangleright f \wedge \text{di } g) \wedge \neg(\text{bs } (\neg(\text{di } f \wedge g))) \longrightarrow \text{ds } f$
 using DsDi by fastforce
 hence 6: $\vdash (\triangleright f \wedge \text{di } g) \wedge \neg(\text{bs } (\neg(\text{di } f \wedge g))) \longrightarrow \neg(\triangleright f)$
 using DsImpNotFst by fastforce
 from 6 show ?thesis by auto
 qed

lemma FstFstOrEqvFstOrL:

$\vdash \triangleright(\triangleright f \vee g) = \triangleright(f \vee g)$

proof –

have 1: $\vdash \triangleright(f \vee g) = ((f \vee g) \wedge \text{bs } (\neg(f \vee g)))$
 by (simp add: first-d-def)
 have 2: $\vdash (\neg(f \vee g)) = (\neg f \wedge \neg g)$

by *auto*
 hence 3: $\vdash bs(\neg(f \vee g)) = bs(\neg f \wedge \neg g)$
 using *BsEqvRule* by *blast*
 have 4: $\vdash bs(\neg f \wedge \neg g) = (bs(\neg f) \wedge bs(\neg g))$
 using *BsAndEqv* by *fastforce*
 hence 5: $\vdash ((f \vee g) \wedge bs(\neg(f \vee g))) = ((f \vee g) \wedge bs(\neg f) \wedge bs(\neg g))$
 using 4 3 by *fastforce*
 have 6: $\vdash ((f \vee g) \wedge bs(\neg f) \wedge bs(\neg g)) =$
 $((f \wedge bs(\neg f)) \vee (g \wedge bs(\neg f))) \wedge bs(\neg g)$
 by *auto*
 have 7: $\vdash (((f \wedge bs(\neg f)) \vee (g \wedge bs(\neg f))) \wedge bs(\neg g)) =$
 $((\triangleright f \vee (g \wedge bs(\neg f))) \wedge bs(\neg g))$
 by (*simp add: first-d-def*)
 have 8: $\vdash ((\triangleright f \vee (g \wedge bs(\neg f))) \wedge bs(\neg g)) =$
 $((\triangleright f \vee g) \wedge (\triangleright f \vee bs(\neg f))) \wedge bs(\neg g)$
 by *auto*
 have 9: $\vdash (((\triangleright f \vee g) \wedge (\triangleright f \vee bs(\neg f))) \wedge bs(\neg g)) =$
 $((\triangleright f \vee g) \wedge ((f \wedge bs(\neg f)) \vee bs(\neg f))) \wedge bs(\neg g)$
 by (*simp add: first-d-def*)
 have 10: $\vdash (((\triangleright f \vee g) \wedge ((f \wedge bs(\neg f)) \vee bs(\neg f))) \wedge bs(\neg g)) =$
 $((\triangleright f \vee g) \wedge bs(\neg f) \wedge bs(\neg g))$
 by *auto*
 have 11: $\vdash ((\triangleright f \vee g) \wedge bs(\neg f) \wedge bs(\neg g)) =$
 $((\triangleright f \vee g) \wedge bs(\neg(\triangleright f)) \wedge bs(\neg g))$
 using *BsNotFstEqvBsNot* by *fastforce*
 have 12: $\vdash ((\triangleright f \vee g) \wedge bs(\neg(\triangleright f)) \wedge bs(\neg g)) =$
 $((\triangleright f \vee g) \wedge bs(\neg(\triangleright f) \wedge \neg g))$
 using *BsAndEqv* by *fastforce*
 have 13: $\vdash (\neg(\triangleright f) \wedge \neg g) = (\neg(\triangleright f \vee g))$
 by *auto*
 hence 14: $\vdash bs(\neg(\triangleright f) \wedge \neg g) = bs(\neg(\triangleright f \vee g))$
 using *BsEqvRule* by *blast*
 hence 15: $\vdash ((\triangleright f \vee g) \wedge bs(\neg(\triangleright f) \wedge \neg g)) = ((\triangleright f \vee g) \wedge bs(\neg(\triangleright f \vee g)))$
 by *auto*
 have 16: $\vdash ((\triangleright f \vee g) \wedge bs(\neg(\triangleright f \vee g))) = \triangleright(\triangleright f \vee g)$
 by (*simp add: first-d-def*)
 from 16 15 12 11 10 9 8 7 6 5 1 show ?thesis by (*metis int-eq*)
 qed

lemma *FstFstOrEqvFstOrR*:

$\vdash \triangleright(f \vee \triangleright g) = \triangleright(f \vee g)$

proof –

have 1: $\vdash (f \vee \triangleright g) = (\triangleright g \vee f)$ by *auto*
 hence 2: $\vdash \triangleright(f \vee \triangleright g) = \triangleright(\triangleright g \vee f)$ using *FstEqvRule* by *blast*
 have 3: $\vdash \triangleright(\triangleright g \vee f) = \triangleright(g \vee f)$ using *FstFstOrEqvFstOrL* by *blast*
 have 4: $\vdash (g \vee f) = (f \vee g)$ by *auto*
 hence 5: $\vdash \triangleright(g \vee f) = \triangleright(f \vee g)$ using *FstEqvRule* by *blast*
 from 2 3 5 show ?thesis by *fastforce*

qed

lemma *FstFstOrEqvFstOr*:

$\vdash \triangleright(\triangleright f \vee \triangleright g) = \triangleright(f \vee g)$

proof –

have 1: $\vdash \triangleright(\triangleright f \vee \triangleright g) = \triangleright(f \vee \triangleright g)$ **using** *FstFstOrEqvFstOrL* **by** *blast*

have 2: $\vdash \triangleright(f \vee \triangleright g) = \triangleright(f \vee g)$ **using** *FstFstOrEqvFstOrR* **by** *blast*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *FstLenEqvLen*:

$\vdash \triangleright(\text{len}(k)) = \text{len}(k)$

proof –

have 1: $\vdash \triangleright(\text{len}(k) \wedge \# \text{True}) = (\text{len}(k) \wedge \# \text{True})$ **using** *FstLenAndEqvLenAnd* **by** *blast*

have 2: $\vdash (\text{len}(k) \wedge \# \text{True}) = \text{len}(k)$ **by** *auto*

hence 3: $\vdash \triangleright(\text{len}(k) \wedge \# \text{True}) = \triangleright(\text{len}(k))$ **using** *FstEqvRule* **by** *blast*

from 1 2 3 **show** *?thesis* **by** *auto*

qed

lemma *FstSkip*:

$\vdash \triangleright \text{skip} = \text{skip}$

proof –

have 1: $\vdash \text{skip} = \text{len}(1)$ **using** *LenOneEqvSkip* **by** *fastforce*

hence 2: $\vdash \triangleright \text{skip} = \triangleright(\text{len}(1))$ **using** *FstEqvRule* **by** *blast*

have 3: $\vdash \triangleright(\text{len}(1)) = \text{len}(1)$ **using** *FstLenEqvLen* **by** *blast*

from 1 2 3 **show** *?thesis* **using** *LenOneEqvSkip* **by** *fastforce*

qed

lemma *HaltStateEqvFstFinState*:

$\vdash \text{halt}(\text{init } w) = \triangleright(\text{fin}(\text{init } w))$

proof –

have 1: $\vdash \text{halt}(\text{init } w) = \square(\text{empty} = (\text{init } w))$ **by** (*simp add: halt-d-def*)

have 21: $\vdash (\text{empty} = (\text{init } w)) = (((\text{empty} \longrightarrow (\text{init } w)) \wedge ((\text{init } w) \longrightarrow \text{empty})))$

by *auto*

hence 2: $\vdash \square(\text{empty} = (\text{init } w)) = (\square((\text{empty} \longrightarrow (\text{init } w)) \wedge ((\text{init } w) \longrightarrow \text{empty})))$

by (*simp add: BoxEqvBox*)

have 3: $\vdash (\square((\text{empty} \longrightarrow (\text{init } w)) \wedge ((\text{init } w) \longrightarrow \text{empty}))) =$

$(\square((\text{empty} \longrightarrow (\text{init } w))) \wedge \square((\text{init } w) \longrightarrow \text{empty}))$

by (*metis 21 BoxAndBoxEqvBoxRule int-eq*)

have 4: $\vdash ((\text{init } w) \longrightarrow \text{empty}) = (\text{more} \longrightarrow \neg(\text{init } w))$

by (*auto simp add: empty-d-def*)

hence 5: $\vdash \square((\text{init } w) \longrightarrow \text{empty}) = \square(\text{more} \longrightarrow \neg(\text{init } w))$ **using** *BoxEqvBox* **by** *blast*

have 6: $\vdash \square(\text{more} \longrightarrow \neg(\text{init } w)) = \text{bs}(\neg(\text{fin}(\text{init } w)))$ **using** *BoxMoreStateEqvBsFinState* **by** *blast*

have 7: $\vdash \square((\text{empty} \longrightarrow (\text{init } w))) = \text{fin}(\text{init } w)$ **by** (*simp add: fin-d-def*)

have 8: $\vdash (\square((\text{empty} \longrightarrow (\text{init } w))) \wedge \square((\text{init } w) \longrightarrow \text{empty})) =$

$(\text{fin}(\text{init } w) \wedge \text{bs}(\neg(\text{fin}(\text{init } w))))$ **using** 5 6 7 **by** *fastforce*

from 1 2 3 8 **show** *?thesis* **by** (*metis first-d-def integ-reflection*)

qed

lemma *FstLenEqvLenFst*:

$\vdash \triangleright(\text{len } k ; f) = \text{len } k ; \triangleright f$

proof –

have 1: $\vdash \text{len } k ; f = \triangleright(\text{len } k) ; f$ **using** *FstLenEqvLen LeftChopEqvChop* **by** *fastforce*
have 2: $\vdash \triangleright(\text{len } k ; f) = \triangleright(\triangleright(\text{len } k) ; f)$ **using** 1 *FstEqvRule* **by** *blast*
have 3: $\vdash \triangleright(\triangleright(\text{len } k) ; f) = \triangleright(\text{len } k) ; \triangleright f$ **using** *FstFstChopEqvFstChopFst* **by** *blast*
have 4: $\vdash \triangleright(\text{len } k) ; \triangleright f = \text{len } k ; \triangleright f$ **using** *FstLenEqvLen LeftChopEqvChop* **by** *fastforce*
from 2 3 4 **show** *?thesis* **by** *fastforce*
qed

lemma *FstNextEqvNextFst*:

$\vdash \triangleright(\circ f) = \circ(\triangleright f)$

proof –

have 1: $\vdash \triangleright(\circ f) = \triangleright(\text{skip} ; f)$ **using** *FstEqvRule* **by** (*simp add: next-d-def*)
have 2: $\vdash \text{skip} ; f = \triangleright \text{skip} ; f$ **using** *FstSkip* **using** *LeftChopEqvChop* **by** *fastforce*
have 3: $\vdash \triangleright(\text{skip} ; f) = \triangleright(\triangleright \text{skip} ; f)$ **using** 2 *FstEqvRule LeftChopEqvChop* **by** *blast*
have 4: $\vdash \triangleright(\triangleright \text{skip} ; f) = \triangleright \text{skip} ; \triangleright f$ **using** 3 *FstFstChopEqvFstChopFst* **by** *blast*
have 5: $\vdash \triangleright \text{skip} ; \triangleright f = \text{skip} ; \triangleright f$ **using** 4 *FstSkip LeftChopEqvChop* **by** *blast*
have 6: $\vdash \text{skip} ; \triangleright f = \circ(\triangleright f)$ **by** (*simp add: next-d-def*)
from 1 2 3 4 5 6 **show** *?thesis* **by** *fastforce*

qed

lemma *FstDiamondStateEqvHalt*:

$\vdash \triangleright(\diamond(\text{init } w)) = \text{halt } (\text{init } w)$

proof –

have 1: $\vdash \diamond(\text{init } w) = \diamond((\text{init } w) \wedge \# \text{True})$ **by** *simp*
have 2: $\vdash \text{fin } (\text{init } w) ; \# \text{True} = \diamond((\text{init } w) \wedge \# \text{True})$ **using** 1 *FinChopEqvDiamond* **by** *blast*
have 3: $\vdash \text{fin } (\text{init } w) ; \# \text{True} = \text{di } (\text{fin } (\text{init } w))$ **by** (*simp add: di-d-def*)
have 4: $\vdash \diamond(\text{init } w) = (\text{di } (\text{fin } (\text{init } w)))$ **using** 1 2 3 **by** *fastforce*
have 5: $\vdash \triangleright(\diamond(\text{init } w)) = \triangleright(\text{di } (\text{fin } (\text{init } w)))$ **using** 4 *FstEqvRule* **by** *blast*
hence 6: $\vdash \triangleright(\diamond(\text{init } w)) = \triangleright(\text{fin } (\text{init } w))$ **using** *FstDiEqvFst* **by** *fastforce*
hence 7: $\vdash \triangleright(\diamond(\text{init } w)) = \text{halt } (\text{init } w)$ **using** *HaltStateEqvFstFinState* **by** *fastforce*
from 7 **show** *?thesis* **by** *simp*

qed

lemma *FstBoxStateEqvStateAndEmpty*:

$\vdash \triangleright(\square(\text{init } w)) = ((\text{init } w) \wedge \text{empty})$

proof –

have 1: $\vdash ((\text{init } w) \wedge (\square(\text{init } w))^*) = \square(\text{init } w)$
using *BoxCSEqvBox* **by** *blast*
have 2: $\vdash \square(\text{init } w) = ((\text{init } w) \wedge (\square(\text{init } w))^*)$
using 1 **by** *auto*
hence 3: $\vdash \square(\text{init } w) = ((\text{init } w) \wedge (\square(\text{init } w))^*)$
by *blast*
have 4: $\vdash ((\text{init } w) \wedge \text{empty}) ; (\square(\text{init } w))^* = ((\text{init } w) \wedge (\square(\text{init } w))^*)$
using *StateAndEmptyChop* **by** *blast*
have 5: $\vdash ((\text{init } w) \wedge (\square(\text{init } w))^*) = ((\text{init } w) \wedge \text{empty}) ; (\square(\text{init } w))^*$
using 4 **by** *fastforce*
have 6: $\vdash \square(\text{init } w) = ((\text{init } w) \wedge \text{empty}) ; (\square(\text{init } w))^*$
using 3 5 **by** *fastforce*
have 7: $\vdash ((\text{init } w) \wedge \text{empty}) ; (\square(\text{init } w))^* = \triangleright(\text{init } w) ; (\square(\text{init } w))^*$
using *FstState* **by** (*metis AndChopCommute int-eq*)
have 8: $\vdash \square(\text{init } w) = \triangleright(\text{init } w) ; (\square(\text{init } w))^*$

```

using 6 7 by fastforce
have 9:  $\vdash \triangleright (\Box (init\ w)) = \triangleright (\triangleright (init\ w) ; (\Box (init\ w))^*)$ 
using 8 FstEqvRule by blast
have 10:  $\vdash \triangleright (\triangleright (init\ w) ; (\Box (init\ w))^*) = \triangleright (init\ w) ; \triangleright ((\Box (init\ w))^*)$ 
using FstFstChopEqvFstChopFst by blast
have 11:  $\vdash \triangleright (init\ w) ; \triangleright ((\Box (init\ w))^*) = \triangleright (init\ w) ; empty$ 
using RightChopEqvChop FstCSeqvEmpty by blast
have 12:  $\vdash \triangleright (init\ w) ; empty = \triangleright (init\ w)$ 
using RightChopEqvChop ChopEmpty by blast
have 13:  $\vdash \triangleright (init\ w) = ((init\ w) \wedge empty)$ 
using FstState by fastforce
from 9 10 11 12 13 show ?thesis by fastforce

```

qed

lemma FstAndFstStarEqvFst:

$\vdash (\triangleright f \wedge (\triangleright f)^*) = \triangleright f$

proof –

```

have 1:  $\vdash (\triangleright f)^* = (empty \vee (\triangleright f);(\triangleright f)^*)$ 
using CSeqvOrChopCS by fastforce
have 2:  $\vdash ((\triangleright f)^* \wedge \triangleright f) = ((empty \vee (\triangleright f);(\triangleright f)^*) \wedge \triangleright f)$ 
using 1 by fastforce
have 3:  $\vdash ((empty \vee (\triangleright f);(\triangleright f)^*) \wedge \triangleright f) = ((empty \wedge \triangleright f) \vee ((\triangleright f);(\triangleright f)^* \wedge \triangleright f))$ 
by auto
have 4:  $\vdash ((\triangleright f)^* \wedge \triangleright f) = ((empty \wedge \triangleright f) \vee ((\triangleright f);(\triangleright f)^* \wedge \triangleright f))$ 
using 2 3 by fastforce
have 5:  $\vdash ((\triangleright f);(\triangleright f)^* \wedge \triangleright f) = ((\triangleright f);(\triangleright f)^* \wedge \triangleright f; empty)$ 
using ChopEmpty by (metis inteq-reflection)
have 6:  $\vdash ((\triangleright f);(\triangleright f)^* \wedge \triangleright f; empty) = (\triangleright f);((\triangleright f)^* \wedge empty)$ 
using LFstAndDistrC by blast
have 7:  $\vdash ((\triangleright f)^* \wedge empty) = empty$ 
using EmptyImpCS by fastforce
have 8:  $\vdash (\triangleright f);((\triangleright f)^* \wedge empty) = \triangleright f$ 
using 7 ChopEmpty by (metis inteq-reflection)
have 9:  $\vdash ((\triangleright f);(\triangleright f)^* \wedge \triangleright f) = \triangleright f$ 
using 5 6 8 by fastforce
have 10:  $\vdash ((\triangleright f)^* \wedge \triangleright f) = ((empty \wedge \triangleright f) \vee \triangleright f)$ 
using 4 9 by fastforce
have 11:  $\vdash ((empty \wedge \triangleright f) \vee \triangleright f) = \triangleright f$ 
by auto
have 12:  $\vdash ((\triangleright f)^* \wedge \triangleright f) = \triangleright f$ 
using 10 11 by fastforce
from 12 show ?thesis by auto

```

qed

lemma HaltStateEqvFstHaltState:

$\vdash halt(init(w)) = \triangleright(halt(init(w)))$

proof –

```

have 1:  $\vdash halt\ (init\ w) = \triangleright (fin\ (init\ w))$ 
by (simp add: HaltStateEqvFstFinState)
have 2:  $\vdash \triangleright (fin\ (init\ w)) = \triangleright (\triangleright (fin\ (init\ w)))$ 

```

```

using FstEqvRule FstFixFst by fastforce
have 3:  $\vdash \triangleright (\triangleright (\text{fin } (\text{init } w))) = \triangleright (\text{halt}(\text{init}(w)))$ 
using FstEqvRule HaltStateEqvFstFinState by fastforce
from 1 2 3 show ?thesis by fastforce
qed

```

lemma *DiHaltAndDiHaltAndEqvDiHaltAndAnd*:

```

 $\vdash (\text{di}(\text{halt } (\text{init } w) \wedge f) \wedge \text{di}(\text{halt } (\text{init } w) \wedge g)) = \text{di}(\text{halt } (\text{init } w) \wedge f \wedge g)$ 
proof –
have 1:  $\vdash (\text{di}(\text{halt } (\text{init } w) \wedge f) \wedge \text{di}(\text{halt } (\text{init } w) \wedge g)) =$ 
 $(\text{di}(\triangleright(\text{fin } (\text{init } w)) \wedge f) \wedge \text{di}(\triangleright(\text{fin } (\text{init } w)) \wedge g))$ 
using HaltStateEqvFstFinState by (metis LFstAndDistrD inteq-reflection)
have 2:  $\vdash (\text{di}(\triangleright(\text{fin } (\text{init } w)) \wedge f) \wedge \text{di}(\triangleright(\text{fin } (\text{init } w)) \wedge g)) =$ 
 $\text{di}(\triangleright(\text{fin } (\text{init } w)) \wedge f \wedge g)$ 
using LFstAndDistrD by fastforce
have 3:  $\vdash \text{di}(\triangleright(\text{fin } (\text{init } w)) \wedge f \wedge g) = \text{di}(\text{halt } (\text{init } w) \wedge f \wedge g)$ 
using HaltStateEqvFstFinState by (metis DiEqvDi int-eq lift-and-com)
from 1 2 3 show ?thesis using int-eq by metis
qed

```

lemma *counter-ex-lhs*:

```

 $\vdash ((\triangleright(\text{len}(5)) \wedge \triangleright(\text{len}(2))) ; (\text{len}(5) \vee \text{len}(2))) = \#False$ 
proof –
have 1:  $\vdash ((\triangleright(\text{len}(5)) \wedge \triangleright(\text{len}(2))) ; (\text{len}(5) \vee \text{len}(2))) =$ 
 $(\text{len}(5) \wedge \text{len}(2)); (\text{len}(5) \vee \text{len}(2)))$ 
by (metis FstLenAndEqvLenAnd FstLenEqvLen LeftChopEqvChop inteq-reflection)
have 2:  $\vdash (\text{len}(5) \wedge \text{len}(2)) = \#False$ 
by (simp add: Valid-def len-defs)
have 3:  $\vdash ((\text{len}(5) \wedge \text{len}(2)); (\text{len}(5) \vee \text{len}(2))) = (\#False; (\text{len}(5) \vee \text{len}(2)))$ 
by (simp add: 2 LeftChopEqvChop)
have 4:  $\vdash (\#False; (\text{len}(5) \vee \text{len}(2))) = \#False$ 
by (simp add: Valid-def chop-defs)
from 1 3 4 show ?thesis by fastforce
qed

```

lemma *counter-ex-rhs*:

```

 $\vdash ((\triangleright(\text{len}(5)) ; (\text{len}(5) \vee \text{len}(2))) \wedge (\triangleright(\text{len}(2)) ; (\text{len}(5) \vee \text{len}(2)))) = \text{len}(7)$ 
proof –
have 1:  $\vdash (\triangleright(\text{len}(5)) ; (\text{len}(5) \vee \text{len}(2))) =$ 
 $\text{len}(5); (\text{len}(5) \vee \text{len}(2))$ 
using FstLenEqvLen LeftChopEqvChop by blast
have 2:  $\vdash (\triangleright(\text{len}(2)) ; (\text{len}(5) \vee \text{len}(2))) =$ 
 $\text{len}(2); (\text{len}(5) \vee \text{len}(2))$ 
using FstLenEqvLen LeftChopEqvChop by blast
have 3:  $\vdash \text{len}(5); (\text{len}(5) \vee \text{len}(2)) =$ 
 $((\text{len}(5); \text{len}(5)) \vee (\text{len}(5); \text{len}(2)))$ 
by (simp add: ChopOrEqv)

```

```

have 4:  $\vdash ((\text{len}(5); \text{len}(5)) \vee (\text{len}(5); \text{len}(2))) =$ 
   $(\text{len}(10) \vee \text{len}(7))$ 
  using LenEqvLenChopLen inteq-reflection by fastforce
have 5:  $\vdash \text{len}(2) ; (\text{len}(5) \vee \text{len}(2)) =$ 
   $((\text{len}(2); \text{len}(5)) \vee (\text{len}(2); \text{len}(2)))$ 
  by (simp add: ChopOrEqv)
have 6:  $\vdash ((\text{len}(2); \text{len}(5)) \vee (\text{len}(2); \text{len}(2))) =$ 
   $(\text{len}(7) \vee \text{len}(4))$ 
  using LenEqvLenChopLen inteq-reflection by fastforce
have 7:  $\vdash ((\text{len}(10) \vee \text{len}(7)) \wedge (\text{len}(7) \vee \text{len}(4))) =$ 
   $((\text{len}(7) \vee \text{len}(10)) \wedge (\text{len}(7) \vee \text{len}(4)))$ 
  by fastforce
have 8:  $\vdash ((\text{len}(7) \vee \text{len}(10)) \wedge (\text{len}(7) \vee \text{len}(4))) =$ 
   $(\text{len}(7) \vee (\text{len}(10) \wedge \text{len}(4)))$ 
  by fastforce
have 9:  $\vdash (\text{len}(10) \wedge \text{len}(4)) = \#False$ 
  by (simp add: Valid-def len-defs)
have 10 :  $\vdash (\text{len}(7) \vee (\text{len}(10) \wedge \text{len}(4))) = \text{len}(7)$ 
  using 9 by auto
have 11:  $\vdash ((\triangleright (\text{len}(5)) ; (\text{len}(5) \vee \text{len}(2))) \wedge (\triangleright (\text{len}(2)) ; (\text{len}(5) \vee \text{len}(2)))) =$ 
   $(\text{len}(5); (\text{len}(5) \vee \text{len}(2)) \wedge \text{len}(2) ; (\text{len}(5) \vee \text{len}(2)))$ 
  using 1 2 by fastforce
have 12:  $\vdash (\text{len}(5); (\text{len}(5) \vee \text{len}(2)) \wedge \text{len}(2) ; (\text{len}(5) \vee \text{len}(2))) = \text{len}(7)$ 
  by (metis 10 4 6 7 8 ChopOrEqv inteq-reflection)
from 11 12 show ?thesis by fastforce
qed

```

end

10 Monitors

theory *Monitor*

imports *First*

begin

The RV monitors language is introduced plus the algebraic properties of the monitor operators.

10.1 Syntax

```

datatype ('a :: world) monitor =
  mFIRST-d 'a formula ((FIRST -) [84] 83)
| mUPTO-d 'a monitor 'a monitor ((- UPTO -) [84,84] 83)
| mTHRU-d 'a monitor 'a monitor ((- THRU -) [84,84] 83)
| mTHEN-d 'a monitor 'a monitor ((- THEN -) [84,84] 83)
| mWITH-d 'a monitor 'a formula ((- WITH -) [84,84] 83)

```

```

fun MON :: ('a::world) monitor  $\Rightarrow$  'a formula
where (MON (FIRST f)) = LIFT( $\triangleright$  f)
      | (MON (a UPTO b)) = LIFT( $\triangleright$ ((MON a)  $\vee$  (MON b) ))
      | (MON (a THRU b)) = LIFT( $\triangleright$ (di(MON a)  $\wedge$  di(MON b)))
      | (MON (a THEN b)) = LIFT((MON a);(MON b))
      | (MON (a WITH f)) = LIFT((MON a)  $\wedge$  f)

syntax
-MON :: 'a monitor  $\Rightarrow$  lift (( $\mathcal{M}$  -) [80] 80)

translations
-MON == CONST MON

definition eq-d :: ('a:: world) monitor  $\Rightarrow$  'a monitor  $\Rightarrow$  bool (( -  $\simeq$  -) [84,84] 83)
where
  eq-d a b  $\equiv$  ( $\vdash$  ( $\mathcal{M}$  a) = ( $\mathcal{M}$  b))

lemma MonEqRefl:
  a  $\simeq$  a
by (simp add: eq-d-def)

lemma MonEqSym:
  assumes a  $\simeq$  b
  shows b  $\simeq$  a
using assms by (metis eq-d-def inteq-reflection)

lemma MonEqTrans:
  assumes a  $\simeq$  b
           b  $\simeq$  c
  shows a  $\simeq$  c
using assms(1) assms(2) by (metis eq-d-def inteq-reflection)

lemma MonEq:
  (a  $\simeq$  b) = ( $\vdash$  ( $\mathcal{M}$  a) = ( $\mathcal{M}$  b))
by (simp add: eq-d-def)

lemma MonEqSubstWith:
  assumes a  $\simeq$  b
  shows (a WITH f)  $\simeq$  (b WITH f)
using assms by (metis MON.simps(5) eq-d-def inteq-reflection lift-and-com)

lemma MonEqSubstThen:
  assumes a1  $\simeq$  b1
           a2  $\simeq$  b2
  shows (a1 THEN a2)  $\simeq$  (b1 THEN b2)
using assms(1) assms(2) by (simp add: ChopEqvChop eq-d-def)

lemma MonEqSubstUpto:
  assumes a1  $\simeq$  b1
           a2  $\simeq$  b2

```

shows $(a1 \text{ UPTO } a2) \simeq (b1 \text{ UPTO } b2)$
using *assms*(1) *assms*(2) **by** (*metis* (*mono-tags*, *lifting*) *MON.simps*(2) *eq-d-def int-eq MonEqRefl*)

lemma *MonEqSubstThru*:

assumes $a1 \simeq b1$

$a2 \simeq b2$

shows $(a1 \text{ THRU } a2) \simeq (b1 \text{ THRU } b2)$

using *assms*(1) *assms*(2) **by** (*metis* (*mono-tags*, *lifting*) *MON.simps*(3) *eq-d-def int-eq MonEqRefl*)

10.2 Derived Monitors

definition *HALT-d* :: ('a :: world) formula \Rightarrow 'a monitor

where *HALT-d* *w* \equiv *FIRST*(*LIFT*(*fn* (*init* *w*)))

definition *LEN-d* :: nat \Rightarrow ('a :: world) monitor

where

LEN-d *k* \equiv *FIRST* (*LIFT*(*len* *k*))

definition *EMPTY-d* :: ('a :: world) monitor

where

EMPTY-d \equiv *FIRST* (*LIFT*(*empty*))

definition *SKIP-d* :: ('a :: world) monitor

where

SKIP-d \equiv *FIRST* (*LIFT* (*skip*))

syntax

-*HALT-d* :: lift \Rightarrow 'a monitor $((\text{HALT } -) [84] 83)$

-*LEN-d* :: nat \Rightarrow 'a monitor $((\text{LEN } -) [84] 83)$

-*EMPTY-d* :: 'a monitor $((\text{EMPTY}))$

-*SKIP-d* :: 'a monitor $((\text{SKIP}))$

syntax (*ASCII*)

-*HALT-d* :: lift \Rightarrow 'a monitor $((\text{HALT } -) [84] 83)$

-*LEN-d* :: nat \Rightarrow 'a monitor $((\text{LEN } -) [84] 83)$

-*EMPTY-d* :: 'a monitor $((\text{EMPTY}))$

-*SKIP-d* :: 'a monitor $((\text{SKIP}))$

translations

-*HALT-d* \equiv *CONST HALT-d*

-*LEN-d* \equiv *CONST LEN-d*

-*EMPTY-d* \equiv *CONST EMPTY-d*

-*SKIP-d* \equiv *CONST SKIP-d*

definition *GUARD-d* :: ('a :: world) formula \Rightarrow 'a monitor

where

GUARD-d *w* \equiv (*EMPTY WITH LIFT*(*init* *w*))

primrec *TIMES-d* :: ('a :: world) monitor \Rightarrow nat \Rightarrow 'a monitor
where

TIMES-0 :: *TIMES-d a 0* = *EMPTY*
| *TIMES-Suc*: *TIMES-d a (Suc k)* = (*a THEN (TIMES-d a k)*)

syntax

-*GUARD-d* :: lift \Rightarrow 'a monitor ((*GUARD -*) [84] 83)
-*TIMES-d* :: ['a monitor, nat] \Rightarrow 'a monitor ((- *TIMES -*) [84,84] 83)

syntax (*ASCII*)

-*GUARD-d* :: lift \Rightarrow 'a monitor ((*GUARD -*) [84] 83)
-*TIMES-d* :: ['a monitor, nat] \Rightarrow 'a monitor ((- *TIMES -*) [84,84] 83)

translations

-*GUARD-d* \equiv *CONST GUARD-d*
-*TIMES-d* \equiv *CONST TIMES-d*

definition *FAIL-d* :: ('a :: world) monitor

where

FAIL-d \equiv *GUARD (#False)*

definition *ALWAYS-d* :: ('a :: world) monitor \Rightarrow 'a formula \Rightarrow 'a monitor

where

ALWAYS-d a w \equiv (*a WITH LIFT((bi (fn (init w))))*)

definition *SOMETIME-d* :: ('a :: world) monitor \Rightarrow 'a formula \Rightarrow 'a monitor

where

SOMETIME-d a w \equiv (*a WITH LIFT((di (fn (init w))))*)

definition *LIMIT-d* :: ('a :: world) formula \Rightarrow 'a formula

where

LIMIT-d f \equiv *LIFT(bs (\neg f))*

definition *UNTIL-d* :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a monitor

where

UNTIL-d w1 w2 \equiv (*HALT w2*) *WITH (LIFT(bm w1))*

syntax

-*FAIL-d* :: 'a monitor (*FAIL*)
-*ALWAYS-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *ALWAYS -*) [84,84] 83)
-*SOMETIME-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *SOMETIME -*) [84,84] 83)
-*LIMIT-d* :: lift \Rightarrow lift ((*Limit -*) [84] 83)
-*UNTIL-d* :: [lift, lift] \Rightarrow 'a monitor ((- *UNTIL -*) [84,84] 83)

syntax (*ASCII*)

-*FAIL-d* :: 'a monitor (*FAIL*)
-*ALWAYS-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *ALWAYS -*) [84,84] 83)
-*SOMETIME-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *SOMETIME -*) [84,84] 83)
-*LIMIT-d* :: lift \Rightarrow lift ((*Limit -*) [84] 83)

$-UNTIL-d \quad :: [lift, lift] \Rightarrow 'a \text{ monitor} \quad ((- \text{ UNTIL } -) [84,84] \ 83)$

translations

$-FAIL-d \quad \Rightarrow \text{CONST FAIL-}d$
 $-ALWAYS-d \quad \Rightarrow \text{CONST ALWAYS-}d$
 $-SOMETIME-d \quad \Rightarrow \text{CONST SOMETIME-}d$
 $-LIMIT-d \quad \Rightarrow \text{CONST LIMIT-}d$
 $-UNTIL-d \quad \Rightarrow \text{CONST UNTIL-}d$

definition $WITHIN-d :: ('a :: world) \text{ monitor} \Rightarrow 'a \text{ formula} \Rightarrow 'a \text{ monitor}$

where

$WITHIN-d \ a \ f \equiv (a \ \text{WITH LIFT}(\text{Limit } f))$

syntax

$-WITHIN-d :: ['a \text{ monitor}, lift] \Rightarrow 'a \text{ monitor} \quad ((- \text{ WITHIN } -) [84,84] \ 83)$

syntax (*ASCII*)

$-WITHIN-d :: ['a \text{ monitor}, lift] \Rightarrow 'a \text{ monitor} \quad ((- \text{ WITHIN } -) [84,84] \ 83)$

translations

$-WITHIN-d \Rightarrow \text{CONST WITHIN-}d$

definition $AND-d :: ('a :: world) \text{ monitor} \Rightarrow 'a \text{ monitor} \Rightarrow 'a \text{ monitor}$

where

$AND-d \ a \ b \equiv (a \ \text{WITH LIFT}(\mathcal{M} \ b))$

definition $ITERATE-d :: ('a :: world) \text{ monitor} \Rightarrow 'a \text{ monitor} \Rightarrow 'a \text{ monitor}$

where

$ITERATE-d \ a \ b \equiv (a \ \text{WITH } (\text{LIFT } (\mathcal{M} \ b)^*))$

syntax

$-AND-d \quad :: ['a \text{ monitor}, 'a \text{ monitor}] \Rightarrow 'a \text{ monitor} \quad ((- \text{ AND } -) [84,84] \ 83)$
 $-ITERATE-d :: ['a \text{ monitor}, 'a \text{ monitor}] \Rightarrow 'a \text{ monitor} \quad ((- \text{ ITERATE } -) [84,84] \ 83)$

syntax (*ASCII*)

$-AND-d \quad :: ['a \text{ monitor}, 'a \text{ monitor}] \Rightarrow 'a \text{ monitor} \quad ((- \text{ AND } -) [84,84] \ 83)$
 $-ITERATE-d :: ['a \text{ monitor}, 'a \text{ monitor}] \Rightarrow 'a \text{ monitor} \quad ((- \text{ ITERATE } -) [84,84] \ 83)$

translations

$-AND-d \quad \Rightarrow \text{CONST AND-}d$
 $-ITERATE-d \Rightarrow \text{CONST ITERATE-}d$

definition $STAR-d :: ('a :: world) \text{ monitor} \Rightarrow 'a \text{ formula} \Rightarrow 'a \text{ monitor}$

where

$STAR-d \ a \ f \equiv ((\text{FIRST LIFT}(\diamond f)) \ \text{ITERATE } (a))$

definition *REPEAT-d* :: ('a :: world) monitor \Rightarrow 'a formula \Rightarrow 'a monitor

where

REPEAT-d a w $\equiv ((\text{HALT } w) \text{ ITERATE } (a \text{ WITH LIFT}(\text{keep}(\neg (\text{init } w))))))$

syntax

-*STAR-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *STAR* -) [84,84] 83)

-*REPEAT-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *REPEATUNTIL* -) [84,84] 83)

syntax (*ASCII*)

-*STAR-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *STAR* -) [84,84] 83)

-*REPEAT-d* :: ['a monitor, lift] \Rightarrow 'a monitor ((- *REPEATUNTIL* -) [84,84] 83)

translations

-*STAR-d* \Rightarrow *CONST STAR-d*

-*REPEAT-d* \Rightarrow *CONST REPEAT-d*

10.3 Monitor Laws

lemma *MFixFst*:

$\vdash (\mathcal{M} \ a) = \triangleright (\mathcal{M} \ a)$

proof

(*induct a*)

case (*mFIRST-d* x)

then show ?*case*

proof –

have 1: $\vdash (\mathcal{M} \ (\text{FIRST } x)) = \triangleright x$ **by** *simp*

have 2: $\vdash \triangleright x = \triangleright (\triangleright x)$ **using** *FstFixFst* **by** *fastforce*

have 3: $\vdash \triangleright (\triangleright x) = \triangleright (\mathcal{M} \ (\text{FIRST } x))$ **by** *simp*

from 1 2 3 **show** ?*thesis* **by** *fastforce*

qed

next

case (*mUPTO-d* a1 a2)

then show ?*case*

proof –

have 1: $\vdash (\mathcal{M} \ (a1 \ \text{UPTO } a2)) = \triangleright ((\mathcal{M} \ a1) \vee (\mathcal{M} \ a2))$

by (*simp*)

have 2: $\vdash \triangleright ((\mathcal{M} \ a1) \vee (\mathcal{M} \ a2)) = \triangleright (\triangleright ((\mathcal{M} \ a1) \vee (\mathcal{M} \ a2)))$

using *FstFixFst* **by** *fastforce*

have 3: $\vdash \triangleright (\triangleright ((\mathcal{M} \ a1) \vee (\mathcal{M} \ a2))) = \triangleright (\mathcal{M} \ (a1 \ \text{UPTO } a2))$

using 2 **by** *simp*

from 1 2 3 **show** ?*thesis* **by** *fastforce*

qed

next

case (*mTHRU-d* a1 a2)

then show ?*case*

proof –

have 1: $\vdash (\mathcal{M} \ (a1 \ \text{THRU } a2)) = \triangleright (di \ (\mathcal{M} \ a1) \wedge di \ (\mathcal{M} \ a2))$

by (*simp*)

have 2: $\vdash \triangleright (di \ (\mathcal{M} \ a1) \wedge di \ (\mathcal{M} \ a2)) = \triangleright (\triangleright (di \ (\mathcal{M} \ a1) \wedge di \ (\mathcal{M} \ a2)))$

using *FstFixFst* **by** *fastforce*

```

  have 3:  $\vdash \triangleright(\triangleright( di(\mathcal{M} a1) \wedge di(\mathcal{M} a2))) = \triangleright(\mathcal{M}(a1 \text{ THRU } a2))$ 
    using 2 by simp
  from 1 2 3 show ?thesis by fastforce
qed
next
  case (mTHEN-d a1 a2)
  then show ?case
  proof -
    have 1:  $\vdash (\mathcal{M}(a1 \text{ THEN } a2)) = (\mathcal{M} a1) ; (\mathcal{M} a2)$ 
      by (simp)
    have 2:  $\vdash (\mathcal{M} a1) ; (\mathcal{M} a2) = \triangleright(\mathcal{M} a1) ; \triangleright(\mathcal{M} a2)$ 
      using ChopEqvChop mTHEN-d.hyps(1) mTHEN-d.hyps(2) by blast
    have 3:  $\vdash \triangleright(\mathcal{M} a1) ; \triangleright(\mathcal{M} a2) = \triangleright(\triangleright(\mathcal{M} a1) ; (\mathcal{M} a2))$ 
      using FstFstChopEqvFstChopFst by fastforce
    have 4:  $\vdash \triangleright(\triangleright(\mathcal{M} a1) ; (\mathcal{M} a2)) = \triangleright((\mathcal{M} a1) ; (\mathcal{M} a2))$ 
      using FstEqvRule LeftChopEqvChop mTHEN-d.hyps(1) by (metis inteq-reflection)
    have 5:  $\vdash \triangleright((\mathcal{M} a1) ; (\mathcal{M} a2)) = \triangleright(\mathcal{M}(a1 \text{ THEN } a2))$ 
      using 4 by simp
    from 1 2 3 4 5 show ?thesis by fastforce
  qed
next
  case (mWITH-d a x2)
  then show ?case
  proof -
    have 1:  $\vdash (\mathcal{M}(a \text{ WITH } x2)) = ((\mathcal{M} a) \wedge (x2))$ 
      by (simp)
    have 2:  $\vdash ((\mathcal{M} a) \wedge (x2)) = (\triangleright(\mathcal{M} a) \wedge (x2))$ 
      using mWITH-d.hyps by fastforce
    have 3:  $\vdash (\triangleright(\mathcal{M} a) \wedge (x2)) = \triangleright(\triangleright(\mathcal{M} a) \wedge (x2))$ 
      using FstFstAndEqvFstAnd by fastforce
    have 4:  $\vdash \triangleright(\triangleright(\mathcal{M} a) \wedge (x2)) = \triangleright((\mathcal{M} a) \wedge (x2))$ 
      using 2 FstEqvRule by fastforce
    have 5:  $\vdash \triangleright((\mathcal{M} a) \wedge (x2)) = \triangleright(\mathcal{M}(a \text{ WITH } x2))$ 
      using 4 by simp
    from 1 2 3 4 5 show ?thesis by (metis inteq-reflection)
  qed
qed

```

lemma MGuardFalseEqvFalse:

$\vdash \mathcal{M}(\text{GUARD } \#False) = \#False$

proof -

```

  have 1:  $\vdash \mathcal{M}(\text{GUARD } \#False) = \mathcal{M}(\text{EMPTY WITH LIFT}(init \#False))$  by (simp add: GUARD-d-def)
  have 2:  $\vdash \mathcal{M}(\text{EMPTY WITH LIFT}(init \#False)) = (\mathcal{M}(\text{EMPTY}) \wedge (init \#False))$  by (simp)
  have 3:  $\vdash \#False = (init \#False)$  by (simp add: init-defs Valid-def)
  have 4:  $\vdash (\mathcal{M}(\text{EMPTY}) \wedge (init \#False)) = (\mathcal{M}(\text{EMPTY}) \wedge \#False)$  using 3 by auto
  have 5:  $\vdash (\mathcal{M}(\text{EMPTY}) \wedge \#False) = \#False$  by simp
  have 6:  $\vdash (\mathcal{M}(\text{EMPTY}) \wedge (init \#False)) = \#False$  using 4 5 by simp
  have 7:  $\vdash \mathcal{M}(\text{EMPTY WITH LIFT}(init \#False)) = \#False$  using 2 6 by fastforce
  have 8:  $\vdash \mathcal{M}(\text{GUARD } \#False) = \#False$  using 1 7 by fastforce
  from 8 show ?thesis by auto

```

qed

lemma *MFirstFalseEqvFalse*:

$\vdash \mathcal{M}(\text{FIRST LIFT } \#False) = \#False$

proof –

have 1: $\vdash \mathcal{M}(\text{FIRST LIFT } \#False) = \triangleright \#False$ **by** (*simp*)

have 2: $\vdash \mathcal{M}(\text{FIRST LIFT } \#False) = \#False$ **using** *FstFalse* **by** *fastforce*

from 2 **show** *?thesis* **by** *auto*

qed

lemma *MFailAlt*:

$\vdash \mathcal{M} \text{ FAIL} = \#False$

proof –

have 1: $\vdash \mathcal{M} \text{ FAIL} = \mathcal{M} (\text{GUARD } (\#False))$ **by** (*simp add: FAIL-d-def*)

have 2: $\vdash \mathcal{M}(\text{GUARD } (\#False)) = \#False$ **using** *MGuardFalseEqvFalse* **by** *auto*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *MFailEqvFirstFalseWithinEmpty*:

$\text{FAIL} \simeq ((\text{FIRST LIFT } \#False) \text{ WITHIN } \text{empty})$

proof –

have 1: $\vdash \mathcal{M} ((\text{FIRST LIFT } \#False) \text{ WITHIN } (\text{empty})) =$
 $\mathcal{M}((\text{FIRST LIFT } \#False) \text{ WITH LIFT}(\text{Limit empty}))$

by (*simp add: WITHIN-d-def*)

have 2: $\vdash \mathcal{M}((\text{FIRST LIFT } \#False) \text{ WITH LIFT}(\text{Limit empty})) =$
 $(\mathcal{M}(\text{FIRST LIFT } \#False) \wedge (\text{Limit empty}))$

by (*simp*)

have 3: $\vdash \mathcal{M}((\text{FIRST LIFT } \#False) \text{ WITH LIFT}(\text{Limit empty})) = \#False$
using *MFirstFalseEqvFalse* **by** *auto*

have 4: $\vdash \mathcal{M} ((\text{FIRST LIFT } \#False) \text{ WITHIN } (\text{empty})) = \#False$
using 1 3 **by** *fastforce*

have 5: $\vdash \mathcal{M} (\text{FAIL}) = \#False$
using *MFailAlt* **by** *simp*

from 4 5 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)

qed

lemma *MEmptyAlt*:

$\vdash \mathcal{M} \text{ EMPTY} = \text{empty}$

proof –

have 1: $\vdash \mathcal{M} (\text{EMPTY}) = \mathcal{M} ((\text{FIRST LIFT } \text{empty}))$ **by** (*simp add: EMPTY-d-def*)

have 2: $\vdash \mathcal{M} ((\text{FIRST LIFT } \text{empty})) = \triangleright \text{empty}$ **by** (*simp*)

have 3: $\vdash \triangleright \text{empty} = \text{empty}$ **using** *FstEmpty* **by** *auto*

from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *MSkipAlt*:

$\vdash \mathcal{M} \text{ SKIP} = \text{skip}$

proof –

have 1: $\vdash \mathcal{M} \text{ SKIP} = \mathcal{M} (\text{FIRST LIFT } \text{skip})$ **by** (*simp add: SKIP-d-def*)

have 2: $\vdash \mathcal{M} (\text{FIRST LIFT } \text{skip}) = \triangleright \text{skip}$ **by** (*simp*)

have 3: $\vdash \triangleright \text{skip} = \text{skip}$ **using** *FstSkip* **by** *simp*
from 1 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *MGuardAlt*:

$\vdash \mathcal{M}(\text{GUARD}(w)) = (\text{empty} \wedge \text{init } w)$

proof –

have 1: $\vdash \mathcal{M}(\text{GUARD}(w)) = \mathcal{M}(\text{EMPTY WITH } (\text{LIFT } (\text{init } w)))$ **by** (*simp add: GUARD-d-def*)

have 2: $\vdash \mathcal{M}(\text{EMPTY WITH } (\text{LIFT } (\text{init } w))) = (\mathcal{M}(\text{EMPTY}) \wedge (\text{init } w))$ **by** (*simp*)

have 3: $\vdash (\mathcal{M}(\text{EMPTY}) \wedge (\text{init } w)) = (\text{empty} \wedge (\text{init } w))$ **using** *MEmpyAlt* **by** *fastforce*

have 4: $\vdash (\text{empty} \wedge (\text{init } w)) = (\text{empty} \wedge \text{init } w)$ **by** *simp*

from 1 2 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *MLengthAlt*:

$\vdash \mathcal{M}(\text{LEN}(k)) = \text{len}(k)$

proof –

have 1: $\vdash \mathcal{M}(\text{LEN}(k)) = \mathcal{M}(\text{FIRST LIFT}(\text{len}(k)))$ **by** (*simp add: LEN-d-def*)

have 2: $\vdash \mathcal{M}(\text{FIRST LIFT}(\text{len}(k))) = \triangleright(\text{len}(k))$ **by** (*simp*)

have 3: $\vdash \triangleright(\text{len}(k)) = \text{len}(k)$ **using** *FstLenEqvLen* **by** *blast*

from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *MAalwaysAlt*:

$\vdash \mathcal{M}(a \text{ ALWAYS } w) = (\mathcal{M}(a) \wedge \square (\text{init } w))$

proof –

have 1: $\vdash \mathcal{M}(a \text{ ALWAYS } w) = \mathcal{M}(a \text{ WITH LIFT}(bi \text{ (fin (init } w)))))$

by (*simp add: ALWAYS-d-def*)

have 2: $\vdash \mathcal{M}(a \text{ WITH LIFT}(bi \text{ (fin (init } w))))) = (\mathcal{M}(a) \wedge (bi \text{ (fin (init } w)))))$

by (*simp*)

have 3: $\vdash (\mathcal{M}(a) \wedge (bi \text{ (fin (init } w))))) = (\mathcal{M}(a) \wedge \square (\text{init } w))$

using *BoxStateEqvBiFinState* **by** *fastforce*

from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *MSometimeAlt*:

$\vdash \mathcal{M}(a \text{ SOMETIME } w) = (\mathcal{M}(a) \wedge \diamond (\text{init } w))$

proof –

have 1: $\vdash \mathcal{M}(a \text{ SOMETIME } w) = \mathcal{M}(a \text{ WITH LIFT}(di \text{ (fin (init } w)))))$

by (*simp add: SOMETIME-d-def*)

have 2: $\vdash \mathcal{M}(a \text{ WITH LIFT}(di \text{ (fin (init } w))))) = (\mathcal{M}(a) \wedge (di \text{ (fin (init } w)))))$

by (*simp*)

have 3: $\vdash \mathcal{M}(a \text{ WITH LIFT}(di \text{ (fin (init } w))))) = (\mathcal{M}(a) \wedge \diamond (\text{init } w))$

using *DiamondStateEqvDiFinState* **by** *fastforce*

from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *MWithinAlt*:

$\vdash \mathcal{M}(a \text{ WITHIN } f) = (\mathcal{M}(a) \wedge (bs \text{ } (\neg f)))$

proof –
have 1: $\vdash \mathcal{M}(a \text{ WITHIN } f) = \mathcal{M}(a \text{ WITH LIFT}(bs(\neg f)))$
by (*simp add: WITHIN-d-def LIMIT-d-def*)
have 2: $\vdash \mathcal{M}(a \text{ WITH LIFT}(bs(\neg f))) = (\mathcal{M}(a) \wedge (bs(\neg f)))$
by (*simp*)
from 1 2 **show** ?thesis **by** fastforce
qed

lemma *MTimesAlt*:

$\vdash \mathcal{M}(a \text{ TIMES } k) = \text{power } (\mathcal{M}(a)) \ k$

proof
 (*induct k*)
case 0
then show ?case
proof –
have 1: $\vdash \mathcal{M}(a \text{ TIMES } 0) = \mathcal{M} \text{ EMPTY}$ **by** *simp*
have 2: $\vdash \mathcal{M} \text{ EMPTY} = \text{empty}$ **using** *MEmptyAlt* **by** *simp*
have 3: $\vdash \text{empty} = \text{power } (\mathcal{M} \ a) \ 0$ **by** *simp*
from 1 2 3 **show** ?thesis **by** auto
qed
next
case (*Suc k*)
then show ?case
proof –
have 1: $\vdash \mathcal{M}(a \text{ TIMES } \text{Suc } k) = \mathcal{M}(a \text{ THEN } (a \text{ TIMES } k))$
by *simp*
have 2: $\vdash \mathcal{M}(a \text{ THEN } (a \text{ TIMES } k)) = (\mathcal{M} \ a);(\mathcal{M}(a \text{ TIMES } k))$
by (*simp*)
have 3: $\vdash (\mathcal{M} \ a);(\mathcal{M}(a \text{ TIMES } k)) = (\mathcal{M} \ a);(\text{power } (\mathcal{M} \ a) \ k)$
using *RightChopEqvChop Suc.hyps* **by** *blast*
have 4: $\vdash (\mathcal{M} \ a);(\text{power } (\mathcal{M} \ a) \ k) = \text{power } (\mathcal{M} \ a) \ (\text{Suc } k)$
by *simp*
from 1 2 3 4 **show** ?thesis **by** fastforce
qed
qed

lemma *MUptoAlt*:

$\vdash \mathcal{M}(a \text{ UPTO } b) = ((\mathcal{M} \ a) \wedge bi(\neg(\mathcal{M} \ b))) \vee ((\mathcal{M} \ b) \wedge bi(\neg(\mathcal{M} \ a))) \vee ((\mathcal{M} \ a) \wedge (\mathcal{M} \ b))$

proof –
have 1: $\vdash \mathcal{M}(a \text{ UPTO } b) = \triangleright((\mathcal{M} \ a) \vee (\mathcal{M} \ b))$
by (*simp*)
have 2: $\vdash \triangleright((\mathcal{M} \ a) \vee (\mathcal{M} \ b)) = ((\triangleright(\mathcal{M} \ a) \wedge (bs(\neg(\mathcal{M} \ b)))) \vee (\triangleright(\mathcal{M} \ b) \wedge (bs(\neg(\mathcal{M} \ a)))))$
using *FstWithOrEqv* **by** *blast*
have 3: $\vdash ((\triangleright(\mathcal{M} \ a) \wedge (bs(\neg(\mathcal{M} \ b)))) \vee (\triangleright(\mathcal{M} \ b) \wedge (bs(\neg(\mathcal{M} \ a))))) =$
 $((\mathcal{M} \ a) \wedge ((\mathcal{M} \ b) \vee \neg(\mathcal{M} \ b)) \wedge (bs(\neg(\mathcal{M} \ b)))) \vee$
 $((\mathcal{M} \ b) \wedge ((\mathcal{M} \ a) \vee \neg(\mathcal{M} \ a)) \wedge (bs(\neg(\mathcal{M} \ a))))$
using *MFixFst* **by** fastforce
have 4: $\vdash (((\mathcal{M} \ a) \wedge ((\mathcal{M} \ b) \vee \neg(\mathcal{M} \ b)) \wedge (bs(\neg(\mathcal{M} \ b)))) \vee$
 $((\mathcal{M} \ b) \wedge ((\mathcal{M} \ a) \vee \neg(\mathcal{M} \ a)) \wedge (bs(\neg(\mathcal{M} \ a))))) =$
 $((\mathcal{M} \ a) \wedge ((\mathcal{M} \ b) \wedge bs(\neg(\mathcal{M} \ b))) \vee (\neg(\mathcal{M} \ b) \wedge bs(\neg(\mathcal{M} \ b)))) \vee$

$((\mathcal{M} \ b) \wedge (((\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a))) \vee (\neg(\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a)))))$
by *auto*
have 5: $\vdash (((\mathcal{M} \ a) \wedge (((\mathcal{M} \ b) \wedge bs \ (\neg(\mathcal{M} \ b))) \vee (\neg(\mathcal{M} \ b) \wedge bs \ (\neg(\mathcal{M} \ b))))) \vee$
 $((\mathcal{M} \ b) \wedge (((\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a))) \vee (\neg(\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a))))) =$
 $((\mathcal{M} \ a) \wedge ((\triangleright(\mathcal{M} \ b)) \vee (\neg(\mathcal{M} \ b) \wedge bs \ (\neg(\mathcal{M} \ b))))) \vee$
 $((\mathcal{M} \ b) \wedge ((\triangleright(\mathcal{M} \ a)) \vee (\neg(\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a)))))$
by (*simp add: first-d-def*)
have 6: $\vdash (((\mathcal{M} \ a) \wedge ((\triangleright(\mathcal{M} \ b)) \vee (\neg(\mathcal{M} \ b) \wedge bs \ (\neg(\mathcal{M} \ b))))) \vee$
 $((\mathcal{M} \ b) \wedge ((\triangleright(\mathcal{M} \ a)) \vee (\neg(\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a))))) =$
 $((\mathcal{M} \ a) \wedge (((\mathcal{M} \ b)) \vee (\neg(\mathcal{M} \ b) \wedge bs \ (\neg(\mathcal{M} \ b))))) \vee$
 $((\mathcal{M} \ b) \wedge (((\mathcal{M} \ a)) \vee (\neg(\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a)))))$
using *MFixFst* **by** *fastforce*
have 7: $\vdash (\neg(\mathcal{M} \ b) \wedge bs \ (\neg(\mathcal{M} \ b))) = bi(\neg(\mathcal{M} \ b))$
using *AndBsEqvBi* **by** *blast*
have 8: $\vdash (\neg(\mathcal{M} \ a) \wedge bs \ (\neg(\mathcal{M} \ a))) = bi(\neg(\mathcal{M} \ a))$
using *AndBsEqvBi* **by** *blast*
have 9: $\vdash (((\mathcal{M} \ a) \wedge (((\mathcal{M} \ b)) \vee ((\neg(\mathcal{M} \ b)) \wedge bs(\neg(\mathcal{M} \ b))))) \vee$
 $((\mathcal{M} \ b) \wedge (((\mathcal{M} \ a)) \vee ((\neg(\mathcal{M} \ a)) \wedge bs(\neg(\mathcal{M} \ a))))) =$
 $((\mathcal{M} \ a) \wedge (((\mathcal{M} \ b)) \vee (bi(\neg(\mathcal{M} \ b)))) \vee$
 $((\mathcal{M} \ b) \wedge (((\mathcal{M} \ a)) \vee (bi(\neg(\mathcal{M} \ a))))$
using 7 8 **by** *fastforce*
have 10: $\vdash (((\mathcal{M} \ a) \wedge (((\mathcal{M} \ b)) \vee (bi(\neg(\mathcal{M} \ b)))) \vee$
 $((\mathcal{M} \ b) \wedge (((\mathcal{M} \ a)) \vee (bi(\neg(\mathcal{M} \ a)))) =$
 $(((\mathcal{M} \ a) \wedge (\mathcal{M} \ b)) \vee ((\mathcal{M} \ a) \wedge bi(\neg(\mathcal{M} \ b))) \vee$
 $(((\mathcal{M} \ b) \wedge (\mathcal{M} \ a)) \vee ((\mathcal{M} \ b) \wedge bi(\neg(\mathcal{M} \ a))))$
by *auto*
have 11: $\vdash ((((\mathcal{M} \ a) \wedge (\mathcal{M} \ b)) \vee ((\mathcal{M} \ a) \wedge bi(\neg(\mathcal{M} \ b))) \vee$
 $(((\mathcal{M} \ b) \wedge (\mathcal{M} \ a)) \vee ((\mathcal{M} \ b) \wedge bi(\neg(\mathcal{M} \ a)))) =$
 $(((\mathcal{M} \ a) \wedge bi \ (\neg(\mathcal{M} \ b))) \vee ((\mathcal{M} \ b) \wedge bi \ (\neg(\mathcal{M} \ a))) \vee ((\mathcal{M} \ a) \wedge (\mathcal{M} \ b)))$
by *auto*
show *?thesis*
by (*metis 10 11 2 3 4 5 6 9 MON.simps(2) int-eq*)
qed

lemma *MThruAlt:*

$\vdash \mathcal{M}(a \text{ THRU } b) = (((\mathcal{M} \ a) \wedge di(\mathcal{M} \ b)) \vee ((\mathcal{M} \ b) \wedge di(\mathcal{M} \ a)))$

proof –

have 1: $\vdash \mathcal{M}(a \text{ THRU } b) = \triangleright(di(\mathcal{M} \ a) \wedge di(\mathcal{M} \ b))$

by (*simp*)

have 2: $\vdash \triangleright(di(\mathcal{M} \ a) \wedge di(\mathcal{M} \ b)) = ((\triangleright(\mathcal{M} \ a) \wedge di(\mathcal{M} \ b)) \vee (\triangleright(\mathcal{M} \ b) \wedge di(\mathcal{M} \ a)))$

using *FstDiAndDiEqv* **by** *auto*

have 3: $\vdash ((\triangleright(\mathcal{M} \ a) \wedge di(\mathcal{M} \ b)) \vee (\triangleright(\mathcal{M} \ b) \wedge di(\mathcal{M} \ a))) =$

$((\mathcal{M} \ a) \wedge di(\mathcal{M} \ b)) \vee ((\mathcal{M} \ b) \wedge di(\mathcal{M} \ a))$

using *MFixFst* **by** *fastforce*

from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma *MHaltAlt:*

$\vdash \mathcal{M}(\text{HALT } w) = \text{halt}(\text{init } w)$

proof –

have 1: $\vdash \mathcal{M}(\text{HALT } w) = \mathcal{M}(\text{FIRST LIFT}(\text{fin } (\text{init } w)))$ **by** (*simp add: HALT-d-def*)
have 2: $\vdash \mathcal{M}(\text{FIRST LIFT}(\text{fin } (\text{init } w))) = \triangleright (\text{fin } (\text{init } w))$ **by** (*simp*)
have 3: $\vdash \triangleright (\text{fin } (\text{init } w)) = \text{halt}(\text{init } w)$ **using** *HaltStateEqvFstFinState* **by** *fastforce*
from 1 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *MFailUpto*:
 $(\text{FAIL UPTO } a) \simeq (a)$

proof –
have 1: $\vdash \mathcal{M}(\text{FAIL UPTO } a) = \triangleright (\mathcal{M} \text{ FAIL} \vee (\mathcal{M} a))$ **by** (*simp*)
have 2: $\vdash (\mathcal{M} \text{ FAIL} \vee \mathcal{M} a) = (\#False \vee \mathcal{M} a)$ **using** *MFailAlt* **by** *auto*
have 3: $\vdash \triangleright (\mathcal{M} \text{ FAIL} \vee (\mathcal{M} a)) = \triangleright (\#False \vee (\mathcal{M} a))$ **using** 2 *FstEqvRule* **by** *blast*
have 4: $\vdash (\#False \vee (\mathcal{M} a)) = \mathcal{M} a$ **by** *simp*
have 5: $\vdash \triangleright (\#False \vee (\mathcal{M} a)) = \triangleright (\mathcal{M} a)$ **using** 4 *FstEqvRule* **by** *blast*
have 6: $\vdash \triangleright (\mathcal{M} a) = \mathcal{M} a$ **using** *MFixFst* **by** *fastforce*
from 1 2 3 4 5 6 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MFailThru*:
 $(\text{FAIL THRU } (a)) \simeq \text{FAIL}$

proof –
have 1: $\vdash \mathcal{M} (\text{FAIL THRU } (a)) = \triangleright (di(\mathcal{M} \text{ FAIL}) \wedge di(\mathcal{M} a))$
by (*simp*)
have 2: $\vdash \triangleright (di(\mathcal{M} \text{ FAIL}) \wedge di(\mathcal{M} a)) = \triangleright (di(\#False) \wedge di(\mathcal{M} a))$
using *MFailAlt* **by** (*metis 1 int-eq*)
have 3: $\vdash di \#False = \#False$
by (*simp add: di-defs Valid-def*)
hence 4: $\vdash \triangleright (di(\#False) \wedge di(\mathcal{M} a)) = \triangleright (\#False \wedge di(\mathcal{M} a))$
by (*metis 2 inteq-reflection*)
have 5: $\vdash \triangleright (\#False \wedge di(\mathcal{M} a)) = \triangleright \#False$
using *FstEqvRule* **by** *fastforce*
have 6: $\vdash \triangleright \#False = \#False$ **using** *FstFalse*
by *auto*
have 7: $\vdash \#False = \mathcal{M} \text{ FAIL}$
using *MFailAlt* **by** *auto*
from 1 2 4 5 6 7 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MFailAnd*:
 $(\text{FAIL AND } a) \simeq \text{FAIL}$

proof –
have 1: $\vdash \mathcal{M} (\text{FAIL AND } a) = (\mathcal{M} \text{ FAIL} \wedge (\mathcal{M} a))$ **by** (*simp add: AND-d-def*)
have 2: $\vdash (\mathcal{M} \text{ FAIL} \wedge (\mathcal{M} a)) = (\#False \wedge (\mathcal{M} a))$ **using** *MFailAlt* **by** *fastforce*
have 3: $\vdash (\#False \wedge (\mathcal{M} a)) = \#False$ **by** *auto*
have 4: $\vdash \mathcal{M}(\text{FAIL AND } a) = \#False$ **using** 1 2 3 **by** *fastforce*
have 5: $\vdash \#False = \mathcal{M} \text{ FAIL}$ **using** *MFailAlt* **by** *auto*
from 1 2 3 4 5 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MThenFail*:

$(a \text{ THEN } \text{FAIL}) \simeq \text{FAIL}$
proof –
 have 1: $\vdash \mathcal{M} (a \text{ THEN } \text{FAIL}) = (\mathcal{M} a);(\mathcal{M} \text{ FAIL})$ **by** (*simp*)
 have 2: $\vdash (\mathcal{M} a);(\mathcal{M} \text{ FAIL}) = (\mathcal{M} a);\#False$ **by** (*simp add: MFailAlt RightChopEqvChop*)
 have 3: $\vdash (\mathcal{M} a);\#False = \#False$ **by** (*simp add: chop-d-def Valid-def*)
 have 4: $\vdash \#False = \mathcal{M} \text{ FAIL}$ **using** *MFailAlt* **by** *auto*
from 1 2 3 4 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MFailThen*:
 $(\text{FAIL THEN } a) \simeq \text{FAIL}$
proof –
 have 1: $\vdash \mathcal{M}(\text{FAIL THEN } a) = (\mathcal{M} \text{ FAIL});(\mathcal{M} a)$ **by** (*simp*)
 have 2: $\vdash (\mathcal{M} \text{ FAIL});(\mathcal{M} a) = \#False;(\mathcal{M} a)$ **using** *MFailAlt* **using** *LeftChopEqvChop* **by** *blast*
 have 3: $\vdash \#False;(\mathcal{M} a) = \#False$ **by** (*simp add: chop-d-def Valid-def*)
 have 4: $\vdash \#False = \mathcal{M} \text{ FAIL}$ **using** *MFailAlt* **by** *auto*
from 1 2 3 4 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MFailWith*:
 $(\text{FAIL WITH } f) \simeq \text{FAIL}$
proof –
 have 1: $\vdash \mathcal{M}(\text{FAIL WITH } f) = ((\mathcal{M} \text{ FAIL}) \wedge f)$ **by** (*simp*)
 have 2: $\vdash ((\mathcal{M} \text{ FAIL}) \wedge f) = (\#False \wedge f)$ **using** *MFailAlt* **by** *auto*
 have 3: $\vdash (\#False \wedge f) = \#False$ **by** *simp*
 have 4: $\vdash \#False = \mathcal{M} \text{ FAIL}$ **using** *MFailAlt* **by** *auto*
from 1 2 3 4 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MWithFalse*:
 $(a \text{ WITH } (\text{LIFT}(\#False))) \simeq \text{FAIL}$
proof –
 have 1: $\vdash \mathcal{M} (a \text{ WITH } \text{LIFT}(\#False)) = ((\mathcal{M} a) \wedge \#False)$ **by** (*simp*)
 have 2: $\vdash ((\mathcal{M} a) \wedge \#False) = \mathcal{M} \text{ FAIL}$ **using** *MFailAlt* **by** *auto*
from 1 2 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MWithTrue*:
 $(a \text{ WITH } (\text{LIFT}(\#True))) \simeq a$
proof –
 have 1: $\vdash \mathcal{M} (a \text{ WITH } \text{LIFT}(\#True)) = ((\mathcal{M} a) \wedge \#True)$ **by** (*simp*)
 have 2: $\vdash ((\mathcal{M} a) \wedge \#True) = \mathcal{M} a$ **by** *simp*
from 1 2 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MEEmptyUpto*:
 $(\text{EMPTY UPTO } a) \simeq \text{EMPTY}$
proof –
 have 1: $\vdash \mathcal{M}(\text{EMPTY UPTO } a) = \triangleright(\mathcal{M} \text{ EMPTY} \vee (\mathcal{M} a))$ **by** (*simp*)
 have 2: $\vdash \mathcal{M} \text{ EMPTY} = \text{empty}$ **using** *MEEmptyAlt* **by** *auto*

hence 3: $\vdash (\mathcal{M} \text{ EMPTY} \vee (\mathcal{M} a)) = (\text{empty} \vee (\mathcal{M} a))$ **by** *auto*
 hence 4: $\vdash \triangleright(\mathcal{M} \text{ EMPTY} \vee \mathcal{M} a) = \triangleright(\text{empty} \vee \mathcal{M} a)$ **using** *FstEqvRule* **by** *blast*
 have 5: $\vdash \triangleright(\text{empty} \vee \mathcal{M} a) = \text{empty}$ **using** *FstEmptyOrEqvEmpty* **by** *blast*
 have 6: $\vdash \text{empty} = \mathcal{M} \text{ EMPTY}$ **using** *MEEmptyAlt* **by** *auto*
 from 1 4 5 6 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MEEmptyThru*:

$(\text{EMPTY THRU } a) \simeq (a)$

proof –

have 1: $\vdash \mathcal{M}(\text{EMPTY THRU } a) = \triangleright(\text{di}(\mathcal{M} \text{ EMPTY}) \wedge \text{di}(\mathcal{M} a))$ **by** (*simp*)
 have 2: $\vdash \text{di}(\mathcal{M} \text{ EMPTY}) = \text{di empty}$ **using** *MEEmptyAlt DiEqvDi* **by** *blast*
 hence 3: $\vdash (\text{di}(\mathcal{M} \text{ EMPTY}) \wedge \text{di}(\mathcal{M} a)) = (\text{di empty} \wedge \text{di}(\mathcal{M} a))$ **by** *auto*
 hence 4: $\vdash (\text{di empty} \wedge \text{di}(\mathcal{M} a)) = \text{di}(\mathcal{M} a)$ **using** *DiEmpty* **by** *auto*
 have 5: $\vdash (\text{di}(\mathcal{M} \text{ EMPTY}) \wedge \text{di}(\mathcal{M} a)) = \text{di}(\mathcal{M} a)$ **using** 3 4 **by** *fastforce*
 hence 6: $\vdash \triangleright(\text{di}(\mathcal{M} \text{ EMPTY}) \wedge \text{di}(\mathcal{M} a)) = \triangleright(\text{di}(\mathcal{M} a))$ **using** *FstEqvRule* **by** *blast*
 have 7: $\vdash \triangleright(\text{di}(\mathcal{M} a)) = \triangleright(\mathcal{M} a)$ **using** *FstDiEqvFst* **by** *blast*
 have 8: $\vdash \triangleright(\mathcal{M} a) = (\mathcal{M} a)$ **using** *MFixFst* **by** *fastforce*
 from 1 6 7 8 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MThenEmpty*:

$(a \text{ THEN EMPTY}) \simeq (a)$

proof –

have 1: $\vdash \mathcal{M}(a \text{ THEN EMPTY}) = (\mathcal{M} a); (\mathcal{M} \text{ EMPTY})$ **by** (*simp*)
 have 2: $\vdash (\mathcal{M} a); (\mathcal{M} \text{ EMPTY}) = (\mathcal{M} a); \text{empty}$ **by** (*simp add: MEEmptyAlt RightChopEqvChop*)
 have 3: $\vdash (\mathcal{M} a); \text{empty} = (\mathcal{M} a)$ **using** *ChopEmpty* **by** *auto*
 from 1 2 3 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MEEmptyThen*:

$(\text{EMPTY THEN } a) \simeq a$

proof –

have 1: $\vdash \mathcal{M}(\text{EMPTY THEN } a) = (\mathcal{M} \text{ EMPTY}); (\mathcal{M} a)$ **by** (*simp*)
 have 2: $\vdash (\mathcal{M} \text{ EMPTY}); (\mathcal{M} a) = \text{empty}; (\mathcal{M} a)$ **by** (*simp add: MEEmptyAlt LeftChopEqvChop*)
 have 3: $\vdash \text{empty}; (\mathcal{M} a) = (\mathcal{M} a)$ **by** (*simp add: EmptyChop*)
 from 1 2 3 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MEEmptyIterate*:

$(\text{EMPTY ITERATE } b) \simeq \text{EMPTY}$

proof –

have 1: $\vdash \mathcal{M}(\text{EMPTY ITERATE } b) = \mathcal{M}(\text{EMPTY WITH LIFT}(\mathcal{M} b)^*)$
 by (*simp add: ITERATE-d-def*)
 have 2: $\vdash \mathcal{M}(\text{EMPTY WITH LIFT}(\mathcal{M} b)^*) = (\mathcal{M} \text{ EMPTY} \wedge (\mathcal{M} b)^*)$
 by (*simp*)
 have 3: $\vdash (\mathcal{M} \text{ EMPTY} \wedge (\mathcal{M} b)^*) = (\text{empty} \wedge (\mathcal{M} b)^*)$
 using *MEEmptyAlt* **by** *auto*
 have 4: $\vdash (\text{empty} \wedge (\mathcal{M} b)^*) = (\text{empty} \wedge (\text{empty} \vee (((\mathcal{M} b) \wedge \text{more}); (\mathcal{M} b)^*)))$
 using *ChopstarEqv* **by** *fastforce*

have 5: $\vdash (\text{empty} \wedge (\text{empty} \vee (((\mathcal{M} \ b) \wedge \text{more}); (\mathcal{M} \ b)^*))) = \text{empty}$
by *auto*
have 6: $\vdash \mathcal{M}(\text{EMPTY ITERATE } b) = \mathcal{M} \ \text{EMPTY}$
using 1 2 3 4 5 *MEmpyAlt* **by** *fastforce*
from 6 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MIterateIdemp*:
 $(a \ \text{ITERATE } a) \simeq (a)$

proof –
have 1: $\vdash \mathcal{M}(a \ \text{ITERATE } a) = \mathcal{M}(a \ \text{WITH LIFT}(\mathcal{M} \ a)^*)$ **by** (*simp add: ITERATE-d-def*)
have 2: $\vdash \mathcal{M}(a \ \text{WITH LIFT}(\mathcal{M} \ a)^*) = ((\mathcal{M} \ a) \wedge (\mathcal{M} \ a)^*)$ **by** (*simp*)
have 3: $\vdash ((\mathcal{M} \ a) \wedge (\mathcal{M} \ a)^*) = (\triangleright(\mathcal{M} \ a) \wedge (\triangleright(\mathcal{M} \ a))^*)$ **using** *MFixFst*
by (*metis ImpCS inteq-reflection Prop10*)
have 4: $\vdash (\triangleright(\mathcal{M} \ a) \wedge (\triangleright(\mathcal{M} \ a))^*) = \triangleright(\mathcal{M} \ a)$ **using** *FstAndFstStarEqvFst* **by** *fastforce*
have 5: $\vdash \triangleright(\mathcal{M} \ a) = \mathcal{M} \ a$ **using** *MFixFst* **by** *fastforce*
from 1 2 3 4 5 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MUptoIdemp*:
 $(a \ \text{UPTO } a) \simeq (a)$

proof –
have 1: $\vdash \mathcal{M}(a \ \text{UPTO } a) = \triangleright((\mathcal{M} \ a) \vee (\mathcal{M} \ a))$ **by** *auto*
have 2: $\vdash \triangleright((\mathcal{M} \ a) \vee (\mathcal{M} \ a)) = \triangleright(\mathcal{M} \ a)$ **using** *FstEqvRule* **by** *fastforce*
have 3: $\vdash \triangleright(\mathcal{M} \ a) = (\mathcal{M} \ a)$ **using** *MFixFst* **by** *fastforce*
from 1 2 3 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MThruIdemp*:
 $(a \ \text{THRU } a) \simeq (a)$

proof –
have 1: $\vdash \mathcal{M}(a \ \text{THRU } a) = \triangleright(\text{di}(\mathcal{M} \ a) \wedge \text{di}(\mathcal{M} \ a))$ **by** *auto*
have 2: $\vdash \triangleright(\text{di}(\mathcal{M} \ a) \wedge \text{di}(\mathcal{M} \ a)) = \triangleright(\text{di}(\mathcal{M} \ a))$ **using** *FstEqvRule* **by** *fastforce*
have 3: $\vdash \triangleright(\text{di}(\mathcal{M} \ a)) = \triangleright(\mathcal{M} \ a)$ **using** *FstDiEqvFst* **by** *blast*
have 4: $\vdash \triangleright(\mathcal{M} \ a) = (\mathcal{M} \ a)$ **using** *MFixFst* **by** *fastforce*
from 1 2 3 4 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MAndIdemp*:
 $(a \ \text{AND } a) \simeq (a)$

proof –
have 1: $\vdash \mathcal{M}(a \ \text{AND } a) = ((\mathcal{M} \ a) \wedge (\mathcal{M} \ a))$ **by** (*simp add: AND-d-def*)
have 2: $\vdash ((\mathcal{M} \ a) \wedge (\mathcal{M} \ a)) = (\mathcal{M} \ a)$ **by** *fastforce*
from 1 2 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MWithIdemp*:
 $((a \ \text{WITH } f) \ \text{WITH } f) \simeq (a \ \text{WITH } f)$

proof –
have 1: $\vdash \mathcal{M}((a \ \text{WITH } f) \ \text{WITH } f) = (((\mathcal{M} \ a) \wedge (f)) \wedge (f))$ **by** *auto*

have 2: $\vdash (((\mathcal{M} a) \wedge (f)) \wedge (f)) = ((\mathcal{M} a) \wedge (f))$ **by** *fastforce*
have 3: $\vdash ((\mathcal{M} a) \wedge (f)) = \mathcal{M}(a \text{ WITH } f)$ **by** *auto*
from 1 2 3 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MUptoCommut*:

$(a \text{ UPTO } b) \simeq (b \text{ UPTO } a)$

proof –

have 1: $\vdash \mathcal{M}(a \text{ UPTO } b) = \triangleright((\mathcal{M} a) \vee (\mathcal{M} b))$ **by** (*simp*)
have 2: $\vdash ((\mathcal{M} a) \vee (\mathcal{M} b)) = ((\mathcal{M} b) \vee (\mathcal{M} a))$ **by** *auto*
hence 3: $\vdash \triangleright((\mathcal{M} a) \vee (\mathcal{M} b)) = \triangleright((\mathcal{M} b) \vee (\mathcal{M} a))$ **using** *FstEqvRule* **by** *blast*
have 4: $\vdash \triangleright((\mathcal{M} b) \vee (\mathcal{M} a)) = \mathcal{M}(b \text{ UPTO } a)$ **by** *auto*
from 1 3 4 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MThruCommut*:

$(a \text{ THRU } b) \simeq (b \text{ THRU } a)$

proof –

have 1: $\vdash \mathcal{M}(a \text{ THRU } b) = \triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b))$ **by** (*simp*)
have 2: $\vdash (di(\mathcal{M} a) \wedge di(\mathcal{M} b)) = (di(\mathcal{M} b) \wedge di(\mathcal{M} a))$ **by** *auto*
hence 3: $\vdash \triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b)) = \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} a))$ **using** *FstEqvRule* **by** *blast*
have 4: $\vdash \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} a)) = \mathcal{M}(b \text{ THRU } a)$ **by** *auto*
from 1 3 4 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MAndCommut*:

$(a \text{ AND } b) \simeq (b \text{ AND } a)$

proof –

have 1: $\vdash \mathcal{M}(a \text{ AND } b) = ((\mathcal{M} a) \wedge (\mathcal{M} b))$ **by** (*simp add: AND-d-def*)
have 2: $\vdash ((\mathcal{M} a) \wedge (\mathcal{M} b)) = ((\mathcal{M} b) \wedge (\mathcal{M} a))$ **by** *auto*
have 3: $\vdash ((\mathcal{M} b) \wedge (\mathcal{M} a)) = \mathcal{M}(b \text{ AND } a)$ **by** (*simp add: AND-d-def*)
from 1 2 3 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MWithCommut*:

$((a \text{ WITH } f) \text{ WITH } g) \simeq ((a \text{ WITH } g) \text{ WITH } f)$

proof –

have 1: $\vdash \mathcal{M}((a \text{ WITH } f) \text{ WITH } g) = (((\mathcal{M} a) \wedge (f)) \wedge (g))$ **by** *auto*
have 2: $\vdash (((\mathcal{M} a) \wedge (f)) \wedge (g)) = (((\mathcal{M} a) \wedge (g)) \wedge (f))$ **by** *auto*
have 3: $\vdash (((\mathcal{M} a) \wedge (g)) \wedge (f)) = \mathcal{M}((a \text{ WITH } g) \text{ WITH } f)$ **by** *auto*
from 1 2 3 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MWithAbsorp*:

$((a \text{ WITH } f) \text{ WITH } g) \simeq (a \text{ WITH } LIFT(f \wedge g))$

proof –

have 1: $\vdash \mathcal{M}((a \text{ WITH } f) \text{ WITH } g) = (((\mathcal{M} a) \wedge (f)) \wedge (g))$ **by** *auto*
have 2: $\vdash (((\mathcal{M} a) \wedge (f)) \wedge (g)) = ((\mathcal{M} a) \wedge (f \wedge g))$ **by** *auto*
from 1 2 **show** *?thesis* **by** (*simp add: MonEq*)
qed

lemma *MUptoAssoc*:

$$((a \text{ UPTO } b) \text{ UPTO } c) \simeq (a \text{ UPTO } (b \text{ UPTO } c))$$

proof –

$$\text{have } 1: \vdash \mathcal{M}((a \text{ UPTO } b) \text{ UPTO } c) = \triangleright(\mathcal{M}(a \text{ UPTO } b) \vee (\mathcal{M} c))$$

by *(simp)*

$$\text{have } 2: \vdash \triangleright(\mathcal{M}(a \text{ UPTO } b) \vee (\mathcal{M} c)) = \triangleright(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b)) \vee (\mathcal{M} c))$$

by *auto*

$$\text{have } 3: \vdash \triangleright(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b)) \vee (\mathcal{M} c)) = \triangleright(((\mathcal{M} a) \vee (\mathcal{M} b)) \vee (\mathcal{M} c))$$

using *FstFstOrEqvFstOrL* **by** *blast*

$$\text{have } 4: \vdash (((\mathcal{M} a) \vee (\mathcal{M} b)) \vee (\mathcal{M} c)) = ((\mathcal{M} a) \vee ((\mathcal{M} b) \vee (\mathcal{M} c)))$$

by *auto*

$$\text{hence } 5: \vdash \triangleright(((\mathcal{M} a) \vee (\mathcal{M} b)) \vee (\mathcal{M} c)) = \triangleright((\mathcal{M} a) \vee ((\mathcal{M} b) \vee (\mathcal{M} c)))$$

using *FstEqvRule* **by** *blast*

$$\text{have } 6: \vdash \triangleright((\mathcal{M} a) \vee ((\mathcal{M} b) \vee (\mathcal{M} c))) = \triangleright((\mathcal{M} a) \vee \triangleright((\mathcal{M} b) \vee (\mathcal{M} c)))$$

using *FstFstOrEqvFstOrR* **by** *fastforce*

$$\text{have } 7: \vdash \triangleright((\mathcal{M} a) \vee \triangleright((\mathcal{M} b) \vee (\mathcal{M} c))) = \triangleright((\mathcal{M} a) \vee \mathcal{M}(b \text{ UPTO } c))$$

by *auto*

$$\text{have } 8: \vdash \triangleright((\mathcal{M} a) \vee \mathcal{M}(b \text{ UPTO } c)) = \mathcal{M}(a \text{ UPTO } (b \text{ UPTO } c))$$

by *auto*

from 1 2 3 5 6 7 8 **show** *?thesis* **using** *MonEq* **by** *(metis int-eq)*

qed

lemma *MThruAssoc*:

$$((a \text{ THRU } b) \text{ THRU } c) \simeq (a \text{ THRU } (b \text{ THRU } c))$$

proof –

$$\text{have } 1: \vdash \mathcal{M}((a \text{ THRU } b) \text{ THRU } c) = \triangleright(di(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))) \wedge di(\mathcal{M} c)$$

by *auto*

$$\text{have } 2: \vdash di(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) = di((di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$$

using *DiEqvDiFst* **by** *fastforce*

$$\text{have } 3: \vdash di((di(\mathcal{M} a) \wedge di(\mathcal{M} b))) = (di(\mathcal{M} a) \wedge di(\mathcal{M} b))$$

using *DiDiAndEqvDi* **by** *blast*

$$\text{have } 4: \vdash di(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) = (di(\mathcal{M} a) \wedge di(\mathcal{M} b))$$

using 2 3 **by** *fastforce*

$$\text{hence } 5: \vdash (di(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) \wedge di(\mathcal{M} c)) = (di(\mathcal{M} a) \wedge di(\mathcal{M} b) \wedge di(\mathcal{M} c))$$

by *auto*

$$\text{have } 6: \vdash (di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = di(di(\mathcal{M} b) \wedge di(\mathcal{M} c))$$

using *DiDiAndEqvDi* **by** *fastforce*

$$\text{have } 7: \vdash di(di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = di(\triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$$

using *DiEqvDiFst* **by** *blast*

$$\text{have } 8: \vdash (di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = di(\triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$$

using 6 7 **by** *fastforce*

$$\text{hence } 9: \vdash (di(\mathcal{M} a) \wedge di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = (di(\mathcal{M} a) \wedge di(\triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c))))$$

by *auto*

$$\text{have } 10: \vdash (di(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) \wedge di(\mathcal{M} c)) =$$

$$(di(\mathcal{M} a) \wedge di(\triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c))))$$

using 5 9 **by** *fastforce*

$$\text{hence } 11: \vdash \triangleright(di(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) \wedge di(\mathcal{M} c)) =$$

$$\triangleright(di(\mathcal{M} a) \wedge di(\triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c))))$$

using *FstEqvRule* **by** *fastforce*

have 12: $\vdash \triangleright (di(\mathcal{M} a) \wedge di(\triangleright (di(\mathcal{M} b) \wedge di(\mathcal{M} c)))) = \mathcal{M}(a \text{ THRU } (b \text{ THRU } c))$
by *auto*
from 1 11 12 show *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MAndAssoc:*

$((a \text{ AND } b) \text{ AND } c) \simeq (a \text{ AND } (b \text{ AND } c))$

proof –

have 1: $\vdash \mathcal{M}((a \text{ AND } b) \text{ AND } c) = ((\mathcal{M} a) \wedge (\mathcal{M} b) \wedge (\mathcal{M} c))$
using *AND-d-def* **by** (*metis MON.simps(5) MWithAbsorp eq-d-def*)
have 2: $\vdash ((\mathcal{M} a) \wedge (\mathcal{M} b) \wedge (\mathcal{M} c)) = \mathcal{M}(a \text{ AND } (b \text{ AND } c))$
using *AND-d-def* **by** (*simp add: AND-d-def*)
from 1 2 show *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MThenAssoc:*

$((a \text{ THEN } b) \text{ THEN } c) \simeq (a \text{ THEN } (b \text{ THEN } c))$

proof –

have 1: $\vdash \mathcal{M}((a \text{ THEN } b) \text{ THEN } c) = ((\mathcal{M} a);(\mathcal{M} b));(\mathcal{M} c)$ **by** *auto*
have 2: $\vdash ((\mathcal{M} a);(\mathcal{M} b));(\mathcal{M} c) = (\mathcal{M} a);((\mathcal{M} b);(\mathcal{M} c))$ **using** *ChopAssocB* **by** *blast*
have 3: $\vdash (\mathcal{M} a);((\mathcal{M} b);(\mathcal{M} c)) = \mathcal{M}(a \text{ THEN } (b \text{ THEN } c))$ **by** *auto*
from 1 2 3 show *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MUptoThruAbsorp:*

$(a \text{ UPTO } (a \text{ THRU } b)) \simeq a$

proof –

have 1: $\vdash \mathcal{M}(a \text{ UPTO } (a \text{ THRU } b)) = \triangleright((\mathcal{M} a) \vee \triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
by *simp*
have 2: $\vdash \triangleright((\mathcal{M} a) \vee \triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) =$
 $\triangleright((\mathcal{M} a) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
using *FstFstOrEqvFstOrR* **by** *auto*
have 3: $\vdash ((\mathcal{M} a) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b))) =$
 $((\mathcal{M} a) \vee di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b))$
by *auto*
have 4: $\vdash (((\mathcal{M} a) \vee di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b))) =$
 $((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))$
using *OrDiEqvDi* **by** *fastforce*
have 5: $\vdash ((\mathcal{M} a) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b))) =$
 $((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))$
using *3 4* **by** *auto*
hence 6: $\vdash \triangleright((\mathcal{M} a) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b))) =$
 $\triangleright((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))$
using *FstEqvRule* **by** *blast*
have 7: $\vdash \triangleright((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b))) =$
 $((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs \neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b))))$
by (*auto simp add: first-d-def*)
have 8: $\vdash ((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b))) =$
 $((di(\mathcal{M} a) \wedge (\mathcal{M} a)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$

by *auto*
 hence 9: $\vdash (\neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))) =$
 $(\neg((di(\mathcal{M} a) \wedge (\mathcal{M} a)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b))))$
 by *fastforce*
 have 10: $\vdash (\neg((di(\mathcal{M} a) \wedge (\mathcal{M} a)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b)))) =$
 $(\neg(((\mathcal{M} a)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b))))$
 using *AndDiEqv* using 5 by *auto*
 have 11: $\vdash (\neg(((\mathcal{M} a)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} b)))) =$
 $(\neg(\mathcal{M} a) \wedge \neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
 by *auto*
 have 12: $\vdash (\neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))) =$
 $(\neg(\mathcal{M} a) \wedge \neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
 using 9 10 11 by *auto*
 hence 13: $\vdash bs(\neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))) =$
 $bs(\neg(\mathcal{M} a) \wedge \neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
 using *BsEqvRule* by *blast*
 have 14: $\vdash bs(\neg(\mathcal{M} a) \wedge \neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) =$
 $(bs(\neg(\mathcal{M} a))) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
 using *BsAndEqv* by *fastforce*
 have 141: $\vdash bs(\neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))) =$
 $(bs(\neg(\mathcal{M} a))) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
 using 13 14 by *fastforce*
 hence 15: $\vdash ((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))) =$
 $((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(\mathcal{M} a))) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
 by *auto*
 have 16: $\vdash ((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(\mathcal{M} a))) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) =$
 $((bs(\neg(\mathcal{M} a))) \wedge di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))$
 by *auto*
 have 17: $\vdash ((bs(\neg(\mathcal{M} a))) \wedge di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) =$
 $((\triangleright(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))))$
 using *FstEqvBsNotAndDi* by *fastforce*
 have 18: $\vdash ((\triangleright(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))) =$
 $((\mathcal{M} a) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))))$
 using *MFixFst* by *fastforce*
 have 19: $\vdash (((\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))) =$
 $((\mathcal{M} a) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))))$
 by *auto*
 have 20: $\vdash (\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) = (\neg(di(\mathcal{M} a)) \vee \neg(di(\mathcal{M} b)))$
 by *auto*
 have 21: $\vdash (\neg(di(\mathcal{M} a)) \vee \neg(di(\mathcal{M} b))) = ((bi(\neg(\mathcal{M} a))) \vee (bi(\neg(\mathcal{M} b))))$
 by (*simp add: bi-d-def*)

have 22: $\vdash (\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) = ((bi(\neg(\mathcal{M} a))) \vee (bi(\neg(\mathcal{M} b))))$
using 20 21 by auto
hence 23: $\vdash bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) = bs((bi(\neg(\mathcal{M} a))) \vee (bi(\neg(\mathcal{M} b))))$
using BSEquivRule by blast
have 24: $\vdash bs((bi(\neg(\mathcal{M} a))) \vee (bi(\neg(\mathcal{M} b)))) = bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b))$
using BsOrBsEqvBsBiOrBi by fastforce
have 25: $\vdash bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))) = (bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b)))$
using 23 24 using BsOrBsEqvBsBiOrBi by fastforce
hence 26: $\vdash ((\mathcal{M} a) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))) =$
 $((\mathcal{M} a) \wedge (bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b))))$
by auto
have 27: $\vdash ((\mathcal{M} a) \wedge (bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b)))) =$
 $(\triangleright(\mathcal{M} a) \wedge (bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b))))$
using MFixFst by fastforce
have 28: $\vdash (\triangleright(\mathcal{M} a) \wedge (bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b)))) =$
 $((\mathcal{M} a) \wedge bs(\neg(\mathcal{M} a)) \wedge (bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b))))$
by (auto simp add: first-d-def)
have 29: $\vdash ((\mathcal{M} a) \wedge bs(\neg(\mathcal{M} a)) \wedge (bs(\neg(\mathcal{M} a)) \vee bs(\neg(\mathcal{M} b)))) =$
 $((\mathcal{M} a) \wedge bs(\neg(\mathcal{M} a)))$
by auto
have 30: $\vdash ((\mathcal{M} a) \wedge bs(\neg(\mathcal{M} a))) = \triangleright(\mathcal{M} a)$
by (simp add: first-d-def)
have 31: $\vdash \triangleright(\mathcal{M} a) = (\mathcal{M} a)$
using MFixFst by fastforce
have 32: $\vdash \mathcal{M}(a \text{ UPTO } (a \text{ THRU } b)) =$
 $((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b))))$
using 1 2 6 7 by fastforce
have 33: $\vdash ((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge$
 $bs(\neg((di(\mathcal{M} a)) \wedge ((\mathcal{M} a) \vee di(\mathcal{M} b)))) =$
 $((\mathcal{M} a) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b))))$
using 15 16 17 18 19 by (metis int-eq)
have 34: $\vdash (((\mathcal{M} a)) \wedge bs(\neg(di(\mathcal{M} a) \wedge di(\mathcal{M} b)))) = (\mathcal{M} a)$
using 26 27 28 29 30 31 by (metis int-eq)
from 32 33 34 show ?thesis using MonEq by (metis int-eq)
qed

lemma MThruUptoAbsorp:

$(a \text{ THRU } (a \text{ UPTO } b)) \simeq (a)$

proof –

have 1: $\vdash \mathcal{M}(a \text{ THRU } (a \text{ UPTO } b)) = \triangleright(di(\mathcal{M} a) \wedge di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b))))$
by simp
have 2: $\vdash \triangleright(di(\mathcal{M} a) \wedge di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b)))) =$
 $\triangleright(di(\mathcal{M} a) \wedge di(((\mathcal{M} a) \vee (\mathcal{M} b))))$
by (metis 1 DiEqvDiFst int-eq)
have 3: $\vdash \triangleright(di(\mathcal{M} a) \wedge di(((\mathcal{M} a) \vee (\mathcal{M} b)))) =$
 $\triangleright(di(\mathcal{M} a) \wedge (di(\mathcal{M} a) \vee di(\mathcal{M} b)))$
by (metis DiOrEqv FstEqvRule inteq-reflection lift-and-com)
have 4: $\vdash (di(\mathcal{M} a) \wedge (di(\mathcal{M} a) \vee di(\mathcal{M} b))) = (di(\mathcal{M} a))$
by auto

hence 5: $\vdash \triangleright(di(\mathcal{M} a) \wedge (di(\mathcal{M} a) \vee di(\mathcal{M} b))) = \triangleright(di(\mathcal{M} a))$
 using *FstEqvRule* by *blast*
 have 6: $\vdash \triangleright(di(\mathcal{M} a)) = \triangleright(\mathcal{M} a)$
 using *FstDiEqvFst* by *blast*
 have 7: $\vdash \triangleright(\mathcal{M} a) = (\mathcal{M} a)$
 using *MFixFst* by *fastforce*
 from 1 2 3 5 6 7 show ?thesis using *MonEq* by (*metis int-eq*)
 qed

lemma *MUptoThruDistrib*:

$(a \text{ UPTO } (b \text{ THRU } c)) \simeq ((a \text{ UPTO } b) \text{ THRU } (a \text{ UPTO } c))$

proof –

have 1: $\vdash \mathcal{M}((a \text{ UPTO } b) \text{ THRU } (a \text{ UPTO } c)) =$
 $\quad \triangleright(di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b))) \wedge di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} c))))$
 by *simp*
 have 2: $\vdash (di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b))) \wedge di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} c)))) =$
 $\quad (di(((\mathcal{M} a) \vee (\mathcal{M} b))) \wedge di(((\mathcal{M} a) \vee (\mathcal{M} c))))$
 using *DiEqvDiFst* by *fastforce*
 have 3: $\vdash (di(((\mathcal{M} a) \vee (\mathcal{M} b))) \wedge di(((\mathcal{M} a) \vee (\mathcal{M} c)))) =$
 $\quad ((di(\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge (di(\mathcal{M} a) \vee di(\mathcal{M} c)))$
 using *DiOrEqv* by *fastforce*
 have 4: $\vdash ((di(\mathcal{M} a) \vee di(\mathcal{M} b)) \wedge (di(\mathcal{M} a) \vee di(\mathcal{M} c))) =$
 $\quad (di(\mathcal{M} a) \vee (di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$
 by *auto*
 have 5: $\vdash (di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b))) \wedge di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} c)))) =$
 $\quad (di(\mathcal{M} a) \vee (di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$
 using 2 3 4 by *fastforce*
 hence 6: $\vdash \triangleright(di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} b))) \wedge di(\triangleright((\mathcal{M} a) \vee (\mathcal{M} c)))) =$
 $\quad \triangleright(di(\mathcal{M} a) \vee (di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$
 using *FstEqvRule* by *blast*
 have 7: $\vdash \triangleright(di(\mathcal{M} a) \vee (di(\mathcal{M} b) \wedge di(\mathcal{M} c))) =$
 $\quad \triangleright(\triangleright(di(\mathcal{M} a)) \vee \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$
 using *FstFstOrEqvFstOr* by *fastforce*
 have 8: $\vdash \triangleright(di(\mathcal{M} a)) = \triangleright((\mathcal{M} a))$
 using *FstDiEqvFst* by *blast*
 have 9: $\vdash \triangleright((\mathcal{M} a)) = (\mathcal{M} a)$
 using *MFixFst* by *fastforce*
 have 10: $\vdash \triangleright(di(\mathcal{M} a)) = (\mathcal{M} a)$
 using 8 9 by *fastforce*
 hence 11: $\vdash (\triangleright(di(\mathcal{M} a)) \vee \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c))) =$
 $\quad ((\mathcal{M} a) \vee \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$
 by *auto*
 hence 12: $\vdash \triangleright(\triangleright(di(\mathcal{M} a)) \vee \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c))) =$
 $\quad \triangleright((\mathcal{M} a) \vee \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$
 using *FstEqvRule* by *blast*
 have 13: $\vdash \triangleright((\mathcal{M} a) \vee \triangleright(di(\mathcal{M} b) \wedge di(\mathcal{M} c))) = \mathcal{M}(a \text{ UPTO } (b \text{ THRU } c))$
 by *simp*

from 1 6 7 12 13 show ?thesis using *MonEq* by (*metis int-eq*)
 qed

lemma *MThruUptoDistrib*:

$((a \text{ THRU } (b \text{ UPTO } c)) \simeq ((a \text{ THRU } b) \text{ UPTO } (a \text{ THRU } c)))$

proof –

have 1: $\vdash \mathcal{M}((a \text{ THRU } b) \text{ UPTO } (a \text{ THRU } c)) =$
 $\triangleright(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b)) \vee \triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} c)))$
by *simp*
have 2: $\vdash \triangleright(\triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} b)) \vee \triangleright(di(\mathcal{M} a) \wedge di(\mathcal{M} c))) =$
 $\triangleright((di(\mathcal{M} a) \wedge di(\mathcal{M} b)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} c)))$
using *FstFstOrEqvFstOr* **by** *auto*
have 3: $\vdash ((di(\mathcal{M} a) \wedge di(\mathcal{M} b)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} c))) =$
 $(di(\mathcal{M} a) \wedge (di(\mathcal{M} b) \vee di(\mathcal{M} c)))$ **by** *auto*
have 4: $\vdash (di(\mathcal{M} a) \wedge (di(\mathcal{M} b) \vee di(\mathcal{M} c))) =$
 $(di(\mathcal{M} a) \wedge di((\mathcal{M} b) \vee (\mathcal{M} c)))$ **using** *DiOrEqv* **by** *fastforce*
have 5: $\vdash (di(\mathcal{M} a) \wedge di((\mathcal{M} b) \vee (\mathcal{M} c))) =$
 $(di(\mathcal{M} a) \wedge di(\triangleright((\mathcal{M} b) \vee (\mathcal{M} c))))$ **using** *DiEqvDiFst* **by** *fastforce*
have 6: $\vdash ((di(\mathcal{M} a) \wedge di(\mathcal{M} b)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} c))) =$
 $(di(\mathcal{M} a) \wedge di(\triangleright((\mathcal{M} b) \vee (\mathcal{M} c))))$ **using** 3 4 5 **by** *fastforce*
hence 7: $\vdash \triangleright((di(\mathcal{M} a) \wedge di(\mathcal{M} b)) \vee (di(\mathcal{M} a) \wedge di(\mathcal{M} c))) =$
 $\triangleright(di(\mathcal{M} a) \wedge di(\triangleright((\mathcal{M} b) \vee (\mathcal{M} c))))$ **using** *FstEqvRule* **by** *blast*
have 8: $\vdash \triangleright(di(\mathcal{M} a) \wedge di(\triangleright((\mathcal{M} b) \vee (\mathcal{M} c)))) =$
 $\mathcal{M}(a \text{ THRU } (b \text{ UPTO } c))$ **by** *simp*
from 1 2 7 8 **show** *?thesis* **using** *MonEq* **by** (*metis int-eq*)
qed

lemma *MThruUptoRDistrib*:

$((a \text{ THRU } b) \text{ UPTO } c) \simeq ((a \text{ UPTO } c) \text{ THRU } (b \text{ UPTO } c))$

proof –

have 1: $((a \text{ THRU } b) \text{ UPTO } c) \simeq (c \text{ UPTO } (a \text{ THRU } b))$
using *MUptoCommut* **by** *auto*
have 2: $(c \text{ UPTO } (a \text{ THRU } b)) \simeq ((c \text{ UPTO } a) \text{ THRU } (c \text{ UPTO } b))$
using *MUptoThruDistrib* **by** *auto*
have 3: $(c \text{ UPTO } a) \simeq (a \text{ UPTO } c)$
using *MUptoCommut* **by** *auto*
have 4: $(c \text{ UPTO } b) \simeq (b \text{ UPTO } c)$
using *MUptoCommut* **by** *auto*
have 5: $((c \text{ UPTO } a) \text{ THRU } (c \text{ UPTO } b)) \simeq ((a \text{ UPTO } c) \text{ THRU } (c \text{ UPTO } b))$
using 3 **by** (*simp add: MonEqRefl MonEqSubstThru*)
have 6: $((a \text{ UPTO } c) \text{ THRU } (c \text{ UPTO } b)) \simeq ((c \text{ UPTO } b) \text{ THRU } (a \text{ UPTO } c))$
using *MThruCommut* **by** *auto*
have 7: $((c \text{ UPTO } b) \text{ THRU } (a \text{ UPTO } c)) \simeq ((b \text{ UPTO } c) \text{ THRU } (a \text{ UPTO } c))$
using 4 **by** (*simp add: MonEqRefl MonEqSubstThru*)
from 1 2 5 6 7 **show** *?thesis* **using** *MThruCommut MonEq* **by** (*metis int-eq*)
qed

lemma *MUptoThruRDistrib*:

$((a \text{ UPTO } b) \text{ THRU } c) \simeq ((a \text{ THRU } c) \text{ UPTO } (b \text{ THRU } c))$

proof –

have 1: $((a \text{ UPTO } b) \text{ THRU } c) \simeq (c \text{ THRU } (a \text{ UPTO } b))$
using *MThruCommut* **by** *auto*
have 2: $(c \text{ THRU } (a \text{ UPTO } b)) \simeq ((c \text{ THRU } a) \text{ UPTO } (c \text{ THRU } b))$

using *MThruUptoDistrib* by *auto*
 have 3: $(c \text{ THRU } a) \simeq (a \text{ THRU } c)$
 using *MThruCommut* by *auto*
 have 4: $(c \text{ THRU } b) \simeq (b \text{ THRU } c)$
 using *MThruCommut* by *auto*
 have 5: $((c \text{ THRU } a) \text{ UPTO } (c \text{ THRU } b)) \simeq ((a \text{ THRU } c) \text{ UPTO } (c \text{ THRU } b))$
 using 3 by (*simp add: MonEqRefl MonEqSubstUpto*)
 have 6: $((a \text{ THRU } c) \text{ UPTO } (c \text{ THRU } b)) \simeq ((c \text{ THRU } b) \text{ UPTO } (a \text{ THRU } c))$
 using *MUptoCommut* by *auto*
 have 7: $((c \text{ THRU } b) \text{ UPTO } (a \text{ THRU } c)) \simeq ((b \text{ THRU } c) \text{ UPTO } (a \text{ THRU } c))$
 using 4 by (*simp add: MonEqRefl MonEqSubstUpto*)
 from 1 2 5 6 7 show ?thesis using *MUptoCommut MonEq* by (*metis int-eq*)
 qed

lemma *MWithAndDistrib*:

$((a \text{ AND } b) \text{ WITH } f) \simeq ((a \text{ WITH } f) \text{ AND } (b \text{ WITH } f))$

proof –

have 1: $\vdash \mathcal{M}((a \text{ AND } b) \text{ WITH } f) = (\mathcal{M}(a \text{ AND } b) \wedge f)$
 by (*simp*)
 have 2: $\vdash \mathcal{M}(a \text{ AND } b) = \mathcal{M}(a \text{ WITH LIFT}(\mathcal{M} \ b))$
 by (*simp add: AND-d-def*)
 have 3: $\vdash (\mathcal{M}(a \text{ AND } b) \wedge f) = (\mathcal{M}(a \text{ WITH LIFT}(\mathcal{M} \ b)) \wedge f)$
 using 2 by *auto*
 have 4: $\vdash \mathcal{M}(a \text{ WITH } (\text{LIFT}(\mathcal{M} \ b) \wedge f)) = (\mathcal{M}(a) \wedge \mathcal{M}(b) \wedge f)$
 by *simp*
 have 5: $\vdash (\mathcal{M}(a) \wedge \mathcal{M}(b) \wedge f) = ((\mathcal{M}(a) \wedge f) \wedge (\mathcal{M}(b) \wedge f))$
 by *auto*
 have 6: $\vdash ((\mathcal{M}(a) \wedge f) \wedge (\mathcal{M}(b) \wedge f)) = (\mathcal{M}(a \text{ WITH } f) \wedge \mathcal{M}(b \text{ WITH } f))$
 by *simp*
 have 7: $\vdash (\mathcal{M}(a \text{ WITH } f) \wedge \mathcal{M}(b \text{ WITH } f)) = \mathcal{M}((a \text{ WITH } f) \text{ WITH LIFT}(\mathcal{M}(b \text{ WITH } f)))$
 by *simp*
 have 8: $\vdash \mathcal{M}((a \text{ WITH } f) \text{ WITH LIFT}(\mathcal{M}(b \text{ WITH } f))) = \mathcal{M}((a \text{ WITH } f) \text{ AND } (b \text{ WITH } f))$
 by (*simp add: AND-d-def*)
 from 1 2 3 4 5 6 7 8 show ?thesis using *MonEq* by (*metis AND-d-def MWithAbsorp int-eq*)
 qed

lemma *MHaltWithAndDistrib*:

$((\text{HALT } w) \text{ WITH } f) \text{ AND } ((\text{HALT } w) \text{ WITH } g) \simeq ((\text{HALT } w) \text{ WITH LIFT}(f \wedge g))$

proof –

have 1: $\vdash \mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ AND } ((\text{HALT } w) \text{ WITH } g)) =$
 $\mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ WITH LIFT}(\mathcal{M}((\text{HALT } w) \text{ WITH } g))))$
 by (*simp add: AND-d-def*)
 have 2: $\vdash \mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ WITH LIFT}(\mathcal{M}((\text{HALT } w) \text{ WITH } g)))) =$
 $(\mathcal{M}(\text{HALT } w) \wedge f \wedge \mathcal{M}(\text{HALT } w) \wedge g)$
 by *auto*
 have 3: $\vdash (\mathcal{M}(\text{HALT } w) \wedge f \wedge \mathcal{M}(\text{HALT } w) \wedge g) = (\mathcal{M}(\text{HALT } w) \wedge f \wedge g)$
 by *auto*
 have 4: $\vdash (\mathcal{M}(\text{HALT } w) \wedge f \wedge g) = \mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \wedge g))$
 by *auto*
 from 1 2 3 4 show ?thesis using *MonEq* by (*metis int-eq*)

qed

lemma *MHaltWithUptoHaltWithEqvHaltWithOr:*

$$(((\text{HALT } w) \text{ WITH } f) \text{ UPTO } ((\text{HALT } w) \text{ WITH } g)) \simeq ((\text{HALT } w) \text{ WITH LIFT}(f \vee g))$$

proof –

have 1: $\vdash \mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ UPTO } ((\text{HALT } w) \text{ WITH } g)) =$
 $\triangleright(\mathcal{M}((\text{HALT } w) \text{ WITH } f) \vee \mathcal{M}((\text{HALT } w) \text{ WITH } g))$
by (*simp*)
have 2: $\vdash \triangleright(\mathcal{M}((\text{HALT } w) \text{ WITH } f) \vee \mathcal{M}((\text{HALT } w) \text{ WITH } g)) =$
 $\triangleright((\mathcal{M}(\text{HALT } w) \wedge f) \vee (\mathcal{M}(\text{HALT } w) \wedge g))$
by *auto*
have 3: $\vdash ((\mathcal{M}(\text{HALT } w) \wedge f) \vee (\mathcal{M}(\text{HALT } w) \wedge g)) = (\mathcal{M}(\text{HALT } w) \wedge (f \vee g))$
by *auto*
have 4: $\vdash \triangleright((\mathcal{M}(\text{HALT } w) \wedge f) \vee (\mathcal{M}(\text{HALT } w) \wedge g)) = \triangleright(\mathcal{M}(\text{HALT } w) \wedge (f \vee g))$
using 3 *FstEqvRule* **by** *blast*
have 5: $\vdash \triangleright(\mathcal{M}(\text{HALT } w) \wedge (f \vee g)) = \triangleright(\mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \vee g)))$
by *simp*
have 6: $\vdash \mathcal{M}(((\text{HALT } w) \text{ WITH LIFT}(f \vee g))) = \triangleright(\mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \vee g)))$
using *MFixFst* **by** *blast*
from 1 2 3 4 5 6 **show** ?thesis **using** *MonEq* **by** (*metis int-eq*)

qed

lemma *MHaltWithThruHaltWithEqvHaltWithAndHaltWith:*

$$(((\text{HALT } w) \text{ WITH } f) \text{ THRU } ((\text{HALT } w) \text{ WITH } g)) \simeq (((\text{HALT } w) \text{ WITH } f) \text{ AND } ((\text{HALT } w) \text{ WITH } g))$$

proof –

have 1: $\vdash \mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ THRU } ((\text{HALT } w) \text{ WITH } g)) =$
 $\triangleright(\text{di}(\mathcal{M}(\text{HALT } w) \wedge f) \wedge \text{di}(\mathcal{M}(\text{HALT } w) \wedge g))$
by *simp*
have 2: $\vdash (\text{di}(\mathcal{M}(\text{HALT } w) \wedge f) \wedge \text{di}(\mathcal{M}(\text{HALT } w) \wedge g)) =$
 $(\text{di}(\text{halt}(\text{init } w) \wedge f) \wedge \text{di}(\text{halt}(\text{init } w) \wedge g))$
using *MHaltAlt DiEqvDi*
by (*metis (no-types, lifting) inteq-reflection lift-and-com*)
have 3: $\vdash (\text{di}(\text{halt}(\text{init } w) \wedge f) \wedge \text{di}(\text{halt}(\text{init } w) \wedge g)) =$
 $\text{di}(\text{halt}(\text{init } w) \wedge f \wedge g)$
using *DiHaltAndDiHaltAndEqvDiHaltAndAnd* **by** *fastforce*
have 4: $\vdash \text{di}(\text{halt}(\text{init } w) \wedge f \wedge g) = \text{di}(\mathcal{M}(\text{HALT } w) \wedge f \wedge g)$
by (*metis DiEqvDi MHaltAlt inteq-reflection lift-and-com*)
have 5: $\vdash (\text{di}(\mathcal{M}(\text{HALT } w) \wedge f) \wedge \text{di}(\mathcal{M}(\text{HALT } w) \wedge g)) = \text{di}(\mathcal{M}(\text{HALT } w) \wedge f \wedge g)$
using 2 3 4 **by** *fastforce*
have 6: $\vdash \triangleright(\text{di}(\mathcal{M}(\text{HALT } w) \wedge f) \wedge \text{di}(\mathcal{M}(\text{HALT } w) \wedge g)) = \triangleright(\text{di}(\mathcal{M}(\text{HALT } w) \wedge f \wedge g))$
using 5 *FstEqvRule* **by** *blast*
have 7: $\vdash \triangleright(\text{di}(\mathcal{M}(\text{HALT } w) \wedge f \wedge g)) = \triangleright(\mathcal{M}(\text{HALT } w) \wedge f \wedge g)$
using *FstDiEqvFst* **by** *fastforce*
have 8: $\vdash \triangleright(\mathcal{M}(\text{HALT } w) \wedge f \wedge g) = \triangleright(\mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \wedge g)))$
by *simp*
have 9: $\vdash \mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \wedge g)) = \triangleright(\mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \wedge g)))$
using *MFixFst* **by** *blast*
have 10: $\vdash \mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ THRU } ((\text{HALT } w) \text{ WITH } g)) = \mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \wedge g))$

using 1 2 3 4 5 6 7 8 9 int-eq by metis
 have 11: $\vdash \mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ AND } ((\text{HALT } w) \text{ WITH } g)) = \mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \wedge g))$
 using MHaltWithAndDistrib using eq-d-def by blast
 have 12: $\vdash \mathcal{M}((\text{HALT } w) \text{ WITH LIFT}(f \wedge g)) = \mathcal{M}(((\text{HALT } w) \text{ WITH } f) \text{ AND } ((\text{HALT } w) \text{ WITH } g))$
 using 11 by fastforce
 from 10 12 show ?thesis using MonEq by (metis int-eq)
 qed

lemma *MThenAndDistrib*:

$(a \text{ THEN } (b \text{ AND } c)) \simeq ((a \text{ THEN } b) \text{ AND } (a \text{ THEN } c))$

proof –

have 1: $\vdash \mathcal{M}(a \text{ THEN } (b \text{ AND } c)) = (\mathcal{M}(a)) ; (\mathcal{M}(b \text{ AND } c))$
 by simp
 have 2: $\vdash (\mathcal{M}(a)) ; (\mathcal{M}(b \text{ AND } c)) = (\mathcal{M}(a)) ; (\mathcal{M}(b) \wedge \mathcal{M}(c))$
 by (simp add: AND-d-def)
 have 3: $\vdash (\mathcal{M}(a)) ; (\mathcal{M}(b) \wedge \mathcal{M}(c)) = \triangleright (\mathcal{M}(a)) ; (\mathcal{M}(b) \wedge \mathcal{M}(c))$
 using MFixFst LeftChopEqvChop by blast
 have 4: $\vdash \triangleright (\mathcal{M}(a)) ; (\mathcal{M}(b) \wedge \mathcal{M}(c)) = ((\triangleright (\mathcal{M}(a)) ; (\mathcal{M}(b))) \wedge (\triangleright (\mathcal{M}(a)) ; (\mathcal{M}(c))))$
 using LFstAndDistrC by fastforce
 have 5: $\vdash ((\triangleright (\mathcal{M}(a)) ; (\mathcal{M}(b))) \wedge (\triangleright (\mathcal{M}(a)) ; (\mathcal{M}(c)))) =$
 $((\mathcal{M}(a)) ; (\mathcal{M}(b))) \wedge ((\mathcal{M}(a)) ; (\mathcal{M}(c)))$ using MFixFst
 by (metis 4 inteq-reflection)
 have 6: $\vdash ((\mathcal{M}(a)) ; (\mathcal{M}(b))) \wedge ((\mathcal{M}(a)) ; (\mathcal{M}(c))) =$
 $(\mathcal{M}(a \text{ THEN } b) \wedge \mathcal{M}(a \text{ THEN } c))$
 by simp
 have 7: $\vdash (\mathcal{M}(a \text{ THEN } b) \wedge \mathcal{M}(a \text{ THEN } c)) = \mathcal{M}((a \text{ THEN } b) \text{ AND } (a \text{ THEN } c))$
 by (simp add: AND-d-def)
 from 1 2 3 4 5 6 7 show ?thesis using MonEq by (metis int-eq)
 qed

lemma *MThenUptoDistrib*:

$(a \text{ THEN } (b \text{ UPTO } c)) \simeq ((a \text{ THEN } b) \text{ UPTO } (a \text{ THEN } c))$

proof –

have 1: $\vdash (\mathcal{M}(a \text{ THEN } (b \text{ UPTO } c))) = ((\mathcal{M} a); (\triangleright((\mathcal{M} b) \vee (\mathcal{M} c))))$
 by simp
 have 2: $\vdash ((\mathcal{M} a); (\triangleright((\mathcal{M} b) \vee (\mathcal{M} c)))) = (\triangleright(\mathcal{M} a); (\triangleright((\mathcal{M} b) \vee (\mathcal{M} c))))$
 by (simp add: MFixFst LeftChopEqvChop)
 have 3: $\vdash (\triangleright(\mathcal{M} a); (\triangleright((\mathcal{M} b) \vee (\mathcal{M} c)))) = ((\triangleright(\triangleright(\mathcal{M} a); ((\mathcal{M} b) \vee (\mathcal{M} c))))$
 using FstFstChopEqvFstChopFst by fastforce
 have 4: $\vdash \triangleright(\mathcal{M} a); ((\mathcal{M} b) \vee (\mathcal{M} c)) = (\mathcal{M} a); ((\mathcal{M} b) \vee (\mathcal{M} c))$
 using MFixFst by (metis LeftChopEqvChop inteq-reflection)
 have 5: $\vdash (\mathcal{M} a); ((\mathcal{M} b) \vee (\mathcal{M} c)) = ((\mathcal{M} a); (\mathcal{M} b) \vee (\mathcal{M} a); (\mathcal{M} c))$
 by (simp add: ChopOrEqv)
 have 6: $\vdash ((\mathcal{M} a); (\mathcal{M} b) \vee (\mathcal{M} a); (\mathcal{M} c)) = (\mathcal{M}(a \text{ THEN } b) \vee \mathcal{M}(a \text{ THEN } c))$
 by simp
 have 7: $\vdash \triangleright(\mathcal{M} a); ((\mathcal{M} b) \vee (\mathcal{M} c)) = (\mathcal{M}(a \text{ THEN } b) \vee \mathcal{M}(a \text{ THEN } c))$
 using 6 5 4 by fastforce
 have 8: $\vdash \triangleright(\triangleright(\mathcal{M} a); ((\mathcal{M} b) \vee (\mathcal{M} c))) = \triangleright(\mathcal{M}(a \text{ THEN } b) \vee \mathcal{M}(a \text{ THEN } c))$
 using 7 by (simp add: FstEqvRule)
 have 9: $\vdash \triangleright(\mathcal{M}(a \text{ THEN } b) \vee \mathcal{M}(a \text{ THEN } c)) = \mathcal{M}((a \text{ THEN } b) \text{ UPTO } (a \text{ THEN } c))$

by simp
 from 9 7 1 2 3 show ?thesis by (metis eq-d-def integ-reflection)
 qed

lemma *MTheinthroughDistrib*:
 $(a \text{ THEN } (b \text{ THRU } c)) \simeq ((a \text{ THEN } b) \text{ THRU } (a \text{ THEN } c))$
proof –
 have 1: $\vdash \mathcal{M}(a \text{ THEN } (b \text{ THRU } c)) = (\mathcal{M} a); \triangleright (di(\mathcal{M} b) \wedge di(\mathcal{M} c))$
 by simp
 have 2: $\vdash (\mathcal{M} a); \triangleright (di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = \triangleright (\mathcal{M} a); \triangleright (di(\mathcal{M} b) \wedge di(\mathcal{M} c))$
 by (simp add: MFixFst LeftChopEqvChop)
 have 3: $\vdash \triangleright (\mathcal{M} a); \triangleright (di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = \triangleright (\triangleright (\mathcal{M} a); (di(\mathcal{M} b) \wedge di(\mathcal{M} c)))$
 using FstFstChopEqvFstChopFst by fastforce
 have 4: $\vdash \triangleright (\mathcal{M} a); (di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = (\triangleright (\mathcal{M} a); di(\mathcal{M} b) \wedge \triangleright (\mathcal{M} a); di(\mathcal{M} c))$
 by (meson LFstAndDistrC Prop11)
 have 5: $\vdash (\triangleright (\mathcal{M} a); di(\mathcal{M} b) \wedge \triangleright (\mathcal{M} a); di(\mathcal{M} c)) = ((\mathcal{M} a); di(\mathcal{M} b) \wedge (\mathcal{M} a); di(\mathcal{M} c))$
 using MFixFst by (metis 4 int-eq)
 have 6: $\vdash (\mathcal{M} a); di(\mathcal{M} b) = (\mathcal{M} a); ((\mathcal{M} b); \# \text{True})$
 by (simp add: di-d-def)
 have 7: $\vdash (\mathcal{M} a); ((\mathcal{M} b); \# \text{True}) = ((\mathcal{M} a); (\mathcal{M} b)); \# \text{True}$
 by (simp add: ChopAssoc)
 have 8: $\vdash ((\mathcal{M} a); (\mathcal{M} b)); \# \text{True} = di((\mathcal{M} a); (\mathcal{M} b))$
 by (simp add: di-d-def)
 have 9: $\vdash (\mathcal{M} a); di(\mathcal{M} b) = di((\mathcal{M} a); (\mathcal{M} b))$
 using 8 7 6 by fastforce
 have 10: $\vdash (\mathcal{M} a); di(\mathcal{M} c) = (\mathcal{M} a); ((\mathcal{M} c); \# \text{True})$
 by (simp add: di-d-def)
 have 11: $\vdash (\mathcal{M} a); ((\mathcal{M} c); \# \text{True}) = ((\mathcal{M} a); (\mathcal{M} c)); \# \text{True}$
 by (simp add: ChopAssoc)
 have 12: $\vdash ((\mathcal{M} a); (\mathcal{M} c)); \# \text{True} = di((\mathcal{M} a); (\mathcal{M} c))$
 by (simp add: di-d-def)
 have 13: $\vdash (\mathcal{M} a); di(\mathcal{M} c) = di((\mathcal{M} a); (\mathcal{M} c))$
 using 12 11 10 by fastforce
 have 14: $\vdash ((\mathcal{M} a); di(\mathcal{M} b) \wedge (\mathcal{M} a); di(\mathcal{M} c)) = (di((\mathcal{M} a); (\mathcal{M} b)) \wedge di((\mathcal{M} a); (\mathcal{M} c)))$
 using 13 9 by fastforce
 have 15: $\vdash (di((\mathcal{M} a); (\mathcal{M} b)) \wedge di((\mathcal{M} a); (\mathcal{M} c))) = (di(\mathcal{M}(a \text{ THEN } b)) \wedge di(\mathcal{M}(a \text{ THEN } c)))$
 by simp
 have 16: $\vdash \triangleright (\mathcal{M} a); (di(\mathcal{M} b) \wedge di(\mathcal{M} c)) = (di(\mathcal{M}(a \text{ THEN } b)) \wedge di(\mathcal{M}(a \text{ THEN } c)))$
 using 15 14 4 5 by fastforce
 have 17: $\vdash \triangleright (\triangleright (\mathcal{M} a); (di(\mathcal{M} b) \wedge di(\mathcal{M} c))) = \triangleright (di(\mathcal{M}(a \text{ THEN } b)) \wedge di(\mathcal{M}(a \text{ THEN } c)))$
 using 16 by (simp add: FstEqvRule)
 have 18: $\vdash \triangleright (di(\mathcal{M}(a \text{ THEN } b)) \wedge di(\mathcal{M}(a \text{ THEN } c))) = \mathcal{M}((a \text{ THEN } b) \text{ THRU } (a \text{ THEN } c))$
 by simp
 from 18 16 1 2 3 show ?thesis by (metis eq-d-def int-eq)
 qed

end

11 Finite ITL Examples

```

theory Example
imports
  FOTheorems TimeReversal
begin

```

11.1 Example 1

```

definition F1 :: nat statefun  $\Rightarrow$  temporal
where F1 w  $\equiv$  TEMP  $\square$  ( #0  $\leq$  $w )

```

```

definition Init1 :: nat statefun  $\Rightarrow$  temporal
where Init1 w  $\equiv$  TEMP $w = #0

```

```

lemma init1:
  ( $\langle s0, s1, s2 \rangle \models \text{len}(2) \wedge \text{Init1 } w$ ) = ( $(w \ s0) = 0$ )
by (simp add: Init1-def itl-defs)

```

```

lemma exist-test-F1 :
   $\vdash \exists \exists w. F1 \ w$ 
proof -
  have 1:  $\bigwedge w. \vdash F1 \ w$  by (simp add: itl-defs F1-def Valid-def)
  from 1 show ?thesis by (meson EExI MP)
qed

```

11.2 Example 2

```

locale Test =
  fixes v :: state  $\Rightarrow$  nat
  fixes v1 :: state  $\Rightarrow$  nat
  fixes y :: state  $\Rightarrow$  bool
  fixes z :: state  $\Rightarrow$  int
  fixes F2 :: nat statefun  $\Rightarrow$  temporal
  fixes F3 :: bool statefun  $\Rightarrow$  temporal
  fixes F4 :: int statefun  $\Rightarrow$  temporal
  fixes F5 :: nat statefun  $\Rightarrow$  temporal
  fixes Init2 :: nat statefun  $\Rightarrow$  temporal
  fixes Init3 :: bool statefun  $\Rightarrow$  temporal
  defines F2  $\equiv$  ( $\lambda v. \text{TEMP } \square ( \#0 \leq \$v )$ )
  defines F3  $\equiv$  ( $\lambda p. \text{TEMP } \square ( \$p \vee \neg \$p )$ )
  defines F4  $\equiv$  ( $\lambda z. \text{TEMP } \square ( \#0 \leq \$z \vee \$z < \#0 )$ )
  defines F5  $\equiv$  ( $\lambda v. \text{TEMP } \$v = \#0 \wedge v \text{ gets } \$v + \#1$ )
  defines Init2  $\equiv$  ( $\lambda v. \text{TEMP } \$v = \#0$ )
  defines Init3  $\equiv$  ( $\lambda p. \text{TEMP } \$p$ )

```

```

lemma (in Test) currentval-test :
  ( $s \models (\$v = \#0)$ ) = (  $(v \ (\text{inth } s \ 0)) = 0$  )
by (simp add: itl-def)

```

lemma (in *Test*) *nextempty-test* :

$(\langle s0 \rangle \models v\$) = (\epsilon \ x. x=x)$

by (*simp add: itl-def*)

lemma (in *Test*) *nextempty-test-1* :

$(\langle s0 \rangle \models v\$ = v\$)$

by *simp*

lemma (in *Test*) *nextempty-test-2* :

$(\langle s0 \rangle \models v\$ = v1\$)$

by (*simp add: Test.nextempty-test*)

lemma (in *Test*) *nextcurrent-test*:

$(\langle s0, s1 \rangle \models \text{skip} \wedge (\$v = \#0) \wedge (v\$ = \$v + \#1)) = (((v \ s0) = 0) \wedge ((v \ s1) = 1 \))$

unfolding *itl-defs* **by** *auto*

lemma (in *Test*) *nextcurrentfinpenult-test*:

$(\langle s0, s1, s2, s3 \rangle \models \text{len}(3) \wedge v =: !v - \#1 \wedge v \leftarrow \#3 \wedge \$v = \#0 \wedge v := \$v + \#1) =$
 $((v \ s0) = 0) \wedge ((v \ s1) = 1 \wedge (v \ (s2)) = 2 \wedge ((v \ s3) = 3 \))$

by (*simp add: itl-def itl-defs*)

(*metis One-nat-def add-Suc-shift add-diff-cancel-right' le-add1 min.orderE min-OR numeral-2-eq-2*
numeral-3-eq-3 plus-1-eq-Suc zero-neq-one)

lemma (in *Test*) *stable-test*:

$(\langle s0, s1, s2, s3 \rangle \models \text{len}(3) \wedge \text{stable } v \wedge \$v = \#0) =$

$((v \ s0) = 0 \wedge (v \ s1) = 0 \wedge (v \ s2) = 0 \wedge (v \ s3) = 0)$

using *nat-le-linear* **by** (*simp add: itl-defs Nitpick.case-nat-unfold, auto*)

lemma (in *Test*) *revnextcurrentfinpenult-test*:

$(\langle s0, s1, s2, s3 \rangle \models (\text{len } 3 \wedge v! = !v - \#1 \wedge !v = \#3 \wedge \$v = \#0 \wedge v\$ = \$v + \#1)^r) =$
 $((v \ s3) = 0) \wedge ((v \ s2) = 1 \wedge (v \ (s1)) = 2 \wedge ((v \ s0) = 3 \))$

unfolding *reverse-d-def len-defs current-val-d-def next-val-d-def*

penult-val-d-def fin-val-d-def **by** *auto*

lemma (in *Test*) *exist-test-F2* :

$\vdash \exists \exists \ v. F2 \ v$

proof –

have *1*: $\vdash F2 \ v$ **by** (*simp add: itl-defs F2-def Valid-def*)

from *1* **show** *?thesis* **by** (*meson EExI MP*)

qed

lemma (in *Test*) *exist-test-F3* :

$\vdash \exists \exists \ y. F3 \ y$

proof –

have *1*: $\vdash F3 \ y$ **by** (*simp add: itl-defs F3-def Valid-def*)

from *1* **show** *?thesis* **by** (*meson EExI MP*)

qed

11.3 Example 3

```

locale Test1 =
  fixes v :: state  $\Rightarrow$  nat
  fixes F5 :: nat statefun  $\Rightarrow$  nat  $\Rightarrow$  temporal
  defines F5  $\equiv$  ( $\lambda$  v n. TEMP $v=#0  $\wedge$  v gets $v+#1  $\wedge$  fn($v=#n))

```

lemma (in Test1) test-E-F5-1:

```

(
  x ( inth w (0::nat)) = (0::nat)  $\wedge$ 
  ( $\forall$  i<ilen w. x ( inth w (Suc i)) = Suc (x ( inth w i)))  $\wedge$ 
  x ( inth w (ilen w)) = n)  $\longrightarrow$ 
  (
    x ( inth w (0::nat)) = (0::nat)  $\wedge$ 
    ( $\forall$  i $\leq$ ilen w. x ( inth w (i)) = i)  $\wedge$ 
    x ( inth w (ilen w)) = n)
proof auto
show  $\bigwedge$  i. x ( inth w 0) = 0  $\implies$ 
   $\forall$  i<ilen w. x ( inth w (Suc i)) = Suc (x ( inth w i))  $\implies$ 
  n = x ( inth w (ilen w))  $\implies$  i  $\leq$  ilen w  $\implies$  x ( inth w i) = i
proof -
  fix i
  assume 0: x ( inth w 0) = 0
  assume 1:  $\forall$  i<ilen w. x ( inth w (Suc i)) = Suc (x ( inth w i))
  assume 2: n = x ( inth w (ilen w))
  assume 3: i  $\leq$  ilen w
  show x ( inth w i) = i
  using 0 1 2 3
  proof (induct i)
  case 0
  then show ?case by simp
  next
  case (Suc i)
  then show ?case by simp
  qed
qed
qed

```

lemma (in Test1) test-E-F5-2:

```

(
  x ( inth w (0::nat)) = (0::nat)  $\wedge$ 
  ( $\forall$  i $\leq$ ilen w. x ( inth w (i)) = i)  $\wedge$ 
  x ( inth w (ilen w)) = n)  $\longrightarrow$  (
    x ( inth w (0::nat)) = (0::nat)  $\wedge$ 
    ( $\forall$  i<ilen w. x ( inth w (Suc i)) = Suc (x ( inth w i)))  $\wedge$ 
    x ( inth w (ilen w)) = n)

```

by simp

```

lemma (in Test1) test-E-F5-3:
(
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} 0::nat \text{))} = (0::nat) \wedge$ 
   $(\forall i < \textit{ilen} \text{ } w. x \text{ ( } \textit{inth} \text{ } w \text{ (} \textit{Suc } i \text{))} = \textit{Suc } (x \text{ ( } \textit{inth} \text{ } w \text{ } i \text{)))} \wedge$ 
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} \textit{ilen } w \text{))} = n) =$ 
(
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} 0::nat \text{))} = (0::nat) \wedge$ 
   $(\forall i \leq \textit{ilen} \text{ } w. x \text{ ( } \textit{inth} \text{ } w \text{ (} i \text{))} = i) \wedge$ 
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} \textit{ilen } w \text{))} = n)$ 
using test-E-F5-1 test-E-F5-2 by auto

```

```

lemma (in Test1) test-E-F5-4:
( $\exists x::state \Rightarrow nat.$ 
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} 0::nat \text{))} = (0::nat) \wedge$ 
   $(\forall i < \textit{ilen} \text{ } w. x \text{ ( } \textit{inth} \text{ } w \text{ (} \textit{Suc } i \text{))} = \textit{Suc } (x \text{ ( } \textit{inth} \text{ } w \text{ } i \text{)))} \wedge$ 
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} \textit{ilen } w \text{))} = n) =$ 
( $\exists x::state \Rightarrow nat.$ 
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} 0::nat \text{))} = (0::nat) \wedge$ 
   $(\forall i \leq \textit{ilen} \text{ } w. x \text{ ( } \textit{inth} \text{ } w \text{ (} i \text{))} = i) \wedge$ 
   $x \text{ ( } \textit{inth} \text{ } w \text{ (} \textit{ilen } w \text{))} = n)$ 
by (simp add: Test1.test-E-F5-3)

```

```

lemma (in Test1) test-E-F5:
 $\vdash (\exists \exists v. (F5 \text{ } v \text{ } n)) \longrightarrow (\textit{len } n)$ 
by (auto simp add: Valid-def F5-def exist-state-d-def itl-defs sub-def )
(metis Test1.test-E-F5-1 nat-le-linear)

```

11.4 Example 4

```

locale Testrev =
fixes  $x :: state \Rightarrow nat$ 
fixes  $F1 :: nat \text{ statefun} \Rightarrow \textit{temporal}$ 
defines  $F1 \equiv (\lambda v. \textit{TEMP } \$v = \#0 \wedge \textit{skip} \wedge v := \$v + \#1)$ 

```

```

lemma (in Testrev) testrev1:
 $(\sigma \models F1 \text{ (} x \text{))} = (\textit{ilen } \sigma = 1 \wedge (x \text{ ( } \textit{inth } \sigma \text{ } 0 \text{))} = 0 \wedge (x \text{ ( } \textit{inth } \sigma \text{ } 1 \text{))} = 1)$ 
by (simp add: F1-def itl-defs next-assign-d-def, auto)

```

```

lemma (in Testrev) testrev2:
 $(\sigma \models (F1 \text{ (} x \text{))}^r) = (\textit{ilen } \sigma = 1 \wedge (x \text{ ( } \textit{inth } \sigma \text{ } 0 \text{))} = 1 \wedge (x \text{ ( } \textit{inth } \sigma \text{ } 1 \text{))} = 0)$ 
proof –
have  $(\sigma \models (F1 \text{ (} x \text{))}^r) = (\sigma \models (\$x = \#0 \wedge \textit{skip} \wedge x := \$x + \#1)^r)$ 
by (simp add: F1-def)
also have  $\dots =$ 
 $(\sigma \models ((\$x = \#0)^r \wedge \textit{skip}^r \wedge (x := \$x + \#1)^r))$ 
by (simp add: all-rev-eq)
also have  $\dots =$ 
 $(\sigma \models (!x = \#0) \wedge \textit{skip} \wedge (x! = !x + \#1))$ 
using RRAnd all-rev-eq

```

```

    by (simp add: all-rev-eq(1) all-rev-eq(12) all-rev-eq(3) all-rev-eq(8) all-rev-eq(9)
        next-assign-d-def)
  also have ... =
    (σ ⊨ ((x$=#0) ∧ skip ∧ ($x = x$+#1)))
  by (simp add: itl-defs, auto)
  also have ... =
    (ilen σ = 1 ∧ (x (inth σ 0)) = 1 ∧ (x (inth σ 1)) = 0)
  by (simp add: itl-defs, auto)
  finally show (σ ⊨ (F1 (x))r) = (ilen σ = 1 ∧ (x (inth σ 0)) = 1 ∧ (x (inth σ 1)) = 0) .
qed

```

11.5 Example 5

lemma *revnextcurrentfinpenult*:

```

⊢ (v$ = $v)r = (v!=!v)
proof –
  have 1: ⊢ (v$ = $v)r = ( (v$)r = ($v)r) by (simp add: rev-fun2)
  have 2: ⊢ ((v$)r = (v!)) by (simp add: rev-next)
  have 3: ⊢ (($v)r = (!v)) by (simp add: rev-current)
  have 4: ⊢ (((v$)r = ($v)r) = ( (v!) = (!v) )) by (metis 1 2 3 inteq-reflection)
  from 1 4 show ?thesis by fastforce
qed

```

end

12 Monitor Example

theory *MonitorExample*

imports

FOTheorems Monitor

begin

```

locale Test =
  fixes v :: state ⇒ nat
  fixes y :: state ⇒ bool
  fixes z :: state ⇒ nat
  fixes F2 :: nat statefun ⇒ temporal
  fixes F3 :: bool statefun ⇒ temporal
  fixes F4 :: nat statefun ⇒ temporal
  fixes F5 :: nat statefun ⇒ temporal
  fixes Init2 :: nat statefun ⇒ temporal
  fixes Init3 :: bool statefun ⇒ temporal
  fixes Mon1 :: state monitor
  fixes Mon2 :: state monitor
  fixes Mon3 :: state monitor

```

```

fixes Mon4 :: state monitor
fixes Mon5 :: state monitor
fixes Mon6 :: state monitor
defines F2  $\equiv (\lambda v. TEMP \square (\#0 \leq \$v))$ 
defines F3  $\equiv (\lambda p. TEMP \square (\$p \vee \neg \$p))$ 
defines F4  $\equiv (\lambda z. TEMP \$z = \#0 \wedge z \text{ gets } \$z + \#1)$ 
defines F5  $\equiv (\lambda z. TEMP \text{ fin } (\$z = \#4))$ 
defines Init2  $\equiv (\lambda v. TEMP \$v = \#0)$ 
defines Init3  $\equiv (\lambda p. TEMP \$p)$ 
defines Mon1  $\equiv FIRST (F2 v)$ 
defines Mon2  $\equiv EMPTY \text{ UPTO } Mon1$ 
defines Mon3  $\equiv Mon1 \text{ WITH } (F2 v)$ 
defines Mon4  $\equiv Mon2 \text{ THEN } Mon1$ 
defines Mon5  $\equiv Mon3 \text{ THRU } Mon4$ 
defines Mon6  $\equiv (FIRST F4 z) \text{ WITH } (F5 z)$ 

```

lemma (**in** *Test*) *test*:

$\vdash \mathcal{M}(Mon1) = \text{empty}$

proof –

have 1: $\vdash \mathcal{M}(Mon1) = \triangleright(\square (\#0 \leq \$v))$

using *F2-def Mon1-def* **by** *fastforce*

have 2: $\vdash \square (\#0 \leq \$v)$

by (*simp add: Valid-def itl-defs*)

have 3: $\vdash \triangleright(\square (\#0 \leq \$v)) = \text{empty}$

using 2 **by** (*metis FstTrue int-eq int-eq-true*)

from 1 2 3 **show** *?thesis* **by** *fastforce*

qed

lemma (**in** *Test*) *test1*:

$\vdash \mathcal{M}(Mon2) = \text{empty}$

proof –

have 1: $\vdash \mathcal{M}(Mon2) = \mathcal{M}(EMPTY \text{ UPTO } Mon1)$

using *Mon2-def* **by** *fastforce*

have 2: $\vdash \mathcal{M}(EMPTY \text{ UPTO } Mon1) = \triangleright(\mathcal{M}(EMPTY) \vee \mathcal{M}(Mon1))$

by *fastforce*

have 3: $\vdash \triangleright(\mathcal{M}(EMPTY) \vee \mathcal{M}(Mon1)) = \triangleright(\text{empty} \vee \text{empty})$

using *test* **by** (*metis 2 MEmptyAlt int-eq*)

have 4: $\vdash \triangleright(\text{empty} \vee \text{empty}) = \text{empty}$

using *FstEmptyOrEqvEmpty* **by** *blast*

from 1 2 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma (**in** *Test*) *test2*:

$\vdash \mathcal{M}(Mon3) = \text{empty}$

proof –

have 1: $\vdash \mathcal{M}(Mon3) = \mathcal{M}(Mon1 \text{ WITH } (F2 v))$ **using** *Mon3-def* **by** *fastforce*

have 2: $\vdash \mathcal{M}(Mon1 \text{ WITH } (F2 v)) = (\mathcal{M}(Mon1) \wedge (F2 v))$ **by** *fastforce*

have 3: $\vdash (\mathcal{M}(Mon1) \wedge (F2 v)) = (\text{empty} \wedge (F2 v))$ **using** *test* **by** *fastforce*

have 4: $\vdash (F2 v)$ **by** (*simp add: F2-def Valid-def itl-defs*)

have 5: $\vdash (\text{empty} \wedge (F2 v)) = \text{empty}$ **using** 4 **by** *fastforce*

from 1 2 3 5 show ?thesis by fastforce
qed

lemma (in Test) test3:

$\vdash \mathcal{M}(\text{Mon4}) = \text{empty}$

proof –

have 1: $\vdash \mathcal{M}(\text{Mon4}) = \mathcal{M}(\text{Mon2 THEN Mon1})$

using Mon4-def by fastforce

have 2: $\vdash \mathcal{M}(\text{Mon2 THEN Mon1}) = (\mathcal{M}(\text{Mon2})) ; (\mathcal{M}(\text{Mon1}))$

by fastforce

have 3: $\vdash (\mathcal{M}(\text{Mon2})) ; (\mathcal{M}(\text{Mon1})) = \text{empty}; \text{empty}$

using test test1 using ChopEqvChop by blast

have 4: $\vdash \text{empty}; \text{empty} = \text{empty}$

by (simp add: ChopEmpty)

from 1 2 3 4 show ?thesis by fastforce

qed

lemma (in Test) test4:

$\vdash \mathcal{M}(\text{Mon5}) = \text{empty}$

proof –

have 1: $\vdash \mathcal{M}(\text{Mon5}) = \mathcal{M}(\text{Mon3 THRU Mon4})$

using Mon5-def by fastforce

have 2: $\vdash \mathcal{M}(\text{Mon3 THRU Mon4}) = \triangleright(\text{di}(\mathcal{M}(\text{Mon3})) \wedge \text{di}(\mathcal{M}(\text{Mon4})))$

by fastforce

have 3: $\vdash (\text{di}(\mathcal{M}(\text{Mon3})) \wedge \text{di}(\mathcal{M}(\text{Mon4}))) = (\text{di}(\text{empty}) \wedge \text{di}(\text{empty}))$

using test3 test2 by (metis inteq-reflection lift-and-com)

hence 4: $\vdash \triangleright(\text{di}(\mathcal{M}(\text{Mon3})) \wedge \text{di}(\mathcal{M}(\text{Mon4}))) = \triangleright(\text{di}(\text{empty}) \wedge \text{di}(\text{empty}))$

by (simp add: FstEqvRule)

have 5: $\vdash \triangleright(\text{di}(\text{empty}) \wedge \text{di}(\text{empty})) = \triangleright(\text{di}(\text{empty}))$

by simp

have 6: $\vdash \triangleright(\text{di}(\text{empty})) = \text{empty}$

using FstDiEqvFst FstEmpty by fastforce

from 6 5 4 2 1 show ?thesis by fastforce

qed

lemma (in Test) test5:

$\vdash \mathcal{M}(\text{Mon6}) = (\triangleright(\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4))$

proof –

have 1: $\vdash \mathcal{M}(\text{Mon6}) = (\mathcal{M}(\text{FIRST } F4 \ z) \wedge (F5 \ z))$

using Mon6-def by fastforce

have 2: $\vdash (\mathcal{M}(\text{FIRST } F4 \ z) \wedge (F5 \ z)) = (\triangleright(F4 \ z) \wedge \text{fin}(\$z = \#4))$

using F5-def by fastforce

have 3: $\vdash (\triangleright(F4 \ z) \wedge \text{fin}(\$z = \#4)) = (\triangleright(\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4))$

using F4-def by fastforce

from 1 2 3 show ?thesis by fastforce

qed

lemma (in Test) test5-1:

$\vdash \triangleright(\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4) \longrightarrow$

$\triangleright((\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4))$

using *FstWithAndImp* **by** *blast*

lemma (**in** *Test*) *test5-2*:

$(s \models (\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4)) =$
 $(z (\text{inth } s \ 0) = 0 \wedge (\forall i < \text{ilen } s. z (\text{inth } s \ (\text{Suc } i)) = \text{Suc}(z (\text{inth } s \ i))) \wedge$
 $z (\text{inth } s \ (\text{ilen } s)) = 4)$
by (*simp add: itl-defs sub-def*)

lemma (**in** *Test*) *test5-3*:

$(z (\text{inth } s \ 0) = 0 \wedge (\forall i < \text{ilen } s. z (\text{inth } s \ (\text{Suc } i)) = \text{Suc}(z (\text{inth } s \ i))) \wedge$
 $z (\text{inth } s \ (\text{ilen } s)) = 4)$
 \implies
 $(z (\text{inth } s \ 0) = 0 \wedge (\forall i \leq \text{ilen } s. z (\text{inth } s \ i) = i)$
 $\wedge z (\text{inth } s \ (\text{ilen } s)) = 4)$

proof –

assume *0*: $(z (\text{inth } s \ 0) = 0 \wedge (\forall i < \text{ilen } s. z (\text{inth } s \ (\text{Suc } i)) = \text{Suc}(z (\text{inth } s \ i))) \wedge$
 $z (\text{inth } s \ (\text{ilen } s)) = 4)$
show $(z (\text{inth } s \ 0) = 0 \wedge (\forall i \leq \text{ilen } s. z (\text{inth } s \ i) = i)$
 $\wedge z (\text{inth } s \ (\text{ilen } s)) = 4)$

proof –

have *1*: $z (\text{inth } s \ 0) = 0$ **using** *0* **by** *auto*
have *2*: $z (\text{inth } s \ (\text{ilen } s)) = 4$ **using** *0* **by** *auto*
have *3*: $(\forall i \leq \text{ilen } s. z (\text{inth } s \ i) = i)$

proof

fix *i*
show $i \leq \text{ilen } s \longrightarrow z (\text{inth } s \ i) = i$

proof

(*induct i*)
case *0*
then show ?*case* **by** (*simp add: 1*)
next
case (*Suc i*)
then show ?*case* **by** (*simp add: 0*)
qed

qed

from *1 2 3* **show** ?*thesis* **by** *auto*

qed

qed

lemma (**in** *Test*) *test5-4*:

$(z (\text{inth } s \ 0) = 0 \wedge (\forall i \leq \text{ilen } s. z (\text{inth } s \ i) = i)$
 $\wedge z (\text{inth } s \ (\text{ilen } s)) = 4) \implies$
 $(z (\text{inth } s \ 0) = 0 \wedge (\forall i < \text{ilen } s. z (\text{inth } s \ (\text{Suc } i)) = \text{Suc}(z (\text{inth } s \ i))) \wedge$
 $z (\text{inth } s \ (\text{ilen } s)) = 4)$

proof –

assume *0*: $(z (\text{inth } s \ 0) = 0 \wedge (\forall i \leq \text{ilen } s. z (\text{inth } s \ i) = i)$
 $\wedge z (\text{inth } s \ (\text{ilen } s)) = 4)$

show $(z \text{ (inth } s \ 0) = 0 \wedge (\forall i < \text{ilen } s. z \text{ (inth } s \ (\text{Suc } i)) = \text{Suc}(z \text{ (inth } s \ i))) \wedge$
 $z \text{ (inth } s \ (\text{ilen } s)) = 4)$
proof –
have 1: $z \text{ (inth } s \ 0) = 0$ **using** 0 **by** auto
have 2: $z \text{ (inth } s \ (\text{ilen } s)) = 4$ **using** 0 **by** auto
have 3: $(\forall i < \text{ilen } s. z \text{ (inth } s \ (\text{Suc } i)) = \text{Suc}(z \text{ (inth } s \ i)))$ **by** (simp add: 0)
from 1 2 3 **show** ?thesis **by** auto
qed
qed

lemma (in Test) test5-5:
 $(z \text{ (inth } s \ 0) = 0 \wedge (\forall i < \text{ilen } s. z \text{ (inth } s \ (\text{Suc } i)) = \text{Suc}(z \text{ (inth } s \ i))) \wedge$
 $z \text{ (inth } s \ (\text{ilen } s)) = 4)$
 $=$
 $(z \text{ (inth } s \ 0) = 0 \wedge (\forall i \leq \text{ilen } s. z \text{ (inth } s \ i) = i)$
 $\wedge z \text{ (inth } s \ (\text{ilen } s)) = 4)$

using test5-3 test5-4 **by** blast

lemma (in Test) test5-6 :
 $(z \text{ (inth } s \ 0) = 0 \wedge (\forall i \leq \text{ilen } s. z \text{ (inth } s \ i) = i)$
 $\wedge z \text{ (inth } s \ (\text{ilen } s)) = 4) =$
 $(\text{ilen } s = 4 \wedge (\forall i \leq \text{ilen } s. z \text{ (inth } s \ i) = i))$

by auto

lemma (in Test) test5-7 :
 $(s \models (\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4)) =$
 $(\text{ilen } s = 4 \wedge (\forall i \leq \text{ilen } s. z \text{ (inth } s \ i) = i))$
using test5-6 test5-5 test5-2 **by** fastforce

lemma (in Test) test5-8 :
 $(s \models \Diamond((\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4))) =$
 $($
 $(s \models (\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4)) \wedge \text{ilen } s = 0) \vee$
 $(0 < \text{ilen } s \wedge (s \models \$z = \#0 \wedge z \text{ gets } \$z + \#1 \wedge \text{fin}(\$z = \#4)) \wedge$
 $(\forall ia < \text{ilen } s. (\text{prefix } ia \ s \models \neg((\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4))))))$
 $)$

using Fstsem[of TEMP $(\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4)$]
by simp

lemma (in Test) test5-9 :
 $\neg(s \models (\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4)) \wedge \text{ilen } s = 0)$
using test5-7 **by** simp

lemma (in Test) test5-10:
 $(s \models (\$z = \#0 \wedge z \text{ gets } \$z + \#1) \wedge \text{fin}(\$z = \#4))$
 \implies
 $0 < \text{ilen } s \wedge$

```

(∀ ia < ilen s. (prefix ia s ⊨ ¬(($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4))))

proof –
assume 0: s ⊨ ($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4)
show 0 < ilen s ∧
  (∀ ia < ilen s. (prefix ia s ⊨ ¬(($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4))))
proof –
have 1: 0 < ilen s using test5-7 0 by simp
have 2: (∀ ia < ilen s. (prefix ia s ⊨ ¬(($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4))))
proof
fix ia
show ia < ilen s ⟶
  (prefix ia s ⊨ ¬ (($z = #0 ∧ z gets $z + #1) ∧ fin ($z = #4)))
proof –
have 1: (prefix ia s ⊨ ¬ (($z = #0 ∧ z gets $z + #1) ∧ fin ($z = #4))) =
  (¬((prefix ia s ⊨ (($z = #0 ∧ z gets $z + #1) ∧ fin ($z = #4)))))
  by auto
have 2: (prefix ia s ⊨ (($z = #0 ∧ z gets $z + #1) ∧ fin ($z = #4))) =
  (ilen (prefix ia s) = 4 ∧ (∀ i ≤ ilen(prefix ia s) . z (inth (prefix ia s) i) = i))
  using test5-7 by simp
have 3: ia < ilen s ⟶ ¬(ilen (prefix ia s) = 4 ∧
  (∀ i ≤ ilen(prefix ia s) . z (inth (prefix ia s) i) = i))
  using 0 using test5-7 by auto
from 1 2 3 show ?thesis by blast
qed
qed
from 1 2 show ?thesis by auto
qed
qed

lemma (in Test) test5-11 :
  (s ⊨ ▷(($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4))) =
  (s ⊨ ($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4))
using test5-8 test5-9 test5-10 by fastforce

lemma (in Test) test5-12 :
  ⊢ ▷(($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4)) = (($z=#0 ∧ z gets $z+#1) ∧ fin($z=#4))
using test5-11 by (simp add: Valid-def)

```

end

13 Filter on Intervals

```

theory IFilter
imports
  Interval
begin

```

The ifilter operator on intervals is defined. The definition of ifilter is slightly more complicated than

the one for lists as an interval has at least one state and one needs to ensure that the ifilter operator always returns an interval. The lemmas involving the ifilter on intervals are similar to those for the filter operator on lists only a bit more complicated.

13.1 Definitions

definition $opfx :: 'a\ interval \Rightarrow 'a\ interval \Rightarrow bool$
where $opfx\ xs\ ys = (\exists\ zs.\ ys = xs \ominus zs \vee ys = xs)$

definition $sopfx :: 'a\ interval \Rightarrow 'a\ interval \Rightarrow bool$
where $sopfx\ xs\ ys \longleftrightarrow opfx\ xs\ ys \wedge xs \neq ys$

interpretation $opfx\text{-}order$: $order\ opfx\ sopfx$

proof *standard*

show $\bigwedge x\ y.\ sopfx\ x\ y = (opfx\ x\ y \wedge \neg opfx\ y\ x)$
by (*metis add-leD1 add-le-same-cancel1 ilen-iapp le-add1 not-one-le-zero opfx-def sopfx-def*)
show $\bigwedge x.\ opfx\ x\ x$
by (*auto simp add: opfx-def sopfx-def*)
show $\bigwedge x\ y\ z.\ opfx\ x\ y \Longrightarrow opfx\ y\ z \Longrightarrow opfx\ x\ z$
by (*auto simp add: opfx-def sopfx-def*)
show $\bigwedge x\ y.\ opfx\ x\ y \Longrightarrow opfx\ y\ x \Longrightarrow x = y$
by (*metis add-leD1 add-le-same-cancel1 ilen-iapp le-add1 not-one-le-zero opfx-def*)
qed

definition $osfx :: 'a\ interval \Rightarrow 'a\ interval \Rightarrow bool$
where $osfx\ xs\ ys = (\exists\ zs.\ ys = zs \ominus xs \vee ys = xs)$

definition $sosfx :: 'a\ interval \Rightarrow 'a\ interval \Rightarrow bool$
where $sosfx\ xs\ ys \longleftrightarrow osfx\ xs\ ys \wedge xs \neq ys$

interpretation $osfx\text{-}order$: $order\ osfx\ sosfx$

proof *standard*

show $\bigwedge x\ y.\ sosfx\ x\ y = (osfx\ x\ y \wedge \neg osfx\ y\ x)$
by (*metis irev-iapp irev-irev-ident opfx-def opfx-order.dual-order.antisym osfx-def sosfx-def*)
show $\bigwedge x.\ osfx\ x\ x$
by (*auto simp add: osfx-def sosfx-def*)
show $\bigwedge x\ y\ z.\ osfx\ x\ y \Longrightarrow osfx\ y\ z \Longrightarrow osfx\ x\ z$
by (*auto simp add: osfx-def sosfx-def*)
(metis iapp-assoc)
show $\bigwedge x\ y.\ osfx\ x\ y \Longrightarrow osfx\ y\ x \Longrightarrow x = y$
using $\langle \bigwedge y\ x.\ sosfx\ x\ y = (osfx\ x\ y \wedge \neg osfx\ y\ x) \rangle\ sosfx\text{-}def$ **by** *blast*
qed

primrec $idistinct :: 'a\ interval \Rightarrow bool$

where $idistinct\ \langle x \rangle \longleftrightarrow True$
 $| idistinct\ (x \odot xs) \longleftrightarrow x \notin iset\ xs \wedge idistinct\ xs$

primrec *iremdups* :: 'a interval \Rightarrow 'a interval
where *iremdups* $\langle x \rangle = \langle x \rangle$
| *iremdups* $(x \odot xs) = (\text{if } x \in \text{iset } xs \text{ then } \text{iremdups } xs \text{ else } x \odot \text{iremdups } xs)$

primrec *ifilter* :: ('a \Rightarrow bool) \Rightarrow 'a interval \Rightarrow 'a interval
where *ifilter* $P \langle x \rangle = \langle x \rangle$
| *ifilter* $P (x \odot xs) = (\text{if } (\exists y \in \text{iset } xs. P y) \text{ then}$
 $(\text{if } P x \text{ then } x \odot \text{ifilter } P xs \text{ else } \text{ifilter } P xs)$
 $\text{else } \langle x \rangle)$

primrec *nfilter* :: ('a \Rightarrow bool) \Rightarrow 'a interval \Rightarrow nat \Rightarrow nat interval
where *nfilter* $P \langle x \rangle n = \langle n \rangle$
| *nfilter* $P (x \odot xs) n = (\text{if } (\exists y \in \text{iset } xs. P y) \text{ then}$
 $(\text{if } P x \text{ then } n \odot (\text{nfilter } P xs (\text{Suc } n))$
 $\text{else } \text{nfilter } P xs (\text{Suc } n))$
 $\text{else } \langle n \rangle)$

primrec *prefixes* :: 'a interval \Rightarrow 'a interval interval
where
prefixes $\langle x \rangle = \langle \langle x \rangle \rangle$
| *prefixes* $(x \odot xs) = \langle x \rangle \odot (\text{imap } ((\odot) x) (\text{prefixes } xs))$

primrec *suffixes* :: 'a interval \Rightarrow 'a interval interval
where
suffixes $\langle x \rangle = \langle \langle x \rangle \rangle$
| *suffixes* $(x \odot xs) = (x \odot xs) \odot (\text{suffixes } xs)$

13.2 Lemmas

13.2.1 opfx and sopfx

lemma *opfxI* [*intro?*]:
assumes $ys = xs \ominus zs \vee ys = xs$
shows $\text{opfx } xs \ ys$
using *assms* **unfolding** *opfx-def* **by** *blast*

lemma *opfxE* [*elim?*]:
assumes $\text{opfx } xs \ ys$
obtains zs **where** $ys = xs \ominus zs \vee ys = xs$
using *assms* **unfolding** *opfx-def* **by** *blast*

lemma *sopfxI'* [*intro?*]:
 $ys = xs \ominus (zs) \Longrightarrow \text{sopfx } xs \ ys$
unfolding *sopfx-def* *opfx-def*
by (*metis add-le-same-cancel1 ilen-iapp le-add1 not-one-le-zero*)

lemma *sopfxE'* [*elim?*]:
assumes $\text{sopfx } xs \ ys$
obtains zs **where** $ys = xs \ominus zs$

using *assms* **unfolding** *sopfx-def opfx-def* **by** *blast*

lemma *opfx-INil* [*simp*]:

opfx $\langle \text{ifirst } xs \rangle xs$

unfolding *opfx-def*

by (*metis iapp-INil ifirst-iapp2 interval.exhaust inth.simps*(1))

lemma *opfx-snoc* [*simp*]:

opfx $xs (ys \ominus \langle y \rangle) \longleftrightarrow xs = ys \ominus \langle y \rangle \vee \text{opfx } xs \text{ } ys$

unfolding *opfx-def*

by (*metis iapp-eq-iapp-conv2 iapp-not-INil*)

lemma *ICons-pfx-ICons* [*simp*]:

opfx $(x \odot xs) (y \odot ys) = (x = y \wedge \text{opfx } xs \text{ } ys)$

by (*auto simp add: opfx-def*)

lemma *opfx-code* [*code*]:

opfx $\langle \text{ifirst } xs \rangle xs \longleftrightarrow \text{True}$

opfx $(x \odot xs) \langle y \rangle \longleftrightarrow \text{False}$

opfx $(x \odot xs) (y \odot ys) \longleftrightarrow (x = y \wedge \text{opfx } xs \text{ } ys)$

proof *simp-all*

show $\neg \text{opfx } (x \odot xs) \langle y \rangle$

by (*simp add: opfx-def*)

qed

lemma *same-opfx-opfx* [*simp*]:

opfx $(xs \ominus ys) (xs \ominus zs) = \text{opfx } ys \text{ } zs$

by (*induct xs simp-all*)

lemma *same-opfx-INil* [*simp*]:

opfx $(xs \ominus ys) (xs \ominus \langle y \rangle) = (ys = \langle y \rangle)$

by (*meson same-iapp-eq opfx-order.less-le-not-le opfx-snoc sopfxI'*)

lemma *opfx-opfx* [*simp*]:

assumes *opfx* $xs \text{ } ys$

shows *opfx* $xs (ys \ominus zs)$

using *assms* **unfolding** *opfx-def* **by** *fastforce*

lemma *iapp-opfxD*:

assumes *opfx* $(xs \ominus ys) \text{ } zs$

shows *opfx* $xs \text{ } zs$

using *assms* **by** (*auto simp add: opfx-def*)

lemma *opfx-ICons*:

opfx $xs (y \odot ys) = (xs = \langle y \rangle \vee (\exists zs. xs = y \odot zs \wedge \text{opfx } zs \text{ } ys))$

by (*case-tac xs*) (*auto simp add: opfx-def*)

lemma *opfx-iapp*:

opfx $xs (ys \ominus zs) = (\text{opfx } xs \text{ } ys \vee (\exists us. xs = ys \ominus us \wedge \text{opfx } us \text{ } zs))$

```

proof (induct zs rule: irev-induct)
case (INil y)
then show ?case by (meson opfx-snoc same-opfx-opfx)
next
case (snoc x xs)
then show ?case by (metis iapp-assoc opfx-snoc)
qed

lemma iapp-one-opfx:
assumes opfx xs ys
         ilen xs < ilen ys
shows opfx (xs ⊖ ⟨inth ys ((ilen xs)+1)⟩) ys
using assms
proof (unfold opfx-def)
assume 1:  $\exists zs. ys = xs \ominus zs \vee ys = xs$ 
then obtain sk where 2:  $ys = xs \ominus sk \vee ys = xs$ 
by fastforce
assume 3: ilen xs < ilen ys
have 4: ilen sk ≥ 0
by auto
have 5:  $\exists v. ys = xs \ominus ((ifirst\ sk) \odot v) \vee ys = xs \ominus \langle ifirst\ sk \rangle$ 
by (metis 2 3 iapp-INil less-irrefl opfx-def opfx-INil)
have 6: ys ≠ xs
using 3 by blast
have 7:  $\langle ifirst\ sk \rangle = inth\ (xs \ominus sk)\ ((ilen\ xs) + 1)$ 
by (simp add: iapp-inth)
have 8:  $ys = xs \ominus sk$ 
using 2 6 by auto
have 9:  $inth\ (xs \ominus sk)\ ((ilen\ xs) + 1) = inth\ (ys)\ ((ilen\ xs) + 1)$ 
using 8 by blast
thus  $\exists zs. ys = (xs \ominus \langle inth\ ys\ ((ilen\ xs) + 1) \rangle) \ominus zs \vee ys = xs \ominus \langle inth\ ys\ ((ilen\ xs) + 1) \rangle$ 
using 5 7 9
by simp
qed

```

```

lemma opfx-ilen-le:
assumes opfx xs ys
shows  $ilen\ xs \leq ilen\ ys$ 
using assms by (auto simp add: opfx-def)

```

```

lemma opfx-same-cases:
assumes opfx xs1 ys
         opfx xs2 ys
shows  $opfx\ xs1\ xs2 \vee opfx\ xs2\ xs1$ 
using assms unfolding opfx-def using iapp-eq-iapp-conv2
by metis

```

```

lemma opfx-ilen-opfx:
assumes opfx ps xs
         opfx qs xs

```

```

       $ilen\ ps \leq\ ilen\ qs$ 
shows    $opfx\ ps\ qs$ 
using  $assms$ 
by ( $auto\ simp: opfx-def$ )
      ( $metis\ assms(2)\ dual-order.antisym\ iapp-eq-iapp-conv\ opfx-iapp\ opfx-ilen-le$ )

```

```

lemma  $iset-mono-opfx$ :
assumes  $opfx\ xs\ ys$ 
shows    $iset\ xs \subseteq iset\ ys$ 
using  $assms$  by ( $auto\ simp: opfx-def$ )

```

```

lemma  $prefix-is-opfx$ :
 $opfx\ (prefix\ n\ xs)\ xs$ 
by ( $metis\ Suc-eq-plus1\ add-diff-inverse-nat\ iapp-prefix-suffix$ 
       $prefix-ilen-gr-1\ le-simps(2)\ less-Suc-eq-0-disj\ not-less\ opfx-def$ )

```

```

lemma  $imap-mono-opfx$ :
assumes  $opfx\ xs\ ys$ 
shows    $opfx\ (imap\ f\ xs)\ (imap\ f\ ys)$ 
using  $assms$  by ( $auto\ simp: opfx-def$ )

```

```

lemma  $opfx-ilen-less$ :
assumes  $sopfx\ xs\ ys$ 
shows    $ilen\ xs < ilen\ ys$ 
using  $assms$  by ( $auto\ simp: sopfx-def\ opfx-def$ )

```

```

lemma  $opfx-snocD$ :
assumes  $opfx\ (xs \ominus \langle x \rangle)\ ys$ 
shows    $sopfx\ xs\ ys$ 
using  $assms$ 
using  $opfx-order.less-le-trans\ sopfxI'$  by  $blast$ 

```

```

lemma  $sopfx-simps\ [simp, code]$ :
 $sopfx\ xs\ \langle y \rangle \longleftrightarrow False$ 
 $sopfx\ \langle x \rangle\ (x \odot xs) \longleftrightarrow True$ 
 $sopfx\ (x \odot xs)\ (y \odot ys) \longleftrightarrow x = y \wedge sopfx\ xs\ ys$ 
proof ( $simp-all\ add: sopfx-def$ )
show  $opfx\ xs\ \langle y \rangle \longrightarrow xs = \langle y \rangle$ 
  by ( $metis\ iapp-not-INil\ opfxE$ )
show  $opfx\ \langle x \rangle\ (x \odot xs)$ 
  by ( $simp\ add: opfx-ICons$ )
show  $(x = y \wedge opfx\ xs\ ys \wedge (x = y \longrightarrow xs \neq ys)) = (x = y \wedge opfx\ xs\ ys \wedge xs \neq ys)$ 
  by  $auto$ 
qed

```

```

lemma  $prefix-sopfx$ :
assumes  $sopfx\ xs\ ys$ 
shows    $sopfx\ (prefix\ n\ xs)\ ys$ 
using  $assms$ 
proof ( $induct\ n\ arbitrary: xs\ ys$ )

```

```

case 0
then show ?case
  proof (cases ys)
    case (INil x1)
    then show ?thesis
      using 0.premys by auto
    next
    case (ICons x21 x22)
    then show ?thesis
      using 0.premys opfx-order.le-less-trans prefix-is-opfx by blast
    qed
  next
  case (Suc n)
  then show ?case
    using opfx-order.order.strict-trans1 prefix-is-opfx by blast
  qed

```

```

lemma osfxI [intro?]:
  assumes ys = zs  $\ominus$  xs  $\vee$  ys = xs
  shows osfx xs ys
using assms unfolding osfx-def by blast

```

```

lemma osfxE [elim?]:
  assumes osfx xs ys
  obtains zs where ys = zs  $\ominus$  xs  $\vee$  ys = xs
  using assms unfolding osfx-def by blast

```

```

lemma osfx-tl [simp]:
  assumes ilen xs > 0
  shows osfx (suffix 1 xs) xs
using assms
proof (induct xs)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case by (auto simp: osfx-def) (metis iapp.simps(1))
qed

```

```

lemma osfx-suffix [simp]:
  shows osfx (suffix i xs) xs
proof –
  have 1:  $i > \text{ilen } xs \implies \text{osfx } (\text{suffix } i \text{ } xs) \text{ } xs$ 
    by (metis ilast-iapp ilen-snoc-1 neq0-conv osfxI suffix-gr suffix-ilen-last suffix-zero)
  have 2:  $i \leq \text{ilen } xs \wedge i = 0 \implies \text{osfx } (\text{suffix } i \text{ } xs) \text{ } xs$ 
    by simp
  have 3:  $i \leq \text{ilen } xs \wedge 0 < i \implies \text{osfx } (\text{suffix } i \text{ } xs) \text{ } xs$ 
    unfolding osfx-def using iapp-prefix-suffix[of i-1 xs]
    by (metis One-nat-def Suc-eq-plus1 Suc-pred less-le-trans)

```

show *?thesis*
using 1 2 3 *leI* **by** *blast*
qed

lemma *osfx-prefix* [*simp*]:
assumes *osfx xs ys*
shows $ys = (\text{prefix } (\text{ilen } ys - \text{ilen } xs - 1) \text{ } ys) \ominus xs \quad \vee \quad ys = xs$
using *assms*
by (*auto simp add: osfx-def*)
(metis diff-zero prefix-iapp prefix-ilen)

lemma *osfx-snoc* [*simp*]:
 $osfx \text{ } xs \text{ } (ys \ominus \langle y \rangle) \longleftrightarrow$
 $xs = \langle y \rangle \vee (\exists zs. xs = zs \ominus \langle y \rangle \wedge osfx \text{ } zs \text{ } ys)$
proof (*cases xs*)
case (*INil x1*)
then show *?thesis* **by** (*metis ilast-iapp ilast-iapp2 osfxI osfx-prefix*)
next
case (*ICons x21 x22*)
then show *?thesis*
proof *auto*
show $xs = x21 \odot x22 \implies osfx \text{ } (x21 \odot x22) \text{ } (ys \ominus \langle y \rangle) \implies \exists zs. x21 \odot x22 = zs \ominus \langle y \rangle \wedge osfx \text{ } zs \text{ } ys$
using *iapp-eq-iapp-conv2 iapp-not-INil unfolding osfx-def*
by (*metis interval.distinct(1)*)
show $\bigwedge zs. xs = zs \ominus \langle y \rangle \implies x21 \odot x22 = zs \ominus \langle y \rangle \implies osfx \text{ } zs \text{ } ys \implies osfx \text{ } (zs \ominus \langle y \rangle) \text{ } (ys \ominus \langle y \rangle)$
unfolding *osfx-def* **by** (*metis iapp-assoc*)
qed
qed

lemma *snoc-osfx-snoc* [*simp*]:
 $osfx \text{ } (xs \ominus \langle x \rangle) \text{ } (ys \ominus \langle y \rangle) = (x = y \wedge osfx \text{ } xs \text{ } ys)$
by (*simp add: osfx-def*)
(metis iapp-assoc iapp-eq-conv)

lemma *same-osfx-osfx* [*simp*]:
 $osfx \text{ } (ys \ominus xs) \text{ } (zs \ominus xs) = osfx \text{ } ys \text{ } zs$
unfolding *osfx-def*
by (*metis iapp-assoc iapp-same-eq*)

lemma *same-suffix-INil* [*simp*]:
 $osfx \text{ } (ys \ominus xs) \text{ } (x \odot xs) = (ys = \langle x \rangle)$
unfolding *osfx-def*
by (*metis iapp-INil iapp-assoc iapp-not-INil iapp-same-eq*)

lemma *osfx-ICons*:
 $osfx \text{ } xs \text{ } (y \odot ys) \longleftrightarrow xs = y \odot ys \vee osfx \text{ } xs \text{ } ys$
unfolding *osfx-def*
by (*auto simp: ICons-eq-iapp-conv*)

lemma *osfx-iapp*:
 $osfx\ xs\ (ys\ \ominus\ zs) \longleftrightarrow$
 $osfx\ xs\ zs\ \vee\ (\exists\ xs'.\ xs = xs' \ominus zs \wedge osfx\ xs'\ ys)$
by (*auto simp: osfx-def iapp-eq-iapp-conv2*)

lemma *osfx-ilen*:
assumes *osfx xs ys*
shows $ilen\ xs \leq ilen\ ys$
using *assms* **by** (*auto simp add: osfx-def*)

lemma *osfx-same-cases*:
assumes *osfx xs₁ ys*
 $osfx\ xs_2\ ys$
shows $osfx\ xs_1\ xs_2 \vee osfx\ xs_2\ xs_1$
using *assms* **unfolding** *osfx-def* **by** (*metis iapp-eq-iapp-conv2*)

lemma *osfx-ilen-osfx*:
assumes *osfx ps xs*
 $osfx\ qs\ xs$
 $ilen\ ps \leq ilen\ qs$
shows $osfx\ ps\ qs$
using *assms*
by (*auto simp: osfx-def iapp-eq-iapp-conv2*)

lemma *osfx-ilen-less*:
assumes *osfx xs ys*
shows $ilen\ xs < ilen\ ys$
using *assms* **by** (*auto simp: osfx-def*)

lemma *osfx-IConsD'*:
assumes *osfx (x \odot xs) ys*
shows *osfx xs ys*
using *assms*
by (*metis One-nat-def add-diff-cancel-right' diff-is-0-eq ilen.simps(2) not-one-le-zero osfx-ICons osfx-ilen osfx-order.dual-order.refl osfx-order.dual-order.trans plus-1-eq-Suc osfx-def*)

lemma *suffix-sosfx*:
assumes *sosfx xs ys*
shows $sosfx\ (suffix\ n\ xs)\ ys$
using *assms*
proof (*induct n arbitrary: xs ys*)
case 0
then show ?*case* **by** *simp*
next
case (*Suc n*)
then show ?*case*
proof (*cases xs*)
case (*INil x1*)
then show ?*thesis* **using** *Suc.prem*s **by** *auto*


```

  next
  case (ICons x21 x22)
  then show ?thesis
  using Suc.hyps Suc.premis osfx-IConsD' osfx-order.less-imp-le by fastforce
qed
qed

```

```

lemma sosfx-tl [simp]:
  assumes ilen xs > 0
  shows   sosfx (suffix 1 xs) xs
using   assms
proof (induct xs)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case using osfx-IConsD' by force
qed

```

```

lemma INil-osfx [simp]:
  osfx ⟨ilast xs⟩ xs
by (metis osfx-suffix suffix-ilen-last)

```

```

lemma osfx-INil [simp]:
  (osfx xs ⟨ilast xs⟩) = (xs = ⟨ilast xs⟩)
by (simp add: osfx-order.antisym-conv)

```

```

lemma osfx-IConsI:
  assumes osfx xs ys
  shows   osfx xs (x⊙ ys)
using   assms
by (simp add: osfx-ICons)

```

```

lemma osfx-IConsD:
  assumes osfx (x⊙xs) ys
  shows   osfx xs ys
using   assms
by (meson osfx-IConsD' osfx-order.less-imp-le)

```

```

lemma osfx-iappI:
  assumes osfx xs ys
  shows   osfx xs (zs ⊖ ys)
using   assms by (metis iapp-assoc osfx-def)

```

```

lemma osfx-iappD:
  assumes osfx (zs ⊖ xs) ys
  shows   osfx xs ys
using   assms osfxI osfx-order.dual-order.trans by blast

```

```

lemma sosfx-iset-subset:
assumes sosfx xs ys
shows  $iset\ xs \subseteq iset\ ys$ 
using assms by (auto simp: sosfx-def osfx-def)

lemma iset-mono-osfx:
assumes osfx xs ys
shows  $iset\ xs \subseteq iset\ ys$ 
using assms by (auto simp: osfx-def)

lemma osfx-IConsD2:
assumes osfx (x⊙xs) (y⊙ys)
shows osfx xs ys
using assms
proof –
  assume osfx (x⊙xs) (y⊙ys)
  then obtain zs where  $y⊙ys = zs ⊖ (x⊙xs) \vee y⊙ys = x⊙xs$ 
    using osfxE by blast
  then show ?thesis
  by (metis assms interval.inject(2) osfx-IConsD osfx-ICons)
qed

lemma osfx-to-opfx [code]:
   $osfx\ xs\ ys \longleftrightarrow opfx\ (irev\ xs)\ (irev\ ys)$ 
unfolding opfx-def
by (metis irev-iapp irev-irev-ident osfx-def)

lemma sosfx-to-sopfx [code]:
   $sosfx\ xs\ ys \longleftrightarrow sopfx\ (irev\ xs)\ (irev\ ys)$ 
by (auto simp: osfx-to-opfx sosfx-def sopfx-def)

lemma imap-mono-osfx:
assumes osfx xs ys
shows osfx (imap f xs) (imap f ys)
using assms by (auto elim!: osfxE intro: osfxI)

lemma prefix-subset:
assumes  $k \leq len\ xs$ 
shows  $iset\ (prefix\ k\ xs) \leq iset\ xs$ 
using assms by (simp add: prefix-is-opfx iset-mono-opfx)

lemma suffix-subset:
assumes  $k \leq len\ xs$ 
shows  $iset\ (suffix\ k\ xs) \leq iset\ xs$ 
using assms by (simp add: iset-mono-osfx)

```

13.2.2 idistinct and iremdups

lemma *idistinct-iapp* [simp]:
 $idistinct\ (xs \ominus ys) = (idistinct\ xs \wedge idistinct\ ys \wedge iset\ xs \cap iset\ ys = \{\})$
by (*induct xs*) *auto*

lemma *idistinct-osfx*:
assumes *idistinct ys*
 $osfx\ xs\ ys$
shows *idistinct xs*
using *assms*
proof (*clarsimp elim!: osfxE*)
 show $\bigwedge zs. idistinct\ ys \implies ys = zs \ominus xs \vee ys = xs \implies idistinct\ xs$
 using *idistinct-iapp* **by** *blast*
qed

lemma *idistinct-tl*:
assumes *idistinct xs*
 $ilen\ xs > 0$
shows *idistinct (suffix 1 xs)*
using *assms*
by (*cases xs*) *auto*

lemma *idistinct-irev* [simp]:
 $idistinct\ (irev\ xs) = idistinct\ xs$
by (*induct xs*) *auto*

lemma *iset-iremdups* [simp]:
 $iset\ (iremdups\ xs) = iset\ xs$
by (*induct xs*) *auto*

lemma *idistinct-iremdups* [iff]:
 $idistinct\ (iremdups\ xs)$
by (*induct xs*) *auto*

lemma *idistinct-iremdups-id*:
assumes *idistinct xs*
shows $iremdups\ xs = xs$
using *assms*
by (*induct xs*) *auto*

lemma *iremdups-id-iff-idistinct* [simp]:
 $iremdups\ xs = xs \longleftrightarrow idistinct\ xs$
by (*metis idistinct-iremdups idistinct-iremdups-id*)

lemma *idistinct-imap*:
 $idistinct\ (imap\ f\ xs) = (idistinct\ xs \wedge inj_on\ f\ (iset\ xs))$
by (*induct xs*) *auto*

lemma *idistinct-prefix* [simp]:
assumes *idistinct xs*

```

shows   idistinct (prefix i xs)
using assms
proof (induct xs arbitrary: i)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases i)
  case 0
  then show ?thesis by auto
  next
  case (Suc nat)
  then show ?thesis
  using ICons.hyps ICons.premys prefix-is-opfx iset-mono-opfx by fastforce
  qed
qed

```

```

lemma idistinct-suffix [simp]:
assumes idistinct xs
shows   idistinct (suffix i xs)
using assms
proof
  (induct xs arbitrary: i)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof (cases i)
    case 0
    then show ?thesis using ICons.premys by auto
    next
    case (Suc nat)
    then show ?thesis using iapp-prefix-suffix ICons.hyps ICons.premys by auto
    qed
  qed

```

```

lemma idistinct-conv-inth:
  idistinct xs =
    ( $\forall i \leq \text{ilen } xs. (\forall j \leq \text{ilen } xs. i \neq j \longrightarrow \text{inth } xs \ i \neq \text{inth } xs \ j)$ )
proof
  (induction xs)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    by (auto simp add: ICons inth-ICons inth-and-iset split: nat.split-asm)

```

blast+
qed

lemma *idistinct-Ex1*:
assumes *idistinct xs*
 $x \in \text{iset } xs$
shows $(\exists !i. i \leq \text{ilen } xs \wedge (\text{inth } xs \ i) = x)$
using *assms*
by (*metis idistinct-conv-inth inth-and-iset*)

lemma *inj-on-inth*:
assumes *idistinct xs*
shows $(\forall i \in I. i \leq \text{ilen } xs \implies \text{inj-on } (\text{inth } xs) \ I)$
using *assms*
by (*meson idistinct-conv-inth inj-onI*)

lemma *bij-betw-inth*:
assumes *idistinct xs*
 $A = \{..
 $B = \text{iset } xs$
shows *bij-betw* $((\text{inth}) \ xs) \ A \ B$
using *assms* **unfolding** *bij-betw-def*
proof (*auto intro!: inj-on-inth simp:*)
show $\bigwedge xa. \text{idistinct } xs \implies$
 $A = \{..
 $B = \text{iset } xs \implies xa < \text{Suc } (\text{ilen } xs) \implies \text{inth } xs \ xa \in \text{iset } xs$
by (*meson inth-and-iset less-Suc-eq-le*)
show $\bigwedge x. \text{idistinct } xs \implies$
 $A = \{..
 $B = \text{iset } xs \implies x \in \text{iset } xs \implies x \in \text{inth } xs \text{ ‘ } \{..
using *inth-and-iset* **by** (*metis image-iff lessThan-iff less-Suc-eq-le*)
qed$$$$

lemma *card-idistinct*:
assumes $\text{card } (\text{iset } xs) = \text{ilen } xs + 1$
shows *idistinct xs*
using *assms*
proof
(induct xs)
case (*INil x*)
then show *?case* **by** *simp*
next
case (*ICons x1a xs*)
then show *?case*
proof (*cases x1a \in iset xs*)
case *True*
then show *?thesis*
by (*metis ICons.premis iremdups.simps(2) iset-iremdups add-Suc card-ilen ilen.simps(2)*
not-less-eq-eq plus-1-eq-Suc)

```

next
case False
then show ?thesis using ICons.hyps ICons.prem by auto
qed
qed

```

```

lemma finite-interval:
  assumes finite A
  shows  $(A \neq \{\}) \longrightarrow (\exists xs. \text{iset } xs = A)$ 
  using assms
  proof (induct rule:finite-induct)
  case empty
  then show ?case by simp
  next
  case (insert x F)
  then show ?case by (metis insert-is-Un interval.set(1) iset-iapp)
  qed

```

```

lemma finite-idistinct-interval:
  assumes finite A
  shows  $A \neq \{\}$ 
  shows  $(\exists xs. \text{iset } xs = A \wedge \text{idistinct } xs)$ 
  using assms by (metis idistinct-iremdups finite-interval iset-iremdups)

```

```

lemma iremdups-eq-INil-iff [simp]:
   $(\text{iremdups } xs = \langle x \rangle) = (\forall i \leq \text{ilen } xs. (\text{inth } xs \ i) = x)$ 
  proof
  (induct xs)
  case (INil x)
  then show ?case by auto
  next
  case (ICons x1a xs)
  then show ?case
  proof -
  have 1:  $(\forall i \leq \text{ilen } (x1a \odot xs). \text{inth } (x1a \odot xs) \ i = x) =$ 
     $(x1a = x \wedge (\forall i \leq \text{ilen } (xs). \text{inth } (xs) \ i = x))$ 
    by auto
    (metis Nitpick.case-nat-unfold Suc-eq-plus1 le-diff-conv)
  have 2:  $(\text{iremdups } (x1a \odot xs) = \langle x \rangle) =$ 
     $((\text{if } x1a \in \text{iset } xs \text{ then } \text{iremdups } xs \text{ else } x1a \odot \text{iremdups } xs) = \langle x \rangle)$ 
    by simp
  have 3:  $((\text{if } x1a \in \text{iset } xs \text{ then } \text{iremdups } xs \text{ else } x1a \odot \text{iremdups } xs) = \langle x \rangle) =$ 
     $(\text{iremdups } xs = \langle x \rangle \wedge x1a \in \text{iset } xs)$ 
    by auto
  have 4:  $(\text{iremdups } xs = \langle x \rangle \wedge x1a \in \text{iset } xs) =$ 
     $(\forall i \leq \text{ilen } (xs). \text{inth } (xs) \ i = x) \wedge x1a \in \text{iset } xs$ 
    using ICons.hyps by blast
  have 5:  $(\forall i \leq \text{ilen } (xs). \text{inth } (xs) \ i = x) \wedge x1a \in \text{iset } xs \longrightarrow x1a = x$ 
    by (metis inth-and-iset)
  

```

```

show ?thesis
by (metis 1 2 3 5 ICons.hyps iset-iremdups interval.set-intros(1))
qed
qed

```

```

lemma iremdups-eq-INil-right-iff [simp]:
  ( $\langle x \rangle = \text{iremdups } xs$ ) = ( $\forall i \leq \text{ilen } xs. (\text{inth } xs \ i) = x$ )
by (metis iremdups-eq-INil-iff)

```

```

lemma ilen-remdups-leq [iff]:
   $\text{ilen}(\text{iremdups } xs) \leq \text{ilen } xs$ 
by (induct xs) auto

```

```

lemma ilen-iremdups-eq[iff]:
  ( $\text{ilen}(\text{iremdups } xs) = \text{ilen } xs$ ) = ( $\text{iremdups } xs = xs$ )
proof
  (induct xs)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
  by (metis ilen-remdups-leq iremdups.simps(2) add-left-cancel ilen.simps(2) not-less-eq-eq
    plus-1-eq-Suc)
qed

```

```

lemma idistinct-card:
  assumes idistinct xs
  shows  $\text{card}(\text{iset } xs) = \text{Suc}(\text{ilen } xs)$ 
using assms
by (induct xs) simp-all

```

13.2.3 prefixes and suffixes

```

lemma in-iset-prefixes [simp]:
   $xs \in \text{iset } (\text{prefixes } ys) \longleftrightarrow \text{opfx } xs \ ys$ 
proof
  (induct xs arbitrary: ys)
  case (INil x)
  then show ?case
  proof (cases ys)
    case (INil x1)
    then show ?thesis using opfxE by force
  next
    case (ICons x21 x22)
    then show ?thesis unfolding opfx-def by auto
  qed
next
case (ICons x1a xs)
then show ?case

```

```

proof (cases ys)
case (INil x1)
then show ?thesis by (simp add: opfx-code(2))
next
case (ICons x21 x22)
then show ?thesis using ICons.hyps by auto
qed
qed

```

```

lemma ilen-prefixes [simp]:
  ilen (prefixes xs) = ilen xs
by (induction xs) auto

```

```

lemma idistinct-prefixes [intro]:
  idistinct (prefixes xs)
proof (induction xs)
case (INil x)
then show ?case by (auto simp: idistinct-imap)
next
case (ICons x1a xs)
then show ?case by (auto simp: idistinct-imap)
  (meson inj-onI interval.inject(2))
qed

```

```

lemma prefixes-snoc [simp]:
  prefixes (xs ⊖ ⟨x⟩) = (prefixes xs) ⊖ ⟨xs ⊖ ⟨x⟩⟩
by (induction xs) auto

```

```

lemma ifirst-prefixes [simp]:
  ifirst (prefixes xs) = ⟨ifirst xs⟩
by (cases xs) auto

```

```

lemma ilast-prefixes [simp]:
  ilast (prefixes xs) = xs
by (induction xs)
  (simp-all add: ilast-imap inth-imap)

```

```

lemma prefixes-iapp:
  prefixes (xs ⊖ ys) =
    prefixes xs ⊖ imap (λys'. xs ⊖ ys') (prefixes ys)
proof
  (induction xs arbitrary: ys)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof –

```



```

have 1:  $\text{prefixes } ((x1a \odot xs) \ominus ys) = \langle x1a \rangle \odot (\text{imap } ((\odot) x1a) (\text{prefixes } (xs \ominus ys)))$ 
  by simp
have 2:  $\text{prefixes } (xs \ominus ys) = \text{prefixes } xs \ominus \text{imap } ((\ominus) xs) (\text{prefixes } ys)$ 
  by (simp add: ICons.IH)
have 3:  $\langle x1a \rangle \odot (\text{imap } ((\odot) x1a) (\text{prefixes } (xs \ominus ys))) =$ 
   $\langle x1a \rangle \odot (\text{imap } ((\odot) x1a) (\text{prefixes } xs \ominus \text{imap } ((\ominus) xs) (\text{prefixes } ys)))$ 

  using 2 by auto
show ?thesis by (simp add: 3)
qed
qed

```

```

lemma prefixes-eq-snoc:
   $\text{prefixes } ys = xs \ominus \langle x \rangle \longleftrightarrow$ 
   $(\exists z \text{ zs. } ys = zs \ominus \langle z \rangle \wedge xs = \text{prefixes } zs) \wedge x = ys$ 
proof (cases ys rule: irev-cases)
case (INil x)
then show ?thesis using iapp-not-INil by (metis prefixes.simps(1))
next
case (snoc ys y)
then show ?thesis by auto
qed

```

```

lemma iset-prefixes-eq:
   $\text{iset } (\text{prefixes } xs) = \{ys. \text{opfx } ys \text{ } xs\}$ 
by auto

```

```

lemma card-iset-prefixes [simp]:
   $\text{card } (\text{iset } (\text{prefixes } xs)) = \text{Suc } (\text{ilen } xs)$ 
by (simp add: idistinct-card idistinct-prefixes)

```

```

lemma iset-prefixes-append:
   $\text{iset } (\text{prefixes } (xs \ominus ys)) = \text{iset } (\text{prefixes } xs) \cup \{xs \ominus ys' \mid ys'. ys' \in \text{iset } (\text{prefixes } ys)\}$ 
by (subst prefixes-iapp) auto

```

```

lemma in-iset-suffixes [simp]:
   $xs \in \text{iset}(\text{suffixes } ys) \longleftrightarrow \text{osfx } xs \text{ } ys$ 
proof (induct ys)
case (INil x)
then show ?case using osfx-prefix by fastforce
next
case (ICons x1a ys)
then show ?case by (simp add: osfx-ICons)
qed

```

```

lemma interval-sfx-INil:
   $\text{iset}(\text{suffixes } \langle x \rangle) = \{\langle x \rangle\}$ 
by simp

```

lemma *interval-sfx-ICons*:

$iset(suffixes (x \odot xs)) = \{x \odot xs\} \cup iset(suffixes xs)$

by *auto*

lemma *iset-suffixes-sfx*:

$iset (suffixes xs) = \{suffix\ i\ xs \mid i. i \leq len\ xs\}$

proof

(*induction xs*)

case (*INil x*)

then show ?*case* **by** *auto*

next

case (*ICons x1a xs*)

then show ?*case*

proof –

have 1: $\{suffix\ i\ (x1a \odot xs) \mid i. i \leq len\ (x1a \odot xs)\} =$
 $\{suffix\ i\ (x1a \odot xs) \mid i. i = 0\} \cup \{suffix\ i\ (x1a \odot xs) \mid i. 0 < i \wedge i \leq len\ (x1a \odot xs)\}$

using *neq0-conv* **by** (*auto, fastforce, force*)

have 2: $\{suffix\ i\ (x1a \odot xs) \mid i. 0 < i \wedge i \leq len\ (x1a \odot xs)\} =$
 $\{suffix\ i\ (xs) \mid i. i \leq len\ (xs)\}$

by (*auto,*

metis Nitpick.case-nat-unfold Suc-eq-plus1 le-diff-conv neq0-conv,

metis Nitpick.case-nat-unfold Suc-le-mono add-diff-cancel-left' gr-implies-not-zero

less-Suc0 not-less-eq plus-1-eq-Suc)

show ?*thesis* **using** 1 2 *ICons.IH* **by** *auto*

qed

qed

lemma *iset-suffixes-osfx*:

$iset(suffixes xs) = \{ys. osfx\ ys\ xs\}$

by *auto*

lemma *idistinct-suffixes [intro]*:

$idistinct(suffixes xs)$

proof (*induct xs*)

case (*INil x*)

then show ?*case* **by** *simp*

next

case (*ICons x1a xs*)

then show ?*case*

by (*metis in-iset-suffixes iremdups.simps(2) iremdups-id-iff-idistinct osfx-IConsI*
osfx-order.dual-order.eq-iff suffixes.simps(2))

qed

lemma *ilen-suffixes [simp]*:

$ilen (suffixes xs) = ilen\ xs$

by (*induct xs*) *simp-all*

lemma *suffixes-snoc [simp]*:

$suffixes (xs \ominus \langle x \rangle) = (imap\ (\lambda\ ys. ys \ominus \langle x \rangle)\ (suffixes\ xs)) \ominus \langle \langle x \rangle \rangle$

by (*induct xs*) *simp-all*

lemma *ifirst-suffixes* [*simp*]:

ifirst (*suffixes xs*) = *xs*

by (*induct xs*) *simp-all*

lemma *ilast-suffixes* [*simp*]:

ilast (*suffixes xs*) = $\langle \text{ilast } xs \rangle$

by (*induct xs*) *simp-all*

lemma *suffixes-iapp*:

suffixes (*xs* \ominus *ys*) = *imap* ($\lambda \text{ xs'}. \text{xs'} \ominus \text{ys}$) (*suffixes xs*) \ominus (*suffixes ys*)

proof

(*induction xs arbitrary: ys*)

case (*INil x*)

then show ?*case* **by** *simp*

next

case (*ICons x1a ys*)

then show ?*case*

proof (*cases ys*)

case (*INil x1*)

then show ?*thesis* **by** *simp*

next

case (*ICons x21 x22*)

then show ?*thesis* **by** (*auto simp add: ICons.IH*)

qed

qed

lemma *card-iset-suffixes* [*simp*]:

card (*iset* (*suffixes xs*)) = *Suc* (*ilen xs*)

by (*simp add: idistinct-card idistinct-suffixes*)

lemma *iset-suffixes-iapp*:

iset (*suffixes* (*xs* \ominus *ys*)) = $\{\text{xs'} \ominus \text{ys} \mid \text{xs'}. \text{xs'} \in \text{iset} (\text{suffixes } \text{xs})\} \cup \text{iset} (\text{suffixes } \text{ys})$

proof (*subst suffixes-iapp*)

show *iset* (*imap* ($\lambda \text{ xs'}. \text{xs'} \ominus \text{ys}$) (*suffixes xs*) \ominus *suffixes ys*) =

$\{\text{xs'} \ominus \text{ys} \mid \text{xs'}. \text{xs'} \in \text{iset} (\text{suffixes } \text{xs})\} \cup \text{iset} (\text{suffixes } \text{ys})$

proof (*cases xs*)

case (*INil x1*)

then show ?*thesis* **by** *simp*

next

case (*ICons x21 x22*)

then show ?*thesis* **by** *force*

qed

qed

lemma *imap-first-suffixes* [*simp*]:

imap ($\lambda \text{ xs}. \text{inth } \text{xs } 0$) (*suffixes xs*) = *xs*

by (*induct xs*) *auto*

lemma *suffixes-conv-prefixes*:
 $(\text{suffixes } xs) = \text{irev } (\text{imap } \text{irev } (\text{prefixes } (\text{irev } xs)))$
by (*induction xs*) *auto*

lemma *prefixes-conv-suffixes*:
 $(\text{prefixes } xs) = \text{irev } (\text{imap } \text{irev } (\text{suffixes } (\text{irev } xs)))$
by (*induction xs*) (*auto simp add: irev-omap*)

lemma *prefixes-irev*:
 $\text{prefixes } (\text{irev } xs) = \text{irev } (\text{imap } \text{irev } (\text{suffixes } xs))$
by (*induction xs*) *auto*

lemma *suffixes-irev*:
 $\text{suffixes } (\text{irev } xs) = \text{irev } (\text{imap } \text{irev } (\text{prefixes } xs))$
by (*induction xs*) (*auto simp add: irev-omap*)

lemma *inth-suffixes*:
assumes $i \leq \text{ilen}(\text{suffixes } xs)$
shows $\text{inth } (\text{suffixes } xs) \ i = (\text{suffix } i \ xs)$
using *assms*
proof (*induct xs arbitrary: i*)
case (*INil x*)
then show ?*case* **by** *simp*
next
case (*ICons x1a xs*)
then show ?*case* **by** (*simp add: Nitpick.case-nat-unfold*)
qed

lemma *suffix-suffixes*:
assumes $i \leq \text{ilen } (\text{suffixes } xs)$
shows $\text{suffix } i \ (\text{suffixes } xs) = \text{suffixes } (\text{suffix } i \ xs)$
using *assms*
proof (*induct xs arbitrary: i*)
case (*INil x*)
then show ?*case* **by** *simp*
next
case (*ICons x1a xs*)
then show ?*case* **by** (*simp add: Nitpick.case-nat-unfold*)
qed

13.2.4 ifilter and nfilter

lemma *sfxfilter-ilen-a*:
assumes $\exists \ ys \in \text{iset } (\text{suffixes } \langle x \rangle). \ P \ ys$
shows $\text{ilen } (\text{ifilter } P \ (\text{suffixes } \langle x \rangle)) = 0$
using *assms* **by** *simp*

lemma *sfxfilter-ilen-b*:

assumes $(\exists \text{ ys} \in \text{iset} (\text{suffixes } (x \odot xs)). P \text{ ys})$
 $(\exists \text{ ys} \in \text{iset} (\text{suffixes } xs). P \text{ ys})$
 $P (x \odot xs)$
shows $\text{ilen} (\text{ifilter } P (\text{suffixes } (x \odot xs))) = \text{ilen}(\text{ifilter } P (\text{suffixes } xs)) + 1$
using *assms* **by** *simp*

lemma *sfxfilter-ilen-c*:

assumes $(\exists \text{ ys} \in \text{iset} (\text{suffixes } (x \odot xs)). P \text{ ys})$
 $(\exists \text{ ys} \in \text{iset} (\text{suffixes } xs). P \text{ ys})$
 $\neg P (x \odot xs)$
shows $\text{ilen} (\text{ifilter } P (\text{suffixes } (x \odot xs))) = \text{ilen}(\text{ifilter } P (\text{suffixes } xs))$
using *assms* **by** *simp*

lemma *sfxfilter-ilen-d*:

assumes $(\exists \text{ ys} \in \text{iset} (\text{suffixes } (x \odot xs)). P \text{ ys})$
 $\neg(\exists \text{ ys} \in \text{iset} (\text{suffixes } xs). P \text{ ys})$
shows $\text{ilen} (\text{ifilter } P (\text{suffixes } (x \odot xs))) = 0$
using *assms* **by** *simp*

lemma *ifilter-ilen-a*:

assumes $\exists \text{ ys} \in \text{iset } \langle x \rangle. P \text{ ys}$
shows $\text{ilen} (\text{ifilter } P \langle x \rangle) = 0$
using *assms* **by** *simp*

lemma *nfilter-ilen-a*:

assumes $\exists \text{ ys} \in \text{iset } \langle x \rangle. P \text{ ys}$
shows $\text{ilen} (\text{nfilter } P \langle x \rangle n) = 0$
using *assms* **by** *simp*

lemma *ifilter-ilen-b*:

assumes $(\exists \text{ ys} \in \text{iset} (x \odot xs). P \text{ ys})$
 $(\exists \text{ ys} \in \text{iset} (xs). P \text{ ys})$
 $P (x)$
shows $\text{ilen} (\text{ifilter } P (x \odot xs)) = \text{ilen}(\text{ifilter } P (xs)) + 1$
using *assms* **by** *simp*

lemma *nfilter-ilen-b*:

assumes $(\exists \text{ ys} \in \text{iset} (x \odot xs). P \text{ ys})$
 $(\exists \text{ ys} \in \text{iset} (xs). P \text{ ys})$
 $P (x)$
shows $\text{ilen} (\text{nfilter } P (x \odot xs) n) = \text{ilen}(\text{nfilter } P (xs) (\text{Suc } n)) + 1$
using *assms* **by** *simp*

lemma *ifilter-ilen-c*:

assumes $(\exists \text{ ys} \in \text{iset} (x \odot xs). P \text{ ys})$
 $(\exists \text{ ys} \in \text{iset} (xs). P \text{ ys})$
 $\neg P (x)$
shows $\text{ilen} (\text{ifilter } P (x \odot xs)) = \text{ilen}(\text{ifilter } P (xs))$
using *assms* **by** *simp*

lemma *nfilter-ilen-c:*

assumes $(\exists \text{ } ys \in \text{iset } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (xs). P \text{ } ys)$
 $\neg P \text{ } (x)$

shows $\text{ilen } (nfilter \text{ } P \text{ } (x \odot xs) \text{ } n) = \text{ilen}(nfilter \text{ } P \text{ } (xs) \text{ } (Suc \text{ } n))$

using *assms* **by** *simp*

lemma *ifilter-ilen-d:*

assumes $(\exists \text{ } ys \in \text{iset } ((x \odot xs)). P \text{ } ys)$
 $\neg(\exists \text{ } ys \in \text{iset } (xs). P \text{ } ys)$

shows $\text{ilen } (ifilter \text{ } P \text{ } ((x \odot xs))) = 0$

using *assms* **by** *simp*

lemma *nfilter-ilen-d:*

assumes $(\exists \text{ } ys \in \text{iset } ((x \odot xs)). P \text{ } ys)$
 $\neg(\exists \text{ } ys \in \text{iset } (xs). P \text{ } ys)$

shows $\text{ilen } (nfilter \text{ } P \text{ } (x \odot xs) \text{ } n) = 0$

using *assms* **by** *simp*

lemma *sfxfilter-inth-a:*

assumes $(\exists \text{ } ys \in \text{iset } (\text{suffixes } \langle x \rangle). P \text{ } ys)$
 $j \leq \text{ilen}(ifilter \text{ } P \text{ } (\text{suffixes } \langle x \rangle))$

shows $\text{inth } (ifilter \text{ } P \text{ } (\text{suffixes } \langle x \rangle)) \text{ } j = \langle x \rangle$

using *assms* **by** *simp*

lemma *sfxfilter-inth-b1:*

assumes $(\exists \text{ } ys \in \text{iset } (\text{suffixes } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs) . P \text{ } ys)$
 $P \text{ } (x \odot xs)$

shows $\text{inth } (ifilter \text{ } P \text{ } (\text{suffixes } (x \odot xs))) \text{ } 0 = x \odot xs$

using *assms* **by** *simp*

lemma *sfxfilter-inth-b2:*

assumes $(\exists \text{ } ys \in \text{iset } (\text{suffixes } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs) . P \text{ } ys)$
 $P \text{ } (x \odot xs)$

$(Suc \text{ } j) \leq \text{ilen } (ifilter \text{ } P \text{ } (\text{suffixes } (x \odot xs)))$

shows $\text{inth } (ifilter \text{ } P \text{ } (\text{suffixes } (x \odot xs))) \text{ } (Suc \text{ } j) = \text{inth } (ifilter \text{ } P \text{ } (\text{suffixes } (xs))) \text{ } j$

using *assms* **by** *auto*

lemma *sfxfilter-inth-c:*

assumes $(\exists \text{ } ys \in \text{iset } (\text{suffixes } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs) . P \text{ } ys)$
 $\neg P \text{ } (x \odot xs)$

$j \leq \text{ilen } (ifilter \text{ } P \text{ } (\text{suffixes } (x \odot xs)))$

shows $\text{inth } (ifilter \text{ } P \text{ } (\text{suffixes } (x \odot xs))) \text{ } j = \text{inth } (ifilter \text{ } P \text{ } (\text{suffixes } (xs))) \text{ } j$

using *assms* **by** *auto*

lemma *sfxfilter-inth-d:*

assumes $(\exists \text{ } ys \in \text{iset } (\text{suffixes } (x \odot xs)). P \text{ } ys)$

$\neg(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). P \text{ } ys)$
 $j \leq \text{ilen } (\text{ifilter } P \text{ } (\text{suffixes } (x \odot xs)))$
shows $\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x \odot xs))) \text{ } j = x \odot xs$
using *assms by auto*

lemma *nfilter-inth-a:*
assumes $(\exists \text{ } ys \in \text{iset } (\langle x \rangle). P \text{ } ys)$
 $j \leq \text{ilen}(\text{nfilter } P \text{ } (\langle x \rangle) \text{ } n)$
shows $\text{inth } (\text{nfilter } P \text{ } (\langle x \rangle) \text{ } n) \text{ } j = n$
using *assms by auto*

lemma *ifilter-inth-a:*
assumes $(\exists \text{ } ys \in \text{iset } (\langle x \rangle). P \text{ } ys)$
 $j \leq \text{ilen}(\text{ifilter } P \text{ } (\langle x \rangle))$
shows $\text{inth } (\text{ifilter } P \text{ } (\langle x \rangle)) \text{ } j = x$
using *assms by simp*

lemma *nfilter-inth-b1:*
assumes $(\exists \text{ } ys \in \text{iset } (\text{ } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{ } xs). P \text{ } ys)$
 $P \text{ } (x)$
shows $\text{inth } (\text{nfilter } P \text{ } (\text{ } (x \odot xs)) \text{ } n) \text{ } 0 = n$
using *assms by simp*

lemma *ifilter-inth-b1:*
assumes $(\exists \text{ } ys \in \text{iset } (\text{ } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{ } xs). P \text{ } ys)$
 $P \text{ } (x)$
shows $\text{inth } (\text{ifilter } P \text{ } (\text{ } (x \odot xs))) \text{ } 0 = x$
using *assms by simp*

lemma *nfilter-inth-b2:*
assumes $(\exists \text{ } ys \in \text{iset } (\text{ } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{ } xs). P \text{ } ys)$
 $P \text{ } (x)$
 $(\text{Suc } j) \leq \text{ilen } (\text{nfilter } P \text{ } (\text{ } (x \odot xs)) \text{ } n)$
shows $\text{inth } (\text{nfilter } P \text{ } (\text{ } (x \odot xs)) \text{ } n) \text{ } (\text{Suc } j) = \text{inth } (\text{nfilter } P \text{ } (\text{ } (xs)) \text{ } (\text{Suc } n)) \text{ } j$
using *assms by auto*

lemma *ifilter-inth-b2:*
assumes $(\exists \text{ } ys \in \text{iset } (\text{ } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{ } xs). P \text{ } ys)$
 $P \text{ } (x)$
 $(\text{Suc } j) \leq \text{ilen } (\text{ifilter } P \text{ } (\text{ } (x \odot xs)))$
shows $\text{inth } (\text{ifilter } P \text{ } (\text{ } (x \odot xs))) \text{ } (\text{Suc } j) = \text{inth } (\text{ifilter } P \text{ } (\text{ } (xs))) \text{ } j$
using *assms by auto*

lemma *nfilter-inth-c:*
assumes $(\exists \text{ } ys \in \text{iset } (\text{ } (x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (\text{ } xs). P \text{ } ys)$

$\neg P (x)$
 $j \leq \text{ilen } (\text{nfilter } P ((x \odot xs)) n)$
shows $\text{inth } (\text{nfilter } P ((x \odot xs)) n) j = \text{inth } (\text{nfilter } P ((xs)) (\text{Suc } n)) j$
using *assms* **by** *auto*

lemma *ifilter-inth-c*:

assumes $(\exists \text{ } ys \in \text{iset } ((x \odot xs)). P \text{ } ys)$
 $(\exists \text{ } ys \in \text{iset } (xs). P \text{ } ys)$
 $\neg P (x)$
 $j \leq \text{ilen } (\text{ifilter } P ((x \odot xs)))$
shows $\text{inth } (\text{ifilter } P ((x \odot xs))) j = \text{inth } (\text{ifilter } P ((xs))) j$
using *assms* **by** *auto*

lemma *nfilter-inth-d*:

assumes $(\exists \text{ } ys \in \text{iset } ((x \odot xs)). P \text{ } ys)$
 $\neg(\exists \text{ } ys \in \text{iset } (xs). P \text{ } ys)$
 $j \leq \text{ilen } (\text{nfilter } P ((x \odot xs)) n)$
shows $\text{inth } (\text{nfilter } P ((x \odot xs)) n) j = n$
using *assms* **by** *auto*

lemma *ifilter-inth-d*:

assumes $(\exists \text{ } ys \in \text{iset } ((x \odot xs)). P \text{ } ys)$
 $\neg(\exists \text{ } ys \in \text{iset } (xs). P \text{ } ys)$
 $j \leq \text{ilen } (\text{ifilter } P ((x \odot xs)))$
shows $\text{inth } (\text{ifilter } P ((x \odot xs))) j = x$
using *assms* **by** *auto*

lemma *sfxfilter-inth-ICons*:

$\text{inth } (\text{ifilter } P (\text{suffixes } (x \odot xs))) j =$
 $(\text{if } (\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). P \text{ } ys) \text{ then}$
 $(\text{if } P (x \odot xs) \text{ then}$
 $(\text{if } j=0 \text{ then } (x \odot xs) \text{ else } \text{inth } (\text{ifilter } P (\text{suffixes } xs)) (j-1))$
 $\text{else } \text{inth } (\text{ifilter } P (\text{suffixes } xs)) j)$
 $\text{else } (x \odot xs))$
by (*simp add: Nitpick.case-nat-unfold*)

lemma *sfxfilter-inth-ICons-a*:

$\text{inth } (\text{ifilter } P (\text{suffixes } (x \odot xs))) j =$
 $(\text{if } (\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). P \text{ } ys) \text{ then}$
 $(\text{if } P (x \odot xs) \text{ then}$
 $(\text{case } j \text{ of } 0 \Rightarrow x \odot xs \mid \text{Suc } m \Rightarrow \text{inth } (\text{ifilter } P (\text{suffixes } xs)) m)$
 $\text{else } \text{inth } (\text{ifilter } P (\text{suffixes } xs)) j)$
 $\text{else } (x \odot xs))$
by (*simp add: Nitpick.case-nat-unfold*)

lemma *nfilter-inth-ICons*:

$\text{inth } (\text{nfilter } P ((x \odot xs)) n) j =$
 $(\text{if } (\exists \text{ } ys \in \text{iset } (xs). P \text{ } ys) \text{ then}$
 $(\text{if } P (x) \text{ then}$
 $(\text{if } j=0 \text{ then } (n) \text{ else } \text{inth } (\text{nfilter } P (xs) (\text{Suc } n)) (j-1))$

$else\ inth\ (nfilter\ P\ (xs)\ (Suc\ n))\ j)$
 $else\ (n))$
by (*simp add: Nitpick.case-nat-unfold*)

lemma *nfilter-inth-ICons-a:*
 $inth\ (nfilter\ P\ (x\odot xs))\ n)\ j =$
 $(if\ (\exists\ ys\ \in\ iset\ (xs).\ P\ ys)\ then$
 $(if\ P\ (x)\ then$
 $(case\ j\ of\ 0\ \Rightarrow\ n\ |\ Suc\ m\ \Rightarrow\ inth\ (nfilter\ P\ (xs)\ (Suc\ n))\ m)$
 $else\ inth\ (nfilter\ P\ (xs)\ (Suc\ n))\ j)$
 $else\ (n))$
by (*simp add: Nitpick.case-nat-unfold*)

lemma *ifilter-inth-ICons:*
 $inth\ (ifilter\ P\ (x\odot xs))\ j =$
 $(if\ (\exists\ ys\ \in\ iset\ (xs).\ P\ ys)\ then$
 $(if\ P\ (x)\ then$
 $(if\ j=0\ then\ (x)\ else\ inth\ (ifilter\ P\ (xs))\ (j-1))$
 $else\ inth\ (ifilter\ P\ (xs))\ j)$
 $else\ (x))$
by (*simp add: Nitpick.case-nat-unfold*)

lemma *ifilter-inth-ICons-a:*
 $inth\ (ifilter\ P\ (x\odot xs))\ j =$
 $(if\ (\exists\ ys\ \in\ iset\ (xs).\ P\ ys)\ then$
 $(if\ P\ (x)\ then$
 $(case\ j\ of\ 0\ \Rightarrow\ x\ |\ (Suc\ m)\ \Rightarrow\ inth\ (ifilter\ P\ (xs))\ m)$
 $else\ inth\ (ifilter\ P\ (xs))\ j)$
 $else\ (x))$
by (*simp add: Nitpick.case-nat-unfold*)

lemma *sfxfilter-inth:*
assumes $(\exists\ ys\ \in\ iset\ (suffixes\ xs).\ P\ ys)$
 $i\ \leq\ ilen\ (ifilter\ P\ (suffixes\ xs))$
shows $P\ (inth\ (ifilter\ P\ (suffixes\ xs))\ i)$
using *assms*
proof
 $(induction\ xs\ arbitrary:\ i)$
case $(INil\ x)$
then show *?case* **by** *simp*
next
case $(ICons\ x1a\ xs)$
then show *?case*
proof $(cases\ \exists\ a\ \in\ iset\ (suffixes\ xs).\ P\ a)$
case *True*
then show *?thesis* **using** *ICons* **by** $(auto\ simp\ add:\ nat.split-sels(2))$
next
case *False*
then show *?thesis* **using** *ICons* **by** *simp*
qed

qed

lemma *nfilter-ilen*:

assumes $(\exists x \in \text{iset } xs. P x)$

shows $\text{ilen}(\text{nfilter } P \text{ } xs \text{ } n) = \text{ilen } (\text{ifilter } P \text{ } xs)$

using *assms*

proof

(*induction xs arbitrary: n*)

case (*INil x*)

then show ?case by *simp*

next

case (*ICons x1a xs*)

then show ?case by *simp*

qed

lemma *nfilter-upper-bound*:

assumes $(\exists x \in \text{iset } xs. P x)$

$i \leq \text{ilen } (\text{nfilter } P \text{ } xs \text{ } n)$

shows $(\text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \text{ } i) \leq n + \text{ilen } xs$

using *assms*

proof

(*induct xs arbitrary: i n*)

case (*INil x*)

then show ?case by *simp*

next

case (*ICons x1a xs*)

then show ?case

proof (*cases $\exists x \in \text{iset } xs. P x$*)

case *True*

then show ?thesis

proof (*cases i*)

case 0

then show ?thesis using *ICons True*

by *simp-all*

(*metis ICons.prem(1) ICons.prem(2) add-Suc nfilter-ilen-c*)

next

case (*Suc nat*)

then show ?thesis using *ICons True* by *simp-all fastforce*

qed

next

case *False*

then show ?thesis using *Cons* by *simp*

qed

qed

lemma *nfilter-lower-bound*:

assumes $(\exists x \in \text{iset } xs. P x)$

$i \leq \text{ilen } (\text{nfilter } P \text{ } xs \text{ } n)$

```

shows     $n \leq (\text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \text{ } i)$ 
using assms
proof
  (induction xs arbitrary: i n)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof (cases  $\exists x \in \text{iset } xs. P x$ )
    case True
    then show ?thesis
      proof (cases i)
      case 0
      then show ?thesis using ICons True Suc-leD by (simp-all, blast)
      next
      case (Suc nat)
      then show ?thesis using ICons True by simp-all fastforce
      qed
    next
    case False
    then show ?thesis using Cons False
    by auto
    qed
  qed

```

lemma *nfilter-ifilter:*

```

assumes ( $\exists x \in \text{iset } xs. P x$ )
           $i \leq \text{ilen } (\text{nfilter } P \text{ } xs \text{ } n)$ 
shows     $(\text{inth } xs ((\text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \text{ } i) - n)) = (\text{inth } (\text{ifilter } P \text{ } xs) \text{ } i)$ 
using assms
proof
  (induct xs arbitrary: i n)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof –
      have 1:  $(\exists x \in \text{iset } xs. P x) \wedge P \text{ } x1a \wedge i = 0 \longrightarrow$ 
         $\text{inth } (x1a \odot xs) (\text{inth } (\text{nfilter } P \text{ } (x1a \odot xs) \text{ } n) \text{ } i - n) =$ 
         $\text{inth } (\text{ifilter } P \text{ } (x1a \odot xs)) \text{ } i$ 
      by auto
      have 2:  $\bigwedge j. (\exists x \in \text{iset } xs. P x) \wedge P \text{ } x1a \wedge j < \text{ilen } (\text{nfilter } P \text{ } (x1a \odot xs) \text{ } n) \longrightarrow$ 
         $\text{inth } (x1a \odot xs) (\text{inth } (\text{nfilter } P \text{ } (x1a \odot xs) \text{ } n) \text{ } (\text{Suc } j) - n) =$ 
         $\text{inth } (\text{ifilter } P \text{ } (x1a \odot xs)) \text{ } (\text{Suc } j)$ 
      by auto
      (metis ICons.hyps inth.simps(2) Suc-diff-Suc inth-Suc less-Suc-eq-le
        less-eq-Suc-le nfilter-lower-bound)
      have 3:  $(\exists x \in \text{iset } xs. P x) \wedge \neg P \text{ } x1a \wedge i = 0 \longrightarrow$ 

```

```

    inth (x1a ⊙ xs) (inth (nfilter P (x1a ⊙ xs) n) i - n) =
    inth (ifilter P (x1a ⊙ xs)) i
  using nfilter-lower-bound[of xs P 0 Suc n] ICons
  by auto
  (metis inth.simps(2) Suc-diff-Suc inth-Suc le0 less-eq-Suc-le)
  have 4:  $\bigwedge j. (\exists x \in \text{iset } xs. P x) \wedge \neg P x1a \wedge j < \text{ilen } (nfilter P (x1a \odot xs) n) \longrightarrow$ 
    inth (x1a ⊙ xs) (inth (nfilter P (x1a ⊙ xs) n) (Suc j) - n) =
    inth (ifilter P (x1a ⊙ xs)) (Suc j)
  by auto
  (metis ICons.hyps inth.simps(2) Suc-diff-Suc inth-Suc less-eq-Suc-le
    nfilter-lower-bound)
  have 5:  $\neg(\exists x \in \text{iset } xs. P x) \longrightarrow \text{inth } (x1a \odot xs) (\text{inth } (nfilter P (x1a \odot xs) n) i - n) =$ 
    inth (ifilter P (x1a ⊙ xs)) i
  by simp
  from 1 2 3 4 5 show ?thesis
  by (metis ICons.premis(2) add-less-cancel-left le-0-eq le-SucE le-imp-less-Suc plus-1-eq-Suc
    zero-induct)
qed
qed

```

```

lemma iset-ifilter [simp]:
  assumes  $(\exists x \in \text{iset } xs. P x)$ 
  shows  $\text{iset } (\text{ifilter } P xs) = \{x. x \in \text{iset } xs \wedge P x\}$ 
  using assms
  proof
    (induct xs)
    case (INil x)
    then show ?case by (simp add: Collect-conv-if)
  next
    case (ICons x1a xs)
    then show ?case by auto
  qed

```

```

lemma iset-nfilter [simp]:
  assumes  $(\exists x \in \text{iset } xs. P x)$ 
  shows  $\text{iset } (nfilter P xs n) = \{n+k \mid k. k \leq \text{ilen } xs \wedge P (\text{inth } xs k)\}$ 
  using assms
  proof
    (induction xs arbitrary: n)
    case (INil x)
    then show ?case by auto
  next
    case (ICons x1a xs)
    then show ?case
      proof -
        have 1:  $(\exists x \in \text{iset } xs. P x) \wedge P x1a \longrightarrow$ 
          iset (nfilter P (x1a ⊙ xs) n) =
           $\{n + k \mid k. k \leq \text{ilen } (x1a \odot xs) \wedge P (\text{inth } (x1a \odot xs) k)\}$ 
        using ICons by auto
        (metis (mono-tags, lifting) Nitpick.case-nat-unfold One-nat-def Suc-le-mono Suc-pred

```

```

      gr-implies-not0)+
have 2: ( $\exists x \in \text{iset } xs. P x$ )  $\wedge \neg P x1a \longrightarrow$ 
       $\text{iset } (\text{nfilter } P (x1a \odot xs) n) =$ 
       $\{n + k \mid k. k \leq \text{ilen } (x1a \odot xs) \wedge P (\text{inth } (x1a \odot xs) k)\}$ 
using ICons by auto
      (metis (mono-tags, lifting) Nitpick.case-nat-unfold One-nat-def Suc-eq-plus1 Suc-pred
      le-0-eq le-diff-conv not-le-imp-less)
have 3:  $\neg(\exists x \in \text{iset } xs. P x) \longrightarrow$ 
       $\text{iset } (\text{nfilter } P (x1a \odot xs) n) =$ 
       $\{n + k \mid k. k \leq \text{ilen } (x1a \odot xs) \wedge P (\text{inth } (x1a \odot xs) k)\}$ 
using ICons by auto
      (metis Nitpick.case-nat-unfold Suc-eq-plus1 le-diff-conv inth-iset)
show ?thesis
using 1 2 3 by linarith
qed
qed

```

```

lemma iset-minus-ifilter-out:
assumes ( $\exists z \in \text{iset } xs. (\lambda x. \neg(x=y)) z$ )
shows  $\text{iset } xs - \{y\} = \text{iset}(\text{ifilter } (\lambda x. \neg(x=y)) xs)$ 
using assms
by (induct xs) auto

```

```

lemma ifilter-ifilter [simp]:
assumes  $\exists x \in \text{iset } (\text{ifilter } Q xs). P x$ 
       $\exists x \in \text{iset } xs. Q x$ 
       $\exists x \in \text{iset } xs. P x \wedge Q x$ 
shows  $\text{ifilter } P (\text{ifilter } Q xs) = \text{ifilter } (\lambda x. P x \wedge Q x) xs$ 
using assms
by (induct xs) auto

```

```

lemma ilen-nfilter-le [simp]:
       $\text{ilen } (\text{nfilter } P xs n) \leq \text{ilen } xs$ 
by (induct xs arbitrary: n) (auto simp add: le-SucI)

```

```

lemma ilen-ifilter-le [simp]:
       $\text{ilen } (\text{ifilter } P xs) \leq \text{ilen } xs$ 
by (induct xs) (auto simp add: le-SucI)

```

```

lemma sfxfilter-bound:
assumes ( $\exists ys \in \text{iset } (\text{suffixes } xs). P ys$ )
shows  $\text{ilen } (\text{ifilter } P (\text{suffixes } xs)) \leq \text{ilen } xs$ 
using assms by (metis ilen-ifilter-le ilen-suffixes)

```

```

lemma ifilter-bound:
assumes ( $\exists ys \in \text{iset } (xs). P ys$ )
shows  $\text{ilen } (\text{ifilter } P (xs)) \leq \text{ilen } xs$ 
using assms by auto

```

```

lemma sfxfilter-inth-bound:

```

assumes $(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } ys)$
 $j \leq \text{ilen } (\text{ifilter } P \text{ } (\text{suffixes } xs))$
shows $\text{ilen } ((\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } xs)) \text{ } j)) \leq \text{ilen } xs$
using *assms iset-ifilter[of suffixes xs P]*
by (*metis (mono-tags, lifting) in-iset-suffixes mem-Collect-eq inth-iset osfx-ilen*)

lemma *sfxfilter-inth-suffix*:

assumes $(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } ys)$
 $j \leq \text{ilen } (\text{ifilter } P \text{ } (\text{suffixes } xs))$
shows $\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } xs)) \text{ } j =$
 $\text{suffix } (\text{ilen } xs - \text{ilen } (\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } xs)) \text{ } j)) \text{ } xs$
using *assms*
proof
(induction xs arbitrary: j)
case (*INil x*)
then show *?case by simp*
next
case (*ICons x1a xs*)
then show *?case*
proof –
have 1: $(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } ys) \wedge P \text{ } (x1a \odot xs) \wedge j=0 \longrightarrow$
 $\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } j =$
 $\text{suffix } (\text{ilen } (x1a \odot xs) - \text{ilen } (\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } j)) \text{ } (x1a \odot xs)$
using *Cons by auto*
have 2: $(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } ys) \wedge \neg P \text{ } (x1a \odot xs) \wedge j=0 \longrightarrow$
 $\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } j =$
 $\text{suffix } (\text{ilen } (x1a \odot xs) - \text{ilen } (\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } j)) \text{ } (x1a \odot xs)$
using *ICons sfxfilter-inth-bound[of xs P 0] Suc-diff-le by auto*
have 3: $\bigwedge x. (\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } ys) \wedge P \text{ } (x1a \odot xs) \wedge$
 $x < \text{ilen } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \longrightarrow$
 $\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } (\text{Suc } x) =$
 $\text{suffix } (\text{ilen } (x1a \odot xs) - \text{ilen } (\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } (\text{Suc } x))) \text{ } (x1a \odot xs)$
using *ICons sfxfilter-inth-bound[of xs P] Suc-diff-le in-iset-suffixes less-Suc-eq-le by auto*
have 4: $\bigwedge x. (\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } ys) \wedge \neg P \text{ } (x1a \odot xs) \wedge$
 $x < \text{ilen } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \longrightarrow$
 $\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } (\text{Suc } x) =$
 $\text{suffix } (\text{ilen } (x1a \odot xs) - \text{ilen } (\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } (\text{Suc } x))) \text{ } (x1a \odot xs)$
using *ICons sfxfilter-inth-bound[of xs P] Suc-diff-le by auto*
have 5: $\neg(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } ys) \longrightarrow$
 $\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } j =$
 $\text{suffix } (\text{ilen } (x1a \odot xs) - \text{ilen } (\text{inth } (\text{ifilter } P \text{ } (\text{suffixes } (x1a \odot xs))) \text{ } j)) \text{ } (x1a \odot xs)$
using *Cons by auto*
show *?thesis*
by (*metis 1 2 3 4 5 ICons.prem(2) Suc-less-SucD le-SucE le-imp-less-Suc le-zero-eq zero-induct*)
qed
qed

lemma *initfilter-sfxfilter-exists*:

$(\exists \text{ } ys \in \text{iset } (\text{suffixes } xs). \text{ } P \text{ } (\text{inth } ys \text{ } 0))) = (\exists \text{ } x \in \text{iset } xs. \text{ } P \text{ } x)$

by (metis inth-and-iset inth-imap ilen-suffixes imap-first-suffixes)

lemma *initfilter-sfxfilter*:

assumes $\exists ys \in \text{iset } (\text{suffixes } xs). P (\text{inth } ys 0)$

shows $\text{ifilter } P \text{ } xs = \text{imap } (\lambda s. (\text{inth } s 0)) (\text{ifilter } (\lambda ys. P(\text{inth } ys 0)) (\text{suffixes } xs))$

using *assms*

proof

(*induction xs*)

case (*INil x*)

then show ?*case* **by** *simp*

next

case (*ICons x1a xs*)

then show ?*case*

proof –

have 1: $(\exists x \in \text{iset } xs. P x) \wedge P \text{ } x1a \longrightarrow$

$\text{ifilter } P (x1a \odot xs) =$

$\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } (\lambda ys. P (\text{inth } ys 0)) (\text{suffixes } (x1a \odot xs)))$

using *ICons initfilter-sfxfilter-exists[of xs P]* **by** *auto*

have 2: $(\exists x \in \text{iset } xs. P x) \wedge \neg P \text{ } x1a \longrightarrow$

$\text{ifilter } P (x1a \odot xs) =$

$\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } (\lambda ys. P (\text{inth } ys 0)) (\text{suffixes } (x1a \odot xs)))$

using *ICons initfilter-sfxfilter-exists[of xs P]* **by** *auto*

have 3: $\neg(\exists x \in \text{iset } xs. P x) \longrightarrow$

$\text{ifilter } P (x1a \odot xs) =$

$\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } (\lambda ys. P (\text{inth } ys 0)) (\text{suffixes } (x1a \odot xs)))$

using *ICons initfilter-sfxfilter-exists[of xs P]* **by** *auto*

show ?*thesis* **using** 1 2 3 **by** *blast*

qed

qed

lemma *ifilter-inth*:

assumes $(\exists x \in \text{iset } xs. P x)$

$i \leq \text{ilen } (\text{ifilter } P \text{ } xs)$

shows $(\exists k \leq \text{ilen } xs. \text{inth}(\text{ifilter } P \text{ } xs) i = \text{inth } xs k)$

using *assms* **using** *iset-ifilter[of xs P]*

by (metis (no-types, lifting) inth-and-iset mem-Collect-eq)

lemma *interval-sfx-inth-zero*:

$\text{iset } xs = \{(\text{inth } ys 0) \mid ys. ys \in \text{iset}(\text{suffixes } xs) \}$

proof

(*induct xs*)

case (*INil x*)

then show ?*case*

by *auto*

next

case (*ICons x1a xs*)

then show ?*case*

by *auto* (metis *inth-zero*)

qed

```

lemma interval-sfx-1:
assumes  $ys \in \text{iset}(\text{suffixes } xs)$ 
shows  $(\text{inth } ys \ 0) \in \text{iset } xs$ 
using assms interval-sfx-inth-zero by fastforce

lemma sum-ilen-ifilter-compl-help:
assumes  $\exists x \in \text{iset } xs. P \ x$ 
          $\exists x \in \text{iset } xs. \neg P \ x$ 
shows  $\text{ilen } xs > 0$ 
using assms
proof
  (induct xs)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    using ilen-ICons-1 by blast
qed

lemma ifilter-id-conv:
assumes  $\exists x \in \text{iset } xs. P \ x$ 
shows  $(\text{ifilter } P \ xs = xs) = (\forall x \in \text{iset } xs. P \ x)$ 
using assms
proof
  (induct xs)
  case (INil x)
  then show ?case by auto
  next
  case (ICons x1a xs)
  then show ?case
    by (auto simp add: interval-iset-nonempty)
        (metis ilen-ifilter-le Suc-n-not-le-n ilen.simps(2) plus-1-eq-Suc)
qed

lemma sum-ilen-ifilter-compl:
assumes  $\exists x \in \text{iset } xs. P \ x$ 
          $\exists x \in \text{iset } xs. \neg P \ x$ 
shows  $\text{ilen}(\text{ifilter } P \ xs) + \text{ilen}(\text{ifilter } (\lambda x. \neg P \ x) \ xs) + 1 = \text{ilen } xs$ 
using assms
proof
  (induct xs)
  case (INil x)
  then show ?case by auto
  next
  case (ICons x1a xs)
  then show ?case
    proof –
      have 1:  $(\exists x \in \text{iset } xs. P \ x) \wedge (\exists x \in \text{iset } xs. \neg P \ x) \wedge P \ x1a \longrightarrow$ 

```



```

      ilen (ifilter P (x1a ⊙ xs)) + ilen (ifilter (λx. ¬ P x) (x1a ⊙ xs)) + 1 =
      ilen (x1a ⊙ xs)
    using ICons by auto
  have 2: (∃ x ∈ iset xs. P x) ∧ (∃ x ∈ iset xs. ¬ P x) ∧ ¬P x1a →
      ilen (ifilter P (x1a ⊙ xs)) + ilen (ifilter (λx. ¬ P x) (x1a ⊙ xs)) + 1 =
      ilen (x1a ⊙ xs)
    using ICons by auto
  have 3: ¬(∃ x ∈ iset xs. P x) →
      ilen (ifilter P (x1a ⊙ xs)) + ilen (ifilter (λx. ¬ P x) (x1a ⊙ xs)) + 1 =
      ilen (x1a ⊙ xs)
    using ICons by auto
      (metis ifilter-id-conv)
  have 4: (∃ x ∈ iset xs. P x) ∧ ¬(∃ x ∈ iset xs. ¬ P x) ∧ P x1a →
      ilen (ifilter P (x1a ⊙ xs)) + ilen (ifilter (λx. ¬ P x) (x1a ⊙ xs)) + 1 =
      ilen (x1a ⊙ xs)
    using ICons by auto
  have 5: (∃ x ∈ iset xs. P x) ∧ ¬(∃ x ∈ iset xs. ¬ P x) ∧ ¬P x1a →
      ilen (ifilter P (x1a ⊙ xs)) + ilen (ifilter (λx. ¬ P x) (x1a ⊙ xs)) + 1 =
      ilen (x1a ⊙ xs)
    using ICons by auto
      (metis ifilter-id-conv)
  show ?thesis
    using 1 2 3 4 5 by blast
qed

```

lemma ifilter-ilen-imp:

```

assumes ∃ x ∈ iset xs. P x ∧ Q x
shows   ilen (ifilter (λx. P x ∧ Q x) xs) ≤ ilen (ifilter P xs)
using   assms
proof (induct xs)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case by force
qed

```

lemma subset-ifilter:

```

assumes (∃ x ∈ iset xs. P x)
shows   iset (ifilter P xs) ≤ iset (ifilter (λ x. P x ∨ Q x) xs)
proof -
  have 1: (∃ x ∈ iset xs. P x ∨ Q x)
    using assms by blast
  have 2: iset (ifilter (λ x. P x ∨ Q x) xs) = {x ∈ iset xs. P x ∨ Q x}
    using assms iset-ifilter[of xs (λ x. P x ∨ Q x)] by blast
  have 3: iset (ifilter P xs) = {x ∈ iset xs. P x}
    using assms iset-ifilter by auto
  have 4: {x ∈ iset xs. P x} ≤ {x ∈ iset xs. P x ∨ Q x}
    by auto

```

show *?thesis* **by** (*simp add: 2 3 4*)
qed

lemma *iset-ifilter-not:*

assumes $\exists x \in \text{iset } xs. P x$
 $\exists x \in \text{iset } xs. \neg (P x)$
shows $\text{iset } (\text{ifilter } (\lambda x. \neg (P x)) xs) = \text{iset } xs - \text{iset } (\text{ifilter } P xs)$
using *assms*
proof (*induct xs*)
case (*INil x*)
then show *?case* **by** *auto*
next
case (*ICons x1a xs*)
then show *?case*
proof –
have 1: $(\exists x \in \text{iset } xs. P x) \wedge (\exists x \in \text{iset } xs. \neg (P x)) \wedge P x1a \longrightarrow$
 $\text{iset } (\text{ifilter } (\lambda x. \neg P x) (x1a \odot xs)) = \text{iset } (x1a \odot xs) - \text{iset } (\text{ifilter } P (x1a \odot xs))$
using *ICons* **by** *auto*
have 2: $(\exists x \in \text{iset } xs. P x) \wedge (\exists x \in \text{iset } xs. \neg (P x)) \wedge \neg P x1a \longrightarrow$
 $\text{iset } (\text{ifilter } (\lambda x. \neg P x) (x1a \odot xs)) = \text{iset } (x1a \odot xs) - \text{iset } (\text{ifilter } P (x1a \odot xs))$
using *ICons* **by** *auto*
have 3: $(\exists x \in \text{iset } xs. P x) \wedge \neg(\exists x \in \text{iset } xs. \neg (P x)) \wedge P x1a \longrightarrow$
 $\text{iset } (\text{ifilter } (\lambda x. \neg P x) (x1a \odot xs)) = \text{iset } (x1a \odot xs) - \text{iset } (\text{ifilter } P (x1a \odot xs))$
using *ICons* **by** *auto*
have 4: $(\exists x \in \text{iset } xs. P x) \wedge \neg(\exists x \in \text{iset } xs. \neg (P x)) \wedge \neg P x1a \longrightarrow$
 $\text{iset } (\text{ifilter } (\lambda x. \neg P x) (x1a \odot xs)) = \text{iset } (x1a \odot xs) - \text{iset } (\text{ifilter } P (x1a \odot xs))$
using *ICons* **by** *auto*
have 5: $\neg(\exists x \in \text{iset } xs. P x) \longrightarrow$
 $\text{iset } (\text{ifilter } (\lambda x. \neg P x) (x1a \odot xs)) = \text{iset } (x1a \odot xs) - \text{iset } (\text{ifilter } P (x1a \odot xs))$
using *ICons* **by** *auto*
show *?thesis* **using** 1 2 3 4 5 **by** *linarith*
qed
qed

lemma *iset-ifilter-cap:*

assumes $\exists x \in \text{iset } xs. f x$
 $\exists x \in \text{iset } xs. \neg f x$
shows $\text{iset } (\text{ifilter } f xs) \cap \text{iset } (\text{ifilter } (\lambda x. \neg f x) xs) = \{\}$
using *assms* **by** *auto*

lemma *ifilter-inth-or:*

assumes $\exists x \in \text{iset } xs. P x$
shows $\exists x \in \text{iset } (\text{ifilter } (\lambda x. P x \vee Q x) xs). P x$
proof –
have 1: $\exists i \leq \text{ilen}(\text{ifilter } (\lambda x. P x \vee Q x) xs). P (\text{inth } (\text{ifilter } (\lambda x. P x) xs) i)$
using *assms* **by** (*metis (mono-tags, lifting) iset-ifilter ilen-gr-zero*
mem-Collect-eq inth-iset)
obtain *i* **where** 2: $i \leq \text{ilen}(\text{ifilter } (\lambda x. P x \vee Q x) xs) \wedge P (\text{inth } (\text{ifilter } (\lambda x. P x) xs) i) \wedge$
 $(\text{inth } (\text{ifilter } (\lambda x. P x) xs) i) \in \text{iset } (\text{ifilter } P xs)$
using 1

```

    by (metis (mono-tags, lifting) iset-ifilter assms ilen-gr-zero
        mem-Collect-eq inth-iset)
  have 3: iset (ifilter P xs) ≤ iset (ifilter (λx. P x ∨ Q x) xs)
    using assms subset-ifilter[of xs P] by auto
  have 4: (inth (ifilter (λx. P x) xs) i) ∈ iset (ifilter P xs)
    using 2 by auto
  have 5: (inth (ifilter (λx. P x) xs) i) ∈ iset (ifilter (λx. P x ∨ Q x) xs)
    using 3 4 by blast
  from 2 5 show ?thesis by blast
qed

```

```

lemma ifilter-iapp1:
  assumes ∀ x ∈ iset xs. ¬ P x
    ∃ y ∈ iset ys. P y
  shows ifilter P (xs ⊖ ys) = ifilter P ys
  using assms
  by (induct xs) auto

```

```

lemma ifilter-iapp2:
  assumes ∀ x ∈ iset xs. ¬ P x
    ∃ y ∈ iset ys. P y
  shows ifilter P (ys ⊖ xs) = ifilter P ys
  using assms
  by (induct ys) auto

```

```

lemma ifilter-iapp [simp]:
  assumes (∃ x ∈ iset (xs ⊖ ys). P x)
    (∃ x ∈ iset xs. P x)
    (∃ x ∈ iset ys. P x)
  shows ifilter P (xs ⊖ ys) = (ifilter P xs) ⊖ (ifilter P ys)
  using assms
  proof
    (induct xs arbitrary: ys)
    case (INil x)
    then show ?case by simp
  next
    case (ICons x1a xs)
    then show ?case
    proof
      (cases (∃ x ∈ iset xs. P x))
      case True
      then show ?thesis using ICons by auto
    next
      case False
      then show ?thesis using ICons using ifilter-iapp1[of xs P]
        using iset-IConsD iset-iapp by auto
    qed
  qed

```

```

lemma ifilter-irev:

```

```

assumes ( $\exists x \in \text{iset } xs. P x$ )
shows  $\text{irev } (\text{ifilter } P \text{ } xs) = \text{ifilter } P \text{ } (\text{irev } xs)$ 
using assms
proof
  (induct xs)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof (cases ( $\exists x \in \text{iset } xs. P x$ ))
    case True
    then show ?thesis
      proof (cases P x1a)
      case True
      then show ?thesis using ICons True
        by (simp add: ifilter-iapp1)
      next
      case False
      then show ?thesis using ICons False  $\langle \exists x \in \text{iset } xs. P x \rangle$ 
        by (simp add: ifilter-iapp2)
      qed
    next
    case False
    then show ?thesis using False ICons
      by (simp add: ifilter-iapp1)
    qed
  qed
qed

```

```

lemma ifilter-True:
assumes  $\forall x \in \text{iset } xs. P x$ 
shows  $\text{ifilter } P \text{ } xs = xs$ 
using assms
by (meson ifilter-id-conv ilen-ifilter-le inth-iset)

```

```

lemma nfilter-imap:
assumes  $\exists x \in \text{iset } (\text{imap } f \text{ } xs). P x$ 
shows  $\text{nfilter } P \text{ } (\text{imap } f \text{ } xs) \text{ } n = (\text{nfilter } (P \circ f) \text{ } xs) \text{ } n$ 
using assms
by (induct xs arbitrary: n) auto

```

```

lemma ifilter-imap:
assumes  $\exists x \in \text{iset } (\text{imap } f \text{ } xs). P x$ 
shows  $\text{ifilter } P \text{ } (\text{imap } f \text{ } xs) = \text{imap } f \text{ } (\text{ifilter } (P \circ f) \text{ } xs)$ 
using assms
by (induct xs) auto

```

lemma *ilen-nfilter-imap[simp]*:
assumes $\exists x \in \text{iset } (\text{imap } f \text{ } xs). P \ x$
shows $\text{ilen } (\text{nfilter } P \ (\text{imap } f \text{ } xs) \ n) = \text{ilen}(\text{nfilter } (P \circ f) \ xs \ n)$
using *assms* **by** (*simp add:nfilter-imap*)

lemma *ilen-ifilter-imap[simp]*:
assumes $\exists x \in \text{iset } (\text{imap } f \text{ } xs). P \ x$
shows $\text{ilen } (\text{ifilter } P \ (\text{imap } f \text{ } xs)) = \text{ilen}(\text{ifilter } (P \circ f) \ xs)$
using *assms* **by** (*simp add:ifilter-imap*)

lemma *nfilter-is-subset [simp]*:
assumes $\exists x \in \text{iset } xs. P \ x$
shows $\text{iset } (\text{nfilter } P \ xs \ n) \leq \{n+k \mid k. k \leq \text{ilen } xs\}$
using *assms* **by** *auto*

lemma *ifilter-is-subset [simp]*:
assumes $\exists x \in \text{iset } xs. P \ x$
shows $\text{iset } (\text{ifilter } P \ xs) \leq \text{iset } xs$
using *assms* **by** *auto*

lemma *ilen-nfilter-less*:
assumes $\exists x \in \text{iset } xs. P \ x$
 $x \in \text{iset } xs$
 $\neg P \ x$
shows $\text{ilen}(\text{nfilter } P \ xs \ n) < \text{ilen } xs$
using *assms*
proof
(induct xs arbitrary: n)
case (*INil x*)
then show ?*case* **by** *simp*
next
case (*ICons x1a xs*)
then show ?*case* **using** *le-imp-less-Suc ilen-nfilter-le* **by** (*simp-all, blast*)
qed

lemma *ilen-ifilter-less*:
assumes $\exists x \in \text{iset } xs. P \ x$
 $x \in \text{iset } xs$
 $\neg P \ x$
shows $\text{ilen}(\text{ifilter } P \ xs) < \text{ilen } xs$
using *assms*
proof
(induct xs)
case (*INil x*)
then show ?*case* **by** *simp*
next
case (*ICons x1a xs*)
then show ?*case*
 $\text{using } \text{ilen-ifilter-le } \text{le-imp-less-Suc } \text{by } (\text{simp-all, blast})$
qed

lemma *nfilter-iset*:

assumes $\exists x \in \text{iset } xs. P x$

shows $\text{iset } (\text{nfilter } P \text{ } xs \text{ } n) = \{n+i \mid i. i \leq \text{ilen } xs \wedge P(\text{inth } xs \text{ } i)\}$

using *assms* **by** *auto*

lemma *nfilter-cong[fundef-cong]*:

assumes $xs = ys$

$(\bigwedge x. x \in \text{iset } ys \implies P x = Q x)$

shows $\text{nfilter } P \text{ } xs \text{ } n = \text{nfilter } Q \text{ } ys \text{ } n$

using *assms* **by** (*induction xs arbitrary: ys n*) *auto*

lemma *ifilter-cong[fundef-cong]*:

assumes $xs = ys$

$(\bigwedge x. x \in \text{iset } ys \implies P x = Q x)$

shows $\text{ifilter } P \text{ } xs = \text{ifilter } Q \text{ } ys$

using *assms* **by** (*induct ys arbitrary: xs*) *auto*

lemma *iremdups-ifilter*:

assumes $\exists x \in \text{iset } xs. P x$

shows $\text{iremdups}(\text{ifilter } P \text{ } xs) = \text{ifilter } P \text{ } (\text{iremdups } xs)$

using *assms*

by (*induct xs*) *auto*

lemma *idistinct-imap-ifilter*:

assumes $\exists x \in \text{iset } xs. P x$

$\text{idistinct } (\text{imap } f \text{ } xs)$

shows $\text{idistinct } (\text{imap } f \text{ } (\text{ifilter } P \text{ } xs))$

using *assms* **by** (*induct xs*) *auto*

lemma *idistinct-nfilter [simp]*:

$\text{idistinct } (\text{nfilter } P \text{ } xs \text{ } n)$

by (*induction xs arbitrary: n*) *auto*

lemma *idistinct-ifilter [simp]*:

assumes $\text{idistinct } xs$

shows $\text{idistinct } (\text{ifilter } P \text{ } xs)$

using *assms* **by** (*induct xs*) *auto*

lemma *idistinct-ilen-ifilter*:

assumes $\exists x \in \text{iset } xs. P x$

$\text{idistinct } xs$

shows $\text{ilen } (\text{ifilter } P \text{ } xs) + 1 = \text{card } (\{x. P x\} \text{ Int } \text{iset } xs)$

using *assms* **by** (*induct xs*) *auto*

lemma *idx-nfilter-mono*:

assumes $\exists x \in \text{iset } xs. P x$

```

      na < ilen (nfilter P xs n)
shows   inth (nfilter P xs n) na < inth (nfilter P xs n) (Suc na)
using   assms
proof   (induct xs arbitrary: n na)
case    (INil x)
then show ?case by simp
next
case    (ICons x1a xs)
then show ?case
  proof -
    have 1: ( $\exists x \in \text{iset } xs . P x$ )  $\wedge P x1a \wedge na = 0 \longrightarrow$ 
      inth (nfilter P (x1a  $\odot$  xs) n) na < inth (nfilter P (x1a  $\odot$  xs) n) (Suc na)
    using ICons by auto
    (meson Suc-le-lessD ilen-gr-zero nfilter-lower-bound)+
    have 5:  $\neg(\exists x \in \text{iset } xs . P x) \longrightarrow$ 
      inth (nfilter P (x1a  $\odot$  xs) n) na < inth (nfilter P (x1a  $\odot$  xs) n) (Suc na)
    using ICons gr-implies-not-zero by (auto, fastforce)
    show ?thesis
    using 1 5 ICons.hyps ICons.prem1(2) less-Suc-eq-0-disj by auto
  qed
qed

```

lemma *idx-nfilter*:

```

assumes  $\exists x \in \text{iset } xs . P x$ 
shows   index-sequence (ifirst((nfilter P xs n))) (nfilter P xs n)
using   assms by (simp add: index-sequence-def idx-nfilter-mono)

```

lemma *idx-nfilter-expand*:

```

assumes  $\exists x \in \text{iset } xs . P x$ 
shows    $\forall na < \text{ilen } (nfilter P xs n). \text{inth } (nfilter P xs n) na < \text{inth } (nfilter P xs n) (Suc na)$ 
using   assms idx-nfilter by (simp add: index-sequence-def)

```

lemma *idx-nfilter-gr-eq*:

```

assumes  $\exists x \in \text{iset } xs . P x$ 
      k  $\leq j$ 
      j  $\leq \text{ilen}(nfilter P xs n)$ 
shows   inth (nfilter P xs n) k  $\leq$  inth (nfilter P xs n) j
using   assms by (meson idx-nfilter idx-less-eq)

```

lemma *idx-nfilter-gr*:

```

assumes  $\exists x \in \text{iset } xs . P x$ 
shows   ( $\forall j . k < j \wedge j \leq \text{ilen}(nfilter P xs n) \longrightarrow$ 
      inth (nfilter P xs n) k < inth (nfilter P xs n) j)
using   assms
by (meson Suc-leI dual-order.strict-trans1 idx-nfilter-expand idx-nfilter-gr-eq)

```

lemma *idx-nfilter-less-eq*:

```

assumes  $\exists x \in \text{iset } xs . P x$ 

```

$k \leq \text{ilen}(\text{nfilter } P \text{ } xs \text{ } n)$
shows $(\forall j \leq k. \text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \text{ } j \leq \text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \text{ } k)$
using *assms* **by** (*simp add: idx-nfilter-gr-eq*)

lemma *idx-nfilter-less*:

assumes $\exists x \in \text{iset } xs . P \text{ } x$
 $k \leq \text{ilen}(\text{nfilter } P \text{ } xs \text{ } n)$
shows $(\forall j < k. \text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \text{ } j < \text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \text{ } k)$
using *assms*
by (*simp add: idx-nfilter-gr*)

lemma *nfilter-prefix-iset-0*:

assumes $k \leq \text{ilen } xs$
 $\exists x \in \text{iset } (\text{prefix } k \text{ } xs) . P \text{ } x$
shows $\text{ilen } (\text{nfilter } P \text{ } (\text{prefix } k \text{ } xs) \text{ } n) \leq \text{ilen } (\text{nfilter } P \text{ } xs \text{ } n)$
using *assms*
proof (*induct xs arbitrary: n k*)
case (*INil x*)
then show *?case* **by** *simp*
next
case (*ICons x1a xs*)
then show *?case*
proof (*cases k*)
case 0
then show *?thesis* **using** *ICons* **by** *simp*
next
case (*Suc nat*)
then show *?thesis* **using** *ICons prefix-subset* **by** *simp blast*
qed
qed

lemma *nfilter-subset*:

assumes $\exists x \in \text{iset } (\text{prefix } k \text{ } xs) . P \text{ } x$
 $k \leq \text{ilen } xs$
shows $\text{iset } (\text{nfilter } P \text{ } (\text{prefix } k \text{ } xs) \text{ } n) \leq \text{iset } (\text{nfilter } P \text{ } xs \text{ } n)$
using *assms*
proof (*induct xs arbitrary:k n*)
case (*INil x*)
then show *?case* **by** *auto*
next
case (*ICons x1a xs*)
then show *?case*
proof (*cases k*)
case 0
then show *?thesis* **using** *ICons* **by** *simp*
next
case (*Suc nat*)
then show *?thesis* **using** *ICons* **by** (*auto simp add: inth-iset*)
qed

qed

lemma *nfilter-prefix-iset*:

assumes $\exists x \in \text{iset } (\text{prefix } k \text{ } xs) . P \ x$

$k \leq \text{ilen } xs$

shows $\text{iset } (\text{nfilter } P \ (\text{prefix } k \text{ } xs) \ n) = \{n+i \mid i. i \leq k \wedge P(\text{inth } xs \ i)\}$

using *assms*

by *auto*

lemma *nfilter-suffix-iset*:

assumes $\exists x \in \text{iset } (\text{suffix } k \text{ } xs) . P \ x$

$k \leq \text{ilen } xs$

shows $\text{iset } (\text{nfilter } P \ (\text{suffix } k \text{ } xs) \ n) = \{n+i \mid i. i \leq \text{ilen } xs - k \wedge P(\text{inth } xs \ (k+i))\}$

using *assms* **by** *auto*

lemma *nfilter-suffix-iset-a*:

assumes $\exists x \in \text{iset } (\text{suffix } k \text{ } xs) . P \ x$

$k \leq \text{ilen } xs$

shows $\text{iset } (\text{nfilter } P \ (\text{suffix } k \text{ } xs) \ n) = \{n+(j-k) \mid j. k \leq j \wedge j \leq \text{ilen } xs \wedge P(\text{inth } xs \ j)\}$

using *assms* *Nat.le-diff-conv2* **by** *auto* *fastforce*

lemma *nfilter-suffix-iset-b*:

assumes $\exists x \in \text{iset } (\text{suffix } k \text{ } xs) . P \ x$

$k \leq \text{ilen } xs$

shows $\text{iset } (\text{nfilter } P \ (\text{suffix } k \text{ } xs) \ n) = \{n+i \mid i. i \leq \text{ilen } xs - k \wedge P(\text{inth } xs \ (i+k))\}$

using *assms* *nfilter-suffix-iset* **by** (*auto* *simp* *add*: *add commute*)

lemma *nfilter-card*:

assumes $\exists x \in \text{iset } xs . P \ x$

shows $\text{ilen } (\text{nfilter } P \ xs \ n) + 1 = \text{card } (\text{iset } (\text{nfilter } P \ xs \ n))$

using *assms*

by (*induct* *xs* *arbitrary*: *n*) *auto*

lemma *nfilter-prefix-iset-1*:

assumes $k \leq \text{ilen } xs$

$\exists x \in \text{iset } (\text{prefix } k \text{ } xs) . P \ x$

shows $\text{iset } (\text{prefix } (\text{ilen}(\text{nfilter } P \ (\text{prefix } k \text{ } xs) \ n)) \ (\text{nfilter } P \ xs \ n)) = \{(\text{inth } (\text{nfilter } P \ xs \ n) \ i) \mid i. i \leq \text{ilen}(\text{nfilter } P \ (\text{prefix } k \text{ } xs) \ n)\}$

using *prefix-iset*[*of* (*ilen*(*nfilter* *P* (*prefix* *k* *xs*) *n*)) (*nfilter* *P* *xs* *n*)]

by (*simp* *add*: *assms*(1) *assms*(2) *nfilter-prefix-iset-0*)

lemma *nfilter-prefix-subset*:

assumes $\exists x \in \text{iset } xs . P \ x$

$k \leq \text{ilen } (\text{nfilter } P \ xs \ n)$

shows $\text{iset } (\text{prefix } k \ (\text{nfilter } P \ xs \ n)) \leq \text{iset } (\text{nfilter } P \ xs \ n)$

using *assms*

using *prefix-subset* **by** *blast*

lemma *ilen-ifilter-conv-card*:

```

assumes  $\exists x \in \text{iset } xs. P x$ 
shows  $\text{ilen} (\text{ifilter } P xs) + 1 = \text{card} \{i. i \leq \text{ilen } xs \wedge P (\text{inth } xs i)\}$ 
proof –
  have 1:  $\text{ilen} (\text{ifilter } P xs) = \text{ilen}(\text{nfilter } P xs 0)$ 
    by (simp add: assms nfilter-ilen)
  have 2:  $\text{ilen} (\text{nfilter } P xs 0) + 1 = \text{card} (\text{iset} (\text{nfilter } P xs 0))$ 
    by (meson assms nfilter-card)
  have 3:  $\text{iset} (\text{nfilter } P xs 0) = \{i | i. i \leq \text{ilen } xs \wedge P(\text{inth } xs i)\}$ 
    using assms by auto
  show ?thesis
  using 1 2 3 by auto
qed

```

```

lemma ifilter-nfilter-prefix-ilen-0:
  assumes  $P (\text{inth } xs \ ( (\text{inth} (\text{nfilter } P xs n) k) - n))$ 
     $k \leq \text{ilen} (\text{ifilter } P xs)$ 
  shows  $(\exists x \in \text{iset } xs. P x)$ 
using assms
by (induction xs arbitrary: k) auto

```

```

lemma nfilter-ilen-n-zero:
  assumes  $(\exists x \in \text{iset } xs. P x)$ 
  shows  $\text{ilen} (\text{nfilter } P xs n) = \text{ilen} (\text{nfilter } P xs 0)$ 
using assms
proof
  (induction xs arbitrary: n)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof (cases n)
    case 0
    then show ?thesis using ICons by blast
    next
    case (Suc nat)
    then show ?thesis using ICons by (simp add: nfilter-ilen)
    qed
  qed

```

```

lemma nfilter-inth-n-zero-a:
  assumes  $(\exists x \in \text{iset } xs. P x)$ 
     $k \leq \text{ilen} (\text{nfilter } P xs n)$ 
  shows  $n \leq (\text{inth} (\text{nfilter } P xs n) k)$ 
using assms by (simp add: nfilter-lower-bound)

```

```

lemma nfilter-inth-n-zero:
  assumes  $(\exists x \in \text{iset } xs. P x)$ 
     $k \leq \text{ilen} (\text{nfilter } P xs n)$ 

```

```

shows (inth (nfilter P xs n) k) - n = inth (nfilter P xs 0) k
using assms
proof
  (induction xs arbitrary: n k)
  case (INil x)
  then show ?case by simp
  next
  case (ICons x1a xs)
  then show ?case
    proof (cases (∃ x ∈ iset xs. P x))
    case True
    then show ?thesis
      proof (cases P x1a)
      case True
      then show ?thesis
        proof (cases k)
        case 0
        then show ?thesis using ICons True by simp
        next
        case (Suc nat)
        then show ?thesis using ICons True ⟨∃ x ∈ iset xs. P x⟩
          proof auto
            fix x
            assume a0: k = Suc nat
            assume a1: (∧ k n. k ≤ ilen (nfilter P xs n) ⇒
              inth (nfilter P xs n) k - n = inth (nfilter P xs 0) k)
            assume a2: nat ≤ ilen (nfilter P xs (Suc n))
            assume a3: P x1a
            assume a4: x ∈ iset xs
            assume a5: P x
            show inth (nfilter P xs (Suc n)) nat - n = inth (nfilter P xs (Suc 0)) nat
            proof -
              have 1: inth (nfilter P xs (Suc n)) nat - (Suc n) = inth (nfilter P xs 0) nat
                using a1 a2 a3 a4 a5 by blast
              have 2: inth (nfilter P xs (Suc 0)) nat - (Suc 0) = inth (nfilter P xs 0) nat
                by (metis a2 a1 a4 a5 nfilter-ilen-n-zero)
              from 1 2 show ?thesis
                by (metis One-nat-def Suc-diff-le ⟨∃ x ∈ iset xs. P x⟩ a2 diff-Suc-Suc
                  nfilter-ilen-n-zero nfilter-lower-bound
                  ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc)
            qed
          qed
        qed
      case False
      then show ?thesis using ICons True
        proof auto
          fix x
          assume b0: ¬ P x1a
          assume b1: (∧ k n. k ≤ ilen (nfilter P xs n) ⇒

```

```

      inth (nfilter P xs n) k - n = inth (nfilter P xs 0) k)
assume b2: k ≤ ilen (nfilter P xs (Suc n))
assume b3: x ∈ iset xs
assume b4: P x
show inth (nfilter P xs (Suc n)) k - n = inth (nfilter P xs (Suc 0)) k
proof -
have 3: inth (nfilter P xs (Suc n)) k - (Suc n) = inth (nfilter P xs 0) k
  using b1 b2 by blast
have 4: inth (nfilter P xs (Suc 0)) k - (Suc 0) = inth (nfilter P xs 0) k
  by (metis True b1 b2 nfilter-ilen-n-zero)
show ?thesis
by (metis 3 4 One-nat-def Suc-diff-le True b2 diff-Suc-Suc nfilter-ilen-n-zero
  nfilter-lower-bound ordered-cancel-comm-monoid-diff-class.add-diff-inverse
  plus-1-eq-Suc)
qed
qed
qed
next
case False
then show ?thesis using ICons False by auto
qed
qed

```

lemma *nfilter-n-zero*:

```

assumes (∃ x ∈ iset xs. P x)
shows (nfilter P xs n) = imap (λi. i+n) (nfilter P xs 0)
using assms
proof -
have 1: ilen (nfilter P xs n) = ilen (imap (λi. i+n) (nfilter P xs 0))
  using assms nfilter-ilen-n-zero by fastforce
have 2: ∧k. k ≤ ilen (nfilter P xs n) →
  inth (nfilter P xs n) k = inth (imap (λi. i+n) (nfilter P xs 0)) k
  using assms nfilter-inth-n-zero[of xs P - n]
  by (metis inth-imap le-add-diff-inverse2 nfilter-lower-bound)
show ?thesis by (simp add: 1 2 interval-eq-inth-eq)
qed

```

lemma *nfilter-n-zero-a*:

```

assumes (∃ x ∈ iset xs. P x)
shows (nfilter P xs 0) = imap (λi. i-n) (nfilter P xs n)
proof -
have 1: ilen(nfilter P xs 0) = ilen( imap (λi. i-n) (nfilter P xs n))
  by (metis assms ilen-imap nfilter-ilen-n-zero)
have 2: ∧k. k ≤ ilen (nfilter P xs 0) →
  inth (nfilter P xs 0) k = inth (imap (λi. i-n) (nfilter P xs n)) k
  using assms
  by (simp add: 1 inth-imap nfilter-inth-n-zero)
show ?thesis
using 1 2 interval-eq-inth-eq by blast
qed

```

lemma *nfilter-count*:

assumes $(\exists x \in \text{iset } xs. P x)$

shows $\text{card } \{(\text{inth } (\text{nfilter } P \text{ } xs \text{ } n) \ k) \mid k. k \leq \text{ilen } (\text{nfilter } P \text{ } xs \text{ } n)\} =$
 $\text{ilen } (\text{nfilter } P \text{ } xs \text{ } n) + 1$

using *assms* *nfilter-card[of xs P n]* *iset-nfilter[of xs P n]*
inth-and-iset **by** (*simp add: iset-inth*)

lemma *nfilter-holds*:

assumes $(\exists x \in \text{iset } xs. P x)$

shows $(\forall x \in \text{iset } (\text{nfilter } P \text{ } xs \text{ } n). P (\text{inth } xs \text{ } (x-n)))$

using *assms* **by** *auto*

lemma *nfilter-holds-not*:

assumes $(\exists x \in \text{iset } xs. P x)$

shows $(\forall x \in (\{i+n \mid i. i \leq \text{ilen } xs\} - (\text{iset } (\text{nfilter } P \text{ } xs \text{ } n)))) . \neg P (\text{inth } xs \text{ } (x-n)))$

using *assms* **by** *auto*

lemma *nfilter-holds-a*:

assumes $(\exists x \in \text{iset } xs. P x)$

shows $(\forall i \leq \text{ilen } xs. (i+n) \in \text{iset } (\text{nfilter } P \text{ } xs \text{ } n) \longrightarrow P (\text{inth } xs \text{ } i))$

using *assms* **by** *auto*

lemma *nfilter-holds-not-a*:

assumes $(\exists x \in \text{iset } xs. P x)$

shows $(\forall i \leq \text{ilen } xs. P (\text{inth } xs \text{ } i) \longrightarrow (i+n) \in \text{iset } (\text{nfilter } P \text{ } xs \text{ } n))$

using *assms* **by** *auto*

lemma *nfilter-holds-b*:

assumes $(\exists x \in \text{iset } xs. P x)$

shows $(\forall i \leq \text{ilen } xs. (i+n) \in \text{iset } (\text{nfilter } P \text{ } xs \text{ } n) = P (\text{inth } xs \text{ } i))$

using *assms* **by** *auto*

lemma *nfilter-holds-c*:

assumes $(\exists x \in \text{iset } xs. P x)$

$i \leq \text{ilen } xs$

shows $(i+n) \in \text{iset } (\text{nfilter } P \text{ } xs \text{ } n) = P (\text{inth } xs \text{ } i)$

by (*simp add: assms(1) assms(2)*)

lemma *nfilter-holds-d*:

assumes $(\exists x \in \text{iset } xs. P x)$

$n \leq i$

$i \leq \text{ilen } xs + n$

shows $i \in \text{iset } (\text{nfilter } P \text{ } xs \text{ } n) = P (\text{inth } xs \text{ } (i-n))$

using *assms*

by (*metis diff-add le-diff-conv nfilter-holds-b*)

lemma *nfilter-holds-not-b*:

assumes $(\exists x \in \text{iset } xs. P x)$
 $n \leq i$
 $i \leq \text{ilen } xs + n$
shows $i \notin \text{iset } (\text{nfilter } P \text{ } xs \text{ } n) = (\neg P (\text{inth } xs (i-n)))$
using *assms* **by** *auto*

lemma *nfilter-disjoint-iset-coset:*

assumes $(\exists x \in \text{iset } xs. P x)$
shows $(\{i+n \mid i. i \leq \text{ilen } xs\} - (\text{iset } (\text{nfilter } P \text{ } xs \text{ } n))) \cap (\text{iset } (\text{nfilter } P \text{ } xs \text{ } n)) = \{\}$
using *assms* **by** *auto*

lemma *nfilter-not-before:*

assumes $(\exists x \in \text{iset } xs. P x)$
 $i < (\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } 0)$
shows $\neg P (\text{inth } xs \text{ } i)$

proof –

have $0: (\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } 0) \leq \text{ilen } xs$
by (*metis add.left-neutral assms(1) ilen-gr-zero nfilter-upper-bound*)
have $1: i \notin \text{iset } (\text{nfilter } P \text{ } xs \text{ } 0)$
using *assms*
proof (*induction xs arbitrary: i*)
case (*INil x*)
then show *?case* **by** *simp*
next
case (*ICons x1a xs*)
then show *?case*
by (*metis idx-nfilter idx-greater inth-and-iset leD*)
qed
have $2: i \notin \text{iset } (\text{nfilter } P \text{ } xs \text{ } 0) \wedge i \leq \text{ilen } xs \longrightarrow \neg P (\text{inth } xs (i))$
by (*metis add.right-neutral nfilter-holds-not-a inth-iset*)
have $3: i \leq \text{ilen } xs$
using 0 *assms(2)* **by** *linarith*
from $0 \ 1 \ 2 \ 3$ **show** *?thesis* **by** *auto*
qed

lemma *nfilter-n-not-before:*

assumes $(\exists x \in \text{iset } (\text{suffix } n \text{ } xs). P x)$
 $n \leq \text{ilen } xs$
 $n \leq i$
 $i < (\text{inth } (\text{nfilter } P (\text{suffix } n \text{ } xs) \text{ } n) \text{ } 0)$
shows $\neg P (\text{inth } xs (i))$

proof –

have $0: (\text{inth } (\text{nfilter } P (\text{suffix } n \text{ } xs) \text{ } n) \text{ } 0) \leq \text{ilen } xs$
by (*metis assms(1) assms(2) ilen-gr-zero suffix-ilen-good le-add-diff-inverse nfilter-upper-bound*)
have $1: i \notin \text{iset } (\text{nfilter } P (\text{suffix } n \text{ } xs) \text{ } n)$
using *assms*
proof (*induction xs arbitrary: i*)
case (*INil x*)
then show *?case* **by** *simp*


```

case (ICons x1a xs)
then show ?case
  by (metis idx-nfilter-gr-eq inth-and-iset leD le-eq-less-or-eq)
qed
have 2:  $i \notin \text{iset } (nfilter\ P\ (\text{suffix}\ n\ xs)\ n) \wedge n \leq i \wedge i \leq \text{ilen}\ xs \longrightarrow \neg P\ (\text{inth}\ xs\ (i))$ 
  using assms nfilter-holds-not-a[of suffix n xs P n]
  by (metis add-le-imp-le-left inth-suffix suffix-ilen le-add-diff-inverse2
    ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 3:  $n \leq i \wedge i \leq \text{ilen}\ xs$ 
  by (meson assms(1) assms(3) assms(4) dual-order.strict-implies-order dual-order.strict-trans2
    nfilter-inth-n-zero-a order-refl)
from 1 2 3 show ?thesis by auto
qed

```

lemma *nfilter-not-between-help-a:*

```

assumes ( $\bigwedge k\ i.$ 
   $\exists a \in \text{iset}\ xs. P\ a \implies$ 
   $k < \text{ilen}\ (nfilter\ P\ xs\ 0) \implies$ 
   $\text{inth}\ (nfilter\ P\ xs\ 0)\ k < i \implies$ 
   $i < \text{inth}\ (nfilter\ P\ xs\ 0)\ (\text{Suc}\ k) \implies$ 
   $\text{inth}\ (nfilter\ P\ xs\ 0)\ (\text{Suc}\ k) \leq \text{ilen}\ xs \implies$ 
   $i \notin \text{iset}\ (nfilter\ P\ xs\ 0))$ 
   $\exists a \in \text{iset}\ (x1a \odot xs). P\ a$ 
   $k < \text{ilen}\ (nfilter\ P\ (x1a \odot xs)\ 0)$ 
   $\text{inth}\ (nfilter\ P\ (x1a \odot xs)\ 0)\ k < i$ 
   $i < \text{inth}\ (nfilter\ P\ (x1a \odot xs)\ 0)\ (\text{Suc}\ k)$ 
   $\text{inth}\ (nfilter\ P\ (x1a \odot xs)\ 0)\ (\text{Suc}\ k) \leq \text{ilen}\ (x1a \odot xs)$ 
   $\exists x \in \text{iset}\ xs. P\ x$ 
   $P\ x1a$ 
shows  $i \notin \text{iset}\ (nfilter\ P\ (x1a \odot xs)\ 0)$ 
proof -
have 1:  $k=0 \implies i \notin \text{iset}\ (nfilter\ P\ (x1a \odot xs)\ 0)$ 
  using assms by auto
  (metis One-nat-def Suc-less-eq2 diff-Suc-1 ilen-gr-zero nfilter-not-before
    nfilter-inth-n-zero)
have 2:  $\bigwedge n. k = (\text{Suc}\ n) \implies i \notin \text{iset}\ (nfilter\ P\ (x1a \odot xs)\ 0)$ 
  using assms
proof auto
fix n
fix x
fix ka
assume a0:  $k = \text{Suc}\ n$ 
assume a1: ( $\bigwedge k\ i. k < \text{ilen}\ (nfilter\ P\ xs\ 0) \implies$ 
   $\text{inth}\ (nfilter\ P\ xs\ 0)\ k < i \implies$ 
   $i < \text{inth}\ (nfilter\ P\ xs\ 0)\ (\text{Suc}\ k) \implies$ 
   $\text{inth}\ (nfilter\ P\ xs\ 0)\ (\text{Suc}\ k) \leq \text{ilen}\ xs \implies$ 
   $\neg P\ (\text{inth}\ xs\ i)$ )
assume a2:  $n < \text{ilen}\ (nfilter\ P\ xs\ (\text{Suc}\ 0))$ 
assume a3:  $\text{inth}\ (nfilter\ P\ xs\ (\text{Suc}\ 0))\ n < \text{Suc}\ ka$ 

```



```

assume a4:  $Suc\ ka < inth\ (nfilter\ P\ xs\ (Suc\ 0))\ (Suc\ n)$ 
assume a5:  $inth\ (nfilter\ P\ xs\ (Suc\ 0))\ (Suc\ n) \leq Suc\ (ilen\ xs)$ 
assume a6:  $P\ x1a$ 
assume a7:  $x \in iset\ xs$ 
assume a8:  $P\ x$ 
assume a9:  $i = Suc\ ka$ 
assume a10:  $P\ (inth\ xs\ ka)$ 
show False
proof –
  have 3:  $n < ilen\ (nfilter\ P\ xs\ 0) \implies$ 
     $inth\ (nfilter\ P\ xs\ 0)\ n < ka \implies$ 
     $ka < inth\ (nfilter\ P\ xs\ 0)\ (Suc\ n) \implies$ 
     $inth\ (nfilter\ P\ xs\ 0)\ (Suc\ n) \leq ilen\ xs \implies$ 
     $\neg P\ (inth\ xs\ ka)$ 
    using a1[of n ka] by auto
  have 4:  $n < ilen\ (nfilter\ P\ xs\ 0)$ 
    by (metis a2 a7 a8 nfilter-ilen-n-zero)
  have 5:  $inth\ (nfilter\ P\ xs\ 0)\ (Suc\ n) \leq ilen\ xs$ 
    by (metis 4 Suc-leI add commute add.right-neutral assms(7) nfilter-upper-bound)
  have 6:  $\exists x \in iset\ xs. P\ x$ 
    using assms(7) by auto
  have 7:  $Suc\ 0 \leq ilen\ (nfilter\ P\ xs\ (Suc\ n))$ 
    by (metis One-nat-def Suc-le-mono Suc-mono a2 a7 a8 ilen-gr-zero le-SucE
      nfilter-ilen-n-zero not-add-less1 plus-1-eq-Suc)
  have 8:  $ka < inth\ (nfilter\ P\ xs\ 0)\ (Suc\ n)$ 
    using nfilter-inth-n-zero[of xs P Suc n Suc 0] a4 6 a2 by linarith
  have 9:  $inth\ (nfilter\ P\ xs\ 0)\ n < ka$ 
    using a2 a3 a7 a8 nfilter-inth-n-zero[of xs P n Suc 0]
    by (metis One-nat-def add commute dual-order.strict-implies-order less-diff-conv2
      nfilter-lower-bound plus-1-eq-Suc)
  have 10:  $\neg P\ (inth\ xs\ ka)$ 
    using 3 4 9 8 5 by auto
  from a10 10 show ?thesis by auto
qed
qed
show ?thesis using 1 2
using gr0-implies-Suc by blast
qed

```

lemma *nfilter-not-between-help-b:*

assumes $(\bigwedge k\ i.$

```

 $\exists a \in iset\ xs. P\ a \implies$ 
 $k < ilen\ (nfilter\ P\ xs\ 0) \implies$ 
 $inth\ (nfilter\ P\ xs\ 0)\ k < i \implies$ 
 $i < inth\ (nfilter\ P\ xs\ 0)\ (Suc\ k) \implies$ 
 $inth\ (nfilter\ P\ xs\ 0)\ (Suc\ k) \leq ilen\ xs \implies$ 
 $i \notin iset\ (nfilter\ P\ xs\ 0))$ 
 $\exists a \in iset\ (x1a \odot xs). P\ a$ 
 $k < ilen\ (nfilter\ P\ (x1a \odot xs)\ 0)$ 
 $inth\ (nfilter\ P\ (x1a \odot xs)\ 0)\ k < i$ 

```

```

i < inth (nfilter P (x1a ⊙ xs) 0) (Suc k)
inth (nfilter P (x1a ⊙ xs) 0) (Suc k) ≤ ilen (x1a ⊙ xs)
∃ x ∈ iset xs. P x
¬ P x1a
shows i ∉ iset (nfilter P (x1a ⊙ xs) 0)
proof –
have 1: k=0 ⇒ i ∉ iset (nfilter P (x1a ⊙ xs) 0)
using assms
proof auto
fix x :: 'a and ka :: nat
assume a1: inth (nfilter P xs (Suc 0)) 0 < Suc ka
assume a2: x ∈ iset xs
assume a3: P x
assume a4: ⋀ k i. [k < ilen (nfilter P xs 0); inth (nfilter P xs 0) k < i;
               i < inth (nfilter P xs 0) (Suc k); inth (nfilter P xs 0) (Suc k) ≤ ilen xs]
               ⇒ ¬ P (inth xs i)
assume a5: P (inth xs ka)
assume a6: 0 < ilen (nfilter P xs (Suc 0))
assume a7: Suc ka < inth (nfilter P xs (Suc 0)) (Suc 0)
assume a8: inth (nfilter P xs (Suc 0)) (Suc 0) ≤ Suc (ilen xs)
have f9: ∃ a. a ∈ iset xs ∧ P a
using a3 a2 by blast
then have f10: 0 ≤ ilen (nfilter P xs (Suc 0)) → inth (nfilter P xs (Suc 0)) 0 ≠ 0
by (metis le-zero-eq nfilter-lower-bound not-less-eq-eq)
have f11: ilen (nfilter P xs (Suc 0)) = ilen (nfilter P xs 0)
using f9 by (meson nfilter-ilen-n-zero)
obtain nn :: nat ⇒ nat ⇒ nat where
f12: inth (nfilter P xs (Suc 0)) (Suc 0) = Suc (nn (inth (nfilter P xs (Suc 0)) (Suc 0)) ka) ∧
      ka < nn (inth (nfilter P xs (Suc 0)) (Suc 0)) ka
using a7 by (meson Suc-less-eq2)
then have inth (nfilter P xs 0) (Suc 0) = nn (inth (nfilter P xs (Suc 0)) (Suc 0)) ka
using f9 a6 by (metis One-nat-def diff-Suc-1 le-zero-eq neq0-conv
                  nfilter-inth-n-zero not-less-eq-eq)
then have ¬ 0 ≤ ilen (nfilter P xs 0)
using f12 f11 f10 f9 a8 a6 a5 a4 a1
by (metis One-nat-def Suc-le-mono diff-Suc-1 less-Suc-eq-0-disj nfilter-inth-n-zero)
then show False
by blast
qed
have 2: ⋀ n. k = (Suc n) ⇒ i ∉ iset (nfilter P (x1a ⊙ xs) 0)
using assms
proof auto
fix n
fix x
fix ka
assume a0: k = Suc n
assume a1: (⋀ k i. k < ilen (nfilter P xs 0) ⇒
            inth (nfilter P xs 0) k < i ⇒
            i < inth (nfilter P xs 0) (Suc k) ⇒
            inth (nfilter P xs 0) (Suc k) ≤ ilen xs) ⇒

```

```

       $\neg P (inth\ xs\ i)$ 
assume a2:  $Suc\ n < ilen\ (nfilter\ P\ xs\ (Suc\ 0))$ 
assume a3:  $inth\ (nfilter\ P\ xs\ (Suc\ 0))\ (Suc\ n) < Suc\ ka$ 
assume a4:  $Suc\ ka < inth\ (nfilter\ P\ xs\ (Suc\ 0))\ (Suc\ (Suc\ n))$ 
assume a5:  $inth\ (nfilter\ P\ xs\ (Suc\ 0))\ (Suc\ (Suc\ n)) \leq Suc\ (ilen\ xs)$ 
assume a6:  $\neg P\ x1a$ 
assume a7:  $x \in iset\ xs$ 
assume a8:  $P\ x$ 
assume a9:  $i = Suc\ ka$ 
assume a10:  $P\ (inth\ xs\ ka)$ 
show False
proof –
  have 3:  $Suc\ n < ilen\ (nfilter\ P\ xs\ 0) \implies$ 
     $inth\ (nfilter\ P\ xs\ 0)\ (Suc\ n) < ka \implies$ 
     $ka < inth\ (nfilter\ P\ xs\ 0)\ (Suc\ (Suc\ n)) \implies$ 
     $inth\ (nfilter\ P\ xs\ 0)\ (Suc\ (Suc\ n)) \leq ilen\ xs \implies$ 
     $\neg P\ (inth\ xs\ ka)$ 
    using a1[of  $Suc\ n\ ka$ ] by auto
  have 4:  $(Suc\ n) < ilen\ (nfilter\ P\ xs\ 0)$ 
    by (metis a2 a7 a8 nfilter-ilen-n-zero)
  have 5:  $inth\ (nfilter\ P\ xs\ 0)\ (Suc\ (Suc\ n)) \leq ilen\ xs$ 
    by (metis 4 Suc-leI a7 a8 add.left-neutral nfilter-upper-bound)
  have 6:  $\exists x \in iset\ xs. P\ x$ 
    by (simp add: assms(7))
  have 7:  $Suc\ 0 \leq ilen\ (nfilter\ P\ xs\ (Suc\ n))$ 
    by (metis One-nat-def Suc-le-mono Suc-mono a2 a7 a8 ilen-gr-zero le-SucE
    nfilter-ilen-n-zero not-add-less1 plus-1-eq-Suc)
  have 8:  $ka < inth\ (nfilter\ P\ xs\ 0)\ (Suc\ (Suc\ n))$ 
    using nfilter-inth-n-zero[of  $xs\ P\ Suc\ (Suc\ n)\ Suc\ 0$ ] a4 6 a2 by linarith
  have 9:  $inth\ (nfilter\ P\ xs\ 0)\ (Suc\ n) < ka$ 
    using a2 a3 a7 a8 nfilter-inth-n-zero[of  $xs\ P\ Suc\ n\ Suc\ 0$ ]
    by (metis One-nat-def add commute dual-order.strict-implies-order less-diff-conv2
    nfilter-lower-bound plus-1-eq-Suc)
  have 10:  $\neg P\ (inth\ xs\ ka)$ 
    using 3 4 9 8 5 by auto
  from a10 10 show ?thesis by auto
qed
qed
show ?thesis using 1 2
using gr0-implies-Suc by blast
qed

```

lemma *nfilter-not-between-help*:

```

assumes  $(\exists\ x \in iset\ xs. P\ x)$ 
     $k < ilen\ (nfilter\ P\ xs\ 0)$ 
     $(inth\ (nfilter\ P\ xs\ 0)\ k) < i$ 
     $i < (inth\ (nfilter\ P\ xs\ 0)\ (Suc\ k))$ 
     $(inth\ (nfilter\ P\ xs\ 0)\ (Suc\ k)) \leq ilen\ xs$ 
shows  $i \notin iset\ (nfilter\ P\ xs\ 0)$ 

```

```

using assms
proof (induction xs arbitrary: i k)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases ( $\exists x \in \text{iset } xs. P x$ ))
  case True
  then show ?thesis
    proof (cases P x1a)
    case True
    then show ?thesis using ICons True using nfilter-not-between-help-a[of xs P x1a k i]
      by fastforce
    next
    case False
    then show ?thesis using ICons False using nfilter-not-between-help-b[of xs P x1a k i]
      by simp
    qed
  next
  case False
  then show ?thesis using ICons False by auto
  qed
qed

```

lemma *nfilter-not-between:*

```

assumes ( $\exists x \in \text{iset } xs. P x$ )
  ( $\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k \ ) < i$ 
   $i < (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ (\text{Suc } k) \ )$ 
   $k < \text{ilen } (\text{nfilter } P \text{ } xs \ 0)$ 
shows  $\neg P (\text{inth } xs \ (i))$ 
proof -
have 0: ( $\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ (\text{Suc } k) \ ) \leq \text{ilen } xs$ 
  by (metis Suc-leI add-cancel-right-left assms(1) assms(4) nfilter-upper-bound)
have 1:  $i \leq \text{ilen } xs$ 
  using 0 assms(3) by linarith
have 2:  $k < \text{ilen } (\text{nfilter } P \text{ } xs \ 0) \wedge (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k \ ) < i \wedge$ 
   $i < (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ (\text{Suc } k) \ ) \wedge (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ (\text{Suc } k) \ ) \leq \text{ilen } xs \longrightarrow$ 
   $i \notin \text{iset } (\text{nfilter } P \text{ } xs \ 0)$ 
using assms(1)
proof (induction xs arbitrary: i k)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case using assms nfilter-not-between-help
  by metis
qed
have 3:  $i \notin \text{iset } (\text{nfilter } P \text{ } xs \ 0) \wedge i \leq \text{ilen } xs \longrightarrow \neg P (\text{inth } xs \ (i))$ 
  by (metis add.right-neutral nfilter-holds-b inth-iset)

```

from 0 1 2 3 show ?thesis using assms by blast
qed

lemma *idx-imp-idistinct*:
assumes *index-sequence* (inth *xs* 0) *xs*
shows *idistinct xs*
using *assms*
proof (*induction xs*)
case (*INil x*)
then show ?*case* **by** *simp*
next
case (*ICons x1a xs*)
then show ?*case*
by (*auto simp add: idx-expand1*)
(*metis idx-expand1 idx-greater-first inth-and-iset le-zero-eq not-less not-less-iff-gr-or-eq*)
qed

lemma *idx-iset-eq*:
assumes *index-sequence* (inth *xs* 0) *xs*
index-sequence (inth *ys* 0) *ys*
iset xs = iset ys
shows *xs = ys*
using *assms*
proof
(*induction xs arbitrary: ys*)
case (*INil x*)
then show ?*case*
by (*metis idistinct.simps(2) iapp-INil iapp-not-INil idx-imp-idistinct*
interval.exhaust interval-in-iset-conv-decomp-last)
next
case (*ICons x1a xs*)
then show ?*case*
proof (*cases ys*)
case (*INil x1*)
then show ?*thesis*
using *ICons.prem1 ICons.prem3 idx-less-last-1 le-eq-less-or-eq inth-iset* **by** *fastforce*
next
case (*ICons x21 x22*)
then show ?*thesis*
proof (*cases x1a = x21*)
case *True*
then show ?*thesis*
proof –
have 1: *ilen (x1a \odot xs) = ilen (x21 \odot x22)*
by (*metis ICons.prem1 ICons.prem2 ICons.prem3 idistinct-card idx-imp-idistinct*
ICons nat.inject)
have 2: *index-sequence* (inth *xs* 0) *xs*
using *ICons.prem1 idx-expand1* **by** *auto*

```

have 3: index-sequence (inth x22 0) x22
  using ICons.prem(2) idx-expand1 ICons by auto
have 4: idistinct xs
  using 2 idx-imp-idistinct by auto
have 5: idistinct x22
  by (simp add: 3 idx-imp-idistinct)
have 6: iset (x1a⊙xs) = {x1a} ∪ iset xs
  by auto
have 7: x1a ∉ iset xs
  by (meson ICons.prem(1) idistinct.simp(2) idx-imp-idistinct)
have 8: iset (x21⊙x22) = {x21} ∪ iset x22
  by auto
have 9: x21 ∉ iset x22
  using ICons.prem(2) idx-imp-idistinct ICons by fastforce
have 10: iset xs = iset x22
  using 7 9 ICons.prem(3) True ICons by fastforce
have 11: xs = x22
  using 10 2 3 ICons.IH by blast
have 12: x1a⊙xs = x21⊙x22
  by (simp add: 11 True)
show ?thesis by (simp add: 12 ICons)
qed
next
case False
then show ?thesis using ICons idx-imp-idistinct interval-hd-in-iset idx-expand1
by (metis (full-types) ICons.prem(1) ICons.prem(2) ICons.prem(3) idistinct.simp(2)
    inth-zero linorder-neqE-nat)
qed
qed
qed

```

```

lemma ifilter-nfilter-prefix-idx-a:
  assumes P (inth xs ( (inth (nfilter P xs 0) k) ) )
    k ≤ ilen (ifilter P xs)
  shows index-sequence (inth (prefix k (nfilter P xs 0)) 0) (prefix k (nfilter P xs 0))
using assms by (auto simp add: index-sequence-def)
    (metis diff-zero ifilter-nfilter-prefix-ilen-0 idx-nfilter-mono)

```

```

lemma ifilter-nfilter-prefix-idx-a-1:
  assumes ∃ x ∈ iset xs. P x
    k ≤ ilen (ifilter P xs)
  shows index-sequence (inth (prefix k (nfilter P xs 0)) 0) (prefix k (nfilter P xs 0))
using assms by (auto simp add: index-sequence-def)
    (meson idx-nfilter-mono)

```

```

lemma ifilter-nfilter-suffix-idx-a:
  assumes P (inth xs ( (inth (nfilter P xs 0) k) ) )
    k ≤ ilen (ifilter P xs)
  shows index-sequence (inth (suffix k (nfilter P xs 0)) 0) (suffix k (nfilter P xs 0))
using assms by (simp add: index-sequence-def)

```

(metis add.commute diff-zero ifilter-nfilter-prefix-ilen-0 idx-nfilter-mono less-diff-conv)

lemma *ifilter-nfilter-suffix-idx-a-1:*

assumes $\exists x \in \text{iset } xs. P x$

$k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$

shows $\text{index-sequence } (\text{inth } (\text{suffix } k (\text{nfilter } P \text{ } xs \ 0)) \ 0) (\text{suffix } k (\text{nfilter } P \text{ } xs \ 0))$

using *assms*

by (*simp add: index-sequence-def idx-nfilter-mono nfilter-ilen*)

lemma *ifilter-nfilter-prefix-idx-b:*

assumes $P (\text{inth } xs \ (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k))$

$k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$

shows $\text{index-sequence } (\text{inth } (\text{nfilter } P \text{ } (\text{prefix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) \ 0) \ 0) \\ (\text{nfilter } P \text{ } (\text{prefix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) \ 0)$

using *assms idx-nfilter[of (prefix (inth (nfilter P xs 0) k) xs) P 0]*

by (*metis add.right-neutral ilast-ifirst inth-suffix le0 le-refl inth-iset*)

lemma *ifilter-nfilter-prefix-idx-b-1:*

assumes $\exists x \in \text{iset } xs. P x$

$k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$

shows $\text{index-sequence } (\text{inth } (\text{nfilter } P \text{ } (\text{prefix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) \ 0) \ 0) \\ (\text{nfilter } P \text{ } (\text{prefix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) \ 0)$

using *assms*

by (*metis ifilter-nfilter-prefix-idx-b nfilter-holds nfilter-ilen nfilter-inth-n-zero inth-iset*)

lemma *ifilter-nfilter-suffix-idx-b:*

assumes $P (\text{inth } xs \ (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k))$

$k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$

shows $\text{index-sequence } (\text{inth } (\text{nfilter } P \text{ } (\text{suffix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) \\ (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k)) \ 0) \\ (\text{nfilter } P \text{ } (\text{suffix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) \\ (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k)))$

using *assms*

by (*auto simp add: index-sequence-def*)

(*metis idx-nfilter-mono ifirst-suffix ilen-gr-zero*

suffix-ilen-code ilen-nfilter-le less-le-trans not-less0 inth-iset)

lemma *ifilter-nfilter-suffix-idx-b-1:*

assumes $\exists x \in \text{iset } xs. P x$

$k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$

shows $\text{index-sequence } (\text{inth } (\text{nfilter } P \text{ } (\text{suffix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) \\ (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k)) \ 0) \\ (\text{nfilter } P \text{ } (\text{suffix } ((\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k) \) \ xs) (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k)))$

using *assms*

by (*metis ifilter-nfilter-suffix-idx-b nfilter-holds nfilter-ilen nfilter-inth-n-zero inth-iset*)

lemma *ifilter-nfilter-prefix-iset-eq:*

assumes $P (\text{inth } xs \ (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ k))$

$k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$

shows $iset \text{ (prefix } k \text{ (nfilter } P \text{ } xs \text{ } 0))} =$
 $iset \text{ (nfilter } P \text{ (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } 0)}$
proof –
have 1: $(inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \leq ilen \text{ } xs$
by (*metis* *assms*(1) *assms*(2) *diff-zero ifilter-nfilter-prefix-ilen-0 ilen-gr-zero*
nfilter-ilen nfilter-upper-bound ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 2: $\exists x \in iset \text{ } xs . P \text{ } x$
using 1 *assms*(1) *inth-iset* **by** *blast*
have 3: $\{ i. i \leq ilen(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i))} =$
 $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge P \text{ (inth } xs \text{ } i) \}$
using 1 **by** *auto*
have 4: $\exists x \in iset(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) . P \text{ } x$
by (*metis* 1 *assms*(1) *inth-prefix prefix-ilen-good inth-iset order-refl*)
have 5: $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge P \text{ (inth } xs \text{ } i) \} =$
 $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge i \in iset(nfilter \text{ } P \text{ } xs \text{ } 0) \}$
using 4 1 2 *nfilter-holds-b* **by** *auto*
have 6: $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge i \in iset(nfilter \text{ } P \text{ } xs \text{ } 0) \} =$
 $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge$
 $i \in \{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq ilen(nfilter \text{ } P \text{ } xs \text{ } 0) \} \}$
by (*auto simp add: inth-and-iset*)
have 7: $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge$
 $i \in \{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq ilen(nfilter \text{ } P \text{ } xs \text{ } 0) \} \} =$
 $\{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq k \}$
using *assms* 2 **by** (*auto simp add: nfilter-ilen,*
metis dual-order.antisym idx-nfilter idx-less-eq le-cases nfilter-ilen,
metis idx-nfilter-less-eq nfilter-ilen)
have 8: $k \leq ilen \text{ (nfilter } P \text{ } xs \text{ } 0)$
by (*simp add: 2 assms*(2) *nfilter-ilen*)
have 9: $\{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq k \} = iset \text{ (prefix } k \text{ (nfilter } P \text{ } xs \text{ } 0))$
using 8 2 **using** *prefix-iset* **by** *force*
have 10: $iset \text{ (nfilter } P \text{ (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } 0) =$
 $\{ i. i \leq ((inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i) \}$
using 4 1 *nfilter-prefix-iset* **by** *auto*
have 11: $ilen(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) = ((inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k))$
using 1 *prefix-ilen-good* **by** *blast*
have 12: $\{ i. i \leq ((inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i) \} =$
 $\{ i. i \leq ilen(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i) \}$
using 11 **by** *auto*
show *?thesis*
using 10 12 3 5 6 7 9 **by** *auto*
qed

lemma *ifilter-nfilter-prefix-iset-eq-1*:
assumes $\exists x \in iset \text{ } xs . P \text{ } x$
 $k \leq ilen \text{ (ifilter } P \text{ } xs)$

shows $iset \text{ (prefix } k \text{ (nfilter } P \text{ } xs \text{ } 0))} =$
 $iset \text{ (nfilter } P \text{ (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } 0)}$
proof –
have 1: $(inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \leq ilen \text{ } xs$
by (*metis* *assms*(1) *assms*(2) *diff-zero* *ilen-gr-zero*
nfilter-ilen *nfilter-upper-bound* *ordered-cancel-comm-monoid-diff-class.add-diff-inverse*)
have 2: $\exists x \in iset \text{ } xs . P \text{ } x$
using 1 *assms*(1) *inth-iset* **by** *blast*
have 3: $\{ i. i \leq ilen(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i)) \} =$
 $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge P \text{ (inth } xs \text{ } i) \}$
using 1 **by** *auto*
have 4: $\exists x \in iset(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) . P \text{ } x$
using *assms* 1 2 *nfilter-holds[of xs P 0]*
by (*metis* *ilast-prefix* *le-refl* *nfilter-ilen* *nfilter-inth-n-zero* *inth-iset*)
have 5: $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge P \text{ (inth } xs \text{ } i) \} =$
 $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge i \in iset(nfilter \text{ } P \text{ } xs \text{ } 0) \}$
using 4 1 2 *nfilter-holds-b* **by** *auto*
have 6: $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge i \in iset(nfilter \text{ } P \text{ } xs \text{ } 0) \} =$
 $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge$
 $i \in \{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq ilen(nfilter \text{ } P \text{ } xs \text{ } 0) \} \}$
by (*auto simp add: inth-and-iset*)
have 7: $\{ i. i \leq (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k) \wedge$
 $i \in \{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq ilen(nfilter \text{ } P \text{ } xs \text{ } 0) \} \} =$
 $\{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq k \}$
using *assms* 2 **by** (*auto simp add: nfilter-ilen,*
metis *dual-order.antisym* *idx-nfilter* *idx-less-eq* *le-cases* *nfilter-ilen,*
metis *idx-nfilter-less-eq* *nfilter-ilen*)
have 8: $k \leq ilen \text{ (nfilter } P \text{ } xs \text{ } 0)$
by (*simp add: 2 assms*(2) *nfilter-ilen*)
have 9: $\{ (inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } j) \mid j. j \leq k \} = iset \text{ (prefix } k \text{ (nfilter } P \text{ } xs \text{ } 0))$
using 8 2 **using** *prefix-iset* **by** *force*
have 10: $iset \text{ (nfilter } P \text{ (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } 0) =$
 $\{ i. i \leq ((inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i) \}$
using 4 1 *nfilter-prefix-iset* **by** *auto*
have 11: $ilen(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) = ((inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k))$
using 1 *prefix-ilen-good* **by** *blast*
have 12: $\{ i. i \leq ((inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i) \} =$
 $\{ i. i \leq ilen(\text{prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \wedge$
 $P \text{ (inth (prefix ((inth (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) \text{ } i) \}$
using 11 **by** *auto*
show *?thesis*
using 10 12 3 5 6 7 9 **by** *auto*
qed

lemma *ifilter-nfilter-suffix-iset-eq*:
assumes $P \text{ (inth } xs \text{ } ((inth \text{ (nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ })$

$k \leq \text{ilen} (\text{ifilter } P \text{ } xs)$
shows $\text{iset} (\text{nfilter } P (\text{suffix} (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \ xs) (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k)) =$
 $\text{iset} (\text{suffix } k (\text{nfilter } P \text{ } xs \ 0))$

proof –

have 1: $(\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq \text{ilen } xs$
by (*metis* *assms*(1) *assms*(2) *diff-zero ifilter-nfilter-prefix-ilen-0 ilen-gr-zero*
nfilter-ilen nfilter-upper-bound ordered-cancel-comm-monoid-diff-class.add-diff-inverse)

have 2: $\exists x \in \text{iset } xs . P \ x$
using 1 *assms*(1) *inth-iset* **by** *blast*

have 4: $\exists x \in \text{iset}(\text{suffix} ((\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k)) \ xs) . P \ x$
using 1 *assms*(1) *inth-and-iset* **by** *force*

have 10: $\text{iset} (\text{nfilter } P (\text{suffix} ((\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \ xs) (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k)) =$
 $\{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) + i \mid i. i \leq \text{ilen } xs - (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \wedge$
 $P (\text{inth } xs (i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k))) \}$

using *nfilter-suffix-iset-b[of ((inth (nfilter P xs 0) k)) xs P (inth (nfilter P xs 0) k)]*
using 1 4 **by** *blast*

have 5: $\{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) + i \mid i. i \leq \text{ilen } xs - (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \wedge$
 $P (\text{inth } xs (i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k))) \}$
 $=$
 $\{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) + i \mid i. i \leq \text{ilen } xs - (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \wedge$
 $i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \in \text{iset}(\text{nfilter } P \text{ } xs \ 0) \}$

using 4 1 2 *nfilter-holds-b* **by** *auto*

have 51: $\{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) + i \mid i. i \leq \text{ilen } xs - (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \wedge$
 $i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \in \text{iset}(\text{nfilter } P \text{ } xs \ 0) \} =$
 $\{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) + i \mid i.$
 $(\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \wedge$
 $i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq \text{ilen } xs \wedge$
 $i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \in \text{iset}(\text{nfilter } P \text{ } xs \ 0) \}$

using 1 **by** *auto*

have 52: $\{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) + i \mid i.$
 $(\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \wedge$
 $i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq \text{ilen } xs \wedge$
 $i + (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \in \text{iset}(\text{nfilter } P \text{ } xs \ 0) \} =$
 $\{ j. (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq j \wedge j \leq \text{ilen } xs \wedge j \in \text{iset}(\text{nfilter } P \text{ } xs \ 0) \}$

by (*metis* (*no-types, lifting*) *add-diff-cancel-right' le0 le-add-diff-inverse*
le-add-same-cancel2)

have 53: $\{ j. (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq j \wedge j \leq \text{ilen } xs \wedge j \in \text{iset}(\text{nfilter } P \text{ } xs \ 0) \} =$
 $\{ j. (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq j \wedge$
 $j \leq \text{ilen } xs \wedge j \in \{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ jj) \mid jj. jj \leq \text{ilen}(\text{nfilter } P \text{ } xs \ 0) \} \}$

by (*auto simp add: inth-and-iset*)

have 54: $\{ j. (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ k) \leq j \wedge$
 $j \leq \text{ilen } xs \wedge j \in \{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ jj) \mid jj. jj \leq \text{ilen}(\text{nfilter } P \text{ } xs \ 0) \} \} =$
 $\{ (\text{inth} (\text{nfilter } P \text{ } xs \ 0) \ j) \mid j. k \leq j \wedge j \leq \text{ilen} (\text{nfilter } P \text{ } xs \ 0) \}$

using *assms* 2 **by** (*auto*,

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metis dual-order.antisym idx-nfilter-less-eq le-cases nfilter-ilen,
metis idx-nfilter-gr-eq,
metis add-cancel-right-left nfilter-upper-bound)
have 8:  $k \leq \text{ilen } (nfilter\ P\ xs\ 0)$ 
  by (simp add: 2 assms(2) nfilter-ilen)
have 9:  $\{ (inth\ (nfilter\ P\ xs\ 0)\ j) \mid j. k \leq j \wedge j \leq \text{ilen } (nfilter\ P\ xs\ 0) \} =$ 
   $\text{iset } (\text{suffix } k\ (nfilter\ P\ xs\ 0))$ 
  using 8 2 using suffix-iset-a by blast
show ?thesis
using 10 5 51 52 53 54 9 by simp
qed

lemma ifilter-nfilter-suffix-iset-eq-1:
  assumes  $\exists x \in \text{iset } xs . P\ x$ 
     $k \leq \text{ilen } (ifilter\ P\ xs)$ 
  shows  $\text{iset } (nfilter\ P\ (\text{suffix } (inth\ (nfilter\ P\ xs\ 0)\ k)\ xs)\ (inth\ (nfilter\ P\ xs\ 0)\ k)) =$ 
     $\text{iset } (\text{suffix } k\ (nfilter\ P\ xs\ 0))$ 

proof –
have 1:  $(inth\ (nfilter\ P\ xs\ 0)\ k) \leq \text{ilen } xs$ 
  by (metis assms(1) assms(2) diff-zero ilen-gr-zero
    nfilter-ilen nfilter-upper-bound ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 2:  $\exists x \in \text{iset } xs . P\ x$ 
  using assms(1) by blast
have 4:  $\exists x \in \text{iset } (\text{suffix } ((inth\ (nfilter\ P\ xs\ 0)\ k))\ xs) . P\ x$ 
  using nfilter-holds[of xs P 0]
  by (metis 1 2 assms(2) ifirst-suffix le0 nfilter-ilen nfilter-inth-n-zero inth-iset)
have 10:  $\text{iset } (nfilter\ P\ (\text{suffix } ((inth\ (nfilter\ P\ xs\ 0)\ k)\ xs)\ (inth\ (nfilter\ P\ xs\ 0)\ k)) =$ 
   $\{ (inth\ (nfilter\ P\ xs\ 0)\ k) + i \mid i. i \leq \text{ilen } xs - (inth\ (nfilter\ P\ xs\ 0)\ k) \wedge$ 
     $P\ (inth\ xs\ (i + (inth\ (nfilter\ P\ xs\ 0)\ k))) \}$ 

  using nfilter-suffix-iset-b[of (inth (nfilter P xs 0) k) xs P (inth (nfilter P xs 0) k)]
  using 1 4 by blast
have 5:  $\{ (inth\ (nfilter\ P\ xs\ 0)\ k) + i \mid i. i \leq \text{ilen } xs - (inth\ (nfilter\ P\ xs\ 0)\ k) \wedge$ 
   $P\ (inth\ xs\ (i + (inth\ (nfilter\ P\ xs\ 0)\ k))) \}$ 
  =
   $\{ (inth\ (nfilter\ P\ xs\ 0)\ k) + i \mid i. i \leq \text{ilen } xs - (inth\ (nfilter\ P\ xs\ 0)\ k) \wedge$ 
     $i + (inth\ (nfilter\ P\ xs\ 0)\ k) \in \text{iset } (nfilter\ P\ xs\ 0) \}$ 
  using 4 1 2 nfilter-holds-b by auto
have 51:  $\{ (inth\ (nfilter\ P\ xs\ 0)\ k) + i \mid i. i \leq \text{ilen } xs - (inth\ (nfilter\ P\ xs\ 0)\ k) \wedge$ 
   $i + (inth\ (nfilter\ P\ xs\ 0)\ k) \in \text{iset } (nfilter\ P\ xs\ 0) \} =$ 
   $\{ (inth\ (nfilter\ P\ xs\ 0)\ k) + i \mid i.$ 
     $(inth\ (nfilter\ P\ xs\ 0)\ k) \leq i + (inth\ (nfilter\ P\ xs\ 0)\ k) \wedge$ 
     $i + (inth\ (nfilter\ P\ xs\ 0)\ k) \leq \text{ilen } xs \wedge$ 
     $i + (inth\ (nfilter\ P\ xs\ 0)\ k) \in \text{iset } (nfilter\ P\ xs\ 0) \}$ 

  using 1 by auto
have 52:  $\{ (inth\ (nfilter\ P\ xs\ 0)\ k) + i \mid i.$ 
     $(inth\ (nfilter\ P\ xs\ 0)\ k) \leq i + (inth\ (nfilter\ P\ xs\ 0)\ k) \wedge$ 
     $i + (inth\ (nfilter\ P\ xs\ 0)\ k) \leq \text{ilen } xs \wedge$ 

```

$$i+(inth (nfilter P xs 0) k) \in iset(nfilter P xs 0) \} = \\ \{ j. (inth (nfilter P xs 0) k) \leq j \wedge j \leq ilen xs \wedge j \in iset(nfilter P xs 0) \}$$

by (*metis* (*no-types*, *lifting*) *add-diff-cancel-right'* *le0* *le-add-diff-inverse* *le-add-same-cancel2*)

have 53: $\{ j. (inth (nfilter P xs 0) k) \leq j \wedge j \leq ilen xs \wedge j \in iset(nfilter P xs 0) \} =$
 $\{ j. (inth (nfilter P xs 0) k) \leq j \wedge$
 $j \leq ilen xs \wedge j \in \{ (inth (nfilter P xs 0) jj) \mid jj. jj \leq ilen(nfilter P xs 0) \} \}$

by (*auto simp add: inth-and-iset*)

have 54: $\{ j. (inth (nfilter P xs 0) k) \leq j \wedge$
 $j \leq ilen xs \wedge j \in \{ (inth (nfilter P xs 0) jj) \mid jj. jj \leq ilen(nfilter P xs 0) \} \} =$
 $\{ (inth (nfilter P xs 0) j) \mid j. k \leq j \wedge j \leq ilen (nfilter P xs 0) \}$

using *assms 2* **by** (*auto*,
metis dual-order.antisym idx-nfilter-less-eq le-cases nfilter-ilen,
simp add: 2 idx-nfilter-less-eq,
metis add-cancel-right-left nfilter-upper-bound)

have 8: $k \leq ilen (nfilter P xs 0)$

by (*simp add: 2 assms(2) nfilter-ilen*)

have 9: $\{ (inth (nfilter P xs 0) j) \mid j. k \leq j \wedge j \leq ilen (nfilter P xs 0) \} =$
 $iset (suffix k (nfilter P xs 0))$

using 8 2 **using** *suffix-iset-a* **by** *blast*

show *?thesis*

using 10 5 51 52 53 54 9 **by** *simp*

qed

lemma *nfilter-nfilter-prefix*:

assumes $P (inth xs \ ((inth (nfilter P xs 0) k)))$
 $k \leq ilen (ifilter P xs)$

shows $(prefix k (nfilter P xs 0)) =$
 $(nfilter P (prefix ((inth (nfilter P xs 0) k)) xs) 0)$

using *assms*

by (*meson ifilter-nfilter-prefix-idx-a ifilter-nfilter-prefix-idx-b*
ifilter-nfilter-prefix-iset-eq idx-iset-eq)

lemma *nfilter-nfilter-prefix-1*:

assumes $\exists x \in iset xs . P x$
 $k \leq ilen (ifilter P xs)$

shows $(prefix k (nfilter P xs 0)) =$
 $(nfilter P (prefix ((inth (nfilter P xs 0) k)) xs) 0)$

using *assms*

by (*meson ifilter-nfilter-prefix-idx-a-1 ifilter-nfilter-prefix-idx-b-1*
ifilter-nfilter-prefix-iset-eq-1 idx-iset-eq)

lemma *nfilter-nfilter-suffix*:

assumes $P (inth xs \ ((inth (nfilter P xs 0) k)))$

$k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$
shows $(\text{suffix } k \text{ } (\text{nfilter } P \text{ } xs \text{ } 0)) =$
 $(\text{nfilter } P \text{ } (\text{suffix } ((\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k) \text{ }) \text{ } xs) \text{ } (\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k))$
using *assms*
by (*metis ifilter-nfilter-suffix-idx-a ifilter-nfilter-suffix-idx-b*
ifilter-nfilter-suffix-iset-eq idx-iset-eq)

lemma *nfilter-nfilter-suffix-1*:
assumes $\exists x \in \text{iset } xs . P \text{ } x$
 $k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$
shows $(\text{suffix } k \text{ } (\text{nfilter } P \text{ } xs \text{ } 0)) =$
 $(\text{nfilter } P \text{ } (\text{suffix } ((\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k) \text{ }) \text{ } xs) \text{ } (\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k))$
proof –
have 1: $P \text{ } (\text{inth } xs \text{ } ((\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k) \text{ })$
by (*metis assms(1) assms(2) nfilter-holds nfilter-ilen nfilter-inth-n-zero inth-iset*)
show ?thesis **using** *assms 1 nfilter-nfilter-suffix* **by** *auto*
qed

lemma *nfilter-imap-ifilter*:
assumes $(\exists x \in \text{iset } xs . P \text{ } x)$
shows $\text{imap } (\lambda n. (\text{inth } xs \text{ } n)) (\text{nfilter } P \text{ } xs \text{ } 0) = \text{ifilter } P \text{ } xs$
proof –
have 1: $\text{ilen } (\text{imap } (\lambda n. (\text{inth } xs \text{ } n)) (\text{nfilter } P \text{ } xs \text{ } 0)) = \text{ilen } (\text{ifilter } P \text{ } xs)$
by (*simp add: assms nfilter-ilen*)
have 2: $\bigwedge i. i \leq \text{ilen } (\text{imap } (\lambda n. (\text{inth } xs \text{ } n)) (\text{nfilter } P \text{ } xs \text{ } 0)) \longrightarrow$
 $(\text{inth } (\text{imap } (\lambda n. (\text{inth } xs \text{ } n)) (\text{nfilter } P \text{ } xs \text{ } 0)) \text{ } i) =$
 $(\text{inth } (\text{ifilter } P \text{ } xs) \text{ } i)$
by (*metis assms diff-zero ilen-imap inth-imap nfilter-ifilter*)
from 1 2 **show** ?thesis
using *interval-eq-inth-eq* **by** *blast*
qed

lemma *ifilter-nfilter-prefix*:
assumes $P \text{ } (\text{inth } xs \text{ } ((\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k) \text{ })$
 $k \leq \text{ilen } (\text{ifilter } P \text{ } xs)$
shows $(\text{prefix } k \text{ } (\text{ifilter } P \text{ } xs)) =$
 $(\text{ifilter } P \text{ } (\text{prefix } ((\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k) \text{ }) \text{ } xs) \text{ })$
proof –
have 1: $\exists x \in \text{iset } xs . P \text{ } x$
by (*metis assms(1) assms(2) diff-zero ifilter-nfilter-prefix-ilen-0*)
have 2: $(\text{ifilter } P \text{ } xs) = \text{imap } (\lambda s. \text{inth } xs \text{ } s) (\text{nfilter } P \text{ } xs \text{ } 0)$
by (*simp add: 1 nfilter-imap-ifilter*)
have 3: $(\text{prefix } k \text{ } (\text{ifilter } P \text{ } xs)) =$
 $(\text{prefix } k \text{ } (\text{imap } (\lambda s. \text{inth } xs \text{ } s) (\text{nfilter } P \text{ } xs \text{ } 0)))$
by (*simp add: 2*)
have 4: $(\text{prefix } k \text{ } (\text{imap } (\lambda s. \text{inth } xs \text{ } s) (\text{nfilter } P \text{ } xs \text{ } 0))) =$
 $\text{imap } (\lambda s. \text{inth } xs \text{ } s) (\text{prefix } k \text{ } (\text{nfilter } P \text{ } xs \text{ } 0))$
by (*simp add: 1 assms(2) imap-prefix nfilter-ilen*)
have 5: $\exists x \in \text{iset}(\text{prefix } ((\text{inth } (\text{nfilter } P \text{ } xs \text{ } 0) \text{ } k)) \text{ } xs) . P \text{ } x$
by (*metis 1 add.left-neutral assms(1) assms(2) ilast-prefix inth-and-iset*)

$nfilter_ilen\ nfilter_upper_bound\ order_refl$
have 6: $(ifilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\) =$
 $imap\ (\lambda s. (inth\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\ s))$
 $(nfilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\ 0)$
by (*simp add: 5 nfilter-imap-ifilter*)
have 7: $imap\ (\lambda s. inth\ xs\ s)\ (prefix\ k\ (nfilter\ P\ xs\ 0)) =$
 $imap\ (\lambda s. inth\ xs\ s)\ (nfilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\ 0)$
by (*simp add: assms(1) assms(2) nfilter-nfilter-prefix*)
have 8: $imap\ (\lambda s. inth\ xs\ s)\ (nfilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\ 0) =$
 $imap\ (\lambda s. (inth\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\ s))$
 $(nfilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\ 0)$

using 1 5 **by** *simp*
show ?thesis
by (*simp add: 3 4 6 7 8*)
qed

lemma *ifilter-nfilter-prefix-1*:
assumes $\exists x \in iset\ xs . P\ x$
 $k \leq ilen\ (ifilter\ P\ xs)$
shows $(prefix\ k\ (ifilter\ P\ xs)) =$
 $(ifilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs)\)$
proof –
have 1: $P\ (inth\ xs\ ((inth\ (nfilter\ P\ xs\ 0)\ k)))$
by (*metis assms(1) assms(2) nfilter-holds nfilter-ilen nfilter-inth-n-zero inth-iset*)
show ?thesis **using** *assms 1 ifilter-nfilter-prefix* **by** *auto*
qed

lemma *ifilter-nfilter-prefix-ilen*:
assumes $P\ (inth\ xs\ ((inth\ (nfilter\ P\ xs\ 0)\ k)))$
 $k \leq ilen\ (ifilter\ P\ xs)$
shows $ilen(prefix\ k\ (ifilter\ P\ xs)) =$
 $ilen(ifilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs))$
using *assms*
by (*simp add: ifilter-nfilter-prefix*)

lemma *ifilter-nfilter-prefix-ilen-1*:
assumes $\exists x \in iset\ xs . P\ x$
 $k \leq ilen\ (ifilter\ P\ xs)$
shows $ilen(prefix\ k\ (ifilter\ P\ xs)) =$
 $ilen(ifilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs))$
using *assms*
by (*simp add: ifilter-nfilter-prefix-1*)

lemma *ifilter-nfilter-prefix-inth*:
assumes $P\ (inth\ xs\ ((inth\ (nfilter\ P\ xs\ 0)\ k)))$
 $k \leq ilen\ (ifilter\ P\ xs)$
 $j \leq ilen(prefix\ k\ (ifilter\ P\ xs))$
shows $inth\ (prefix\ k\ (ifilter\ P\ xs))\ j =$
 $inth\ (ifilter\ P\ (prefix\ ((inth\ (nfilter\ P\ xs\ 0)\ k)\)\ xs))\ j$

using *assms*
 by (simp add: ifilter-nfilter-prefix)

lemma *ifilter-nfilter-prefix-inth-1*:

assumes $\exists x \in \text{iset } xs . P x$
 $k \leq \text{ilen } (\text{ifilter } P \ xs)$
 $j \leq \text{ilen}(\text{prefix } k \ (\text{ifilter } P \ xs))$
 shows $\text{inth } (\text{prefix } k \ (\text{ifilter } P \ xs)) \ j =$
 $\text{inth } (\text{ifilter } P \ (\text{prefix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs)) \ j$

using *assms*
 by (simp add: ifilter-nfilter-prefix-1)

lemma *nfilter-imap-shift*:

assumes $\exists x \in \text{iset } xs . P x$
 shows $\text{imap } (\lambda s. \text{inth } xs \ (s+n)) \ (\text{nfilter } P \ xs \ 0) =$
 $\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ n)$

proof –

have 1: $\text{ilen } (\text{imap } (\lambda s. \text{inth } xs \ (s+n)) \ (\text{nfilter } P \ xs \ 0)) =$
 $\text{ilen } (\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ n))$
 by (metis *assms*(1) *ilen-imap nfilter-ilen-n-zero*)
 have 2: $\bigwedge i. \text{inth } (\text{imap } (\lambda s. \text{inth } xs \ (s+n)) \ (\text{nfilter } P \ xs \ 0)) \ i =$
 $(\text{inth } xs \ ((\text{inth } (\text{nfilter } P \ xs \ 0) \ i) + n))$
 by (simp add: *inth-imap*)
 have 3: $\bigwedge i. \text{inth } (\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ n)) \ i =$
 $(\text{inth } xs \ ((\text{inth } (\text{nfilter } P \ xs \ n) \ i)))$
 by (simp add: *inth-imap*)
 have 4: $\bigwedge i. (\text{inth } (\text{nfilter } P \ xs \ 0) \ i) + n = (\text{inth } (\text{nfilter } P \ xs \ n) \ i)$
 by (metis *assms*(1) *inth-imap nfilter-n-zero*)
 show ?thesis
 by (metis 1 2 3 4 *interval-eq-inth-eq*)

qed

lemma *nfilter-imap-shift-suffix*:

assumes $\exists x \in \text{iset}(\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k)) \ xs) . P x$
 shows $\text{imap } (\lambda s. \text{inth } xs \ (s+(\text{inth } (\text{nfilter } P \ xs \ 0) \ k)))$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0) =$
 $\text{imap } (\lambda s. \text{inth } xs \ s)$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k))$

proof –

have 1: $\text{ilen } (\text{imap } (\lambda s. \text{inth } xs \ (s+(\text{inth } (\text{nfilter } P \ xs \ 0) \ k)))$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0)) =$
 $\text{ilen } (\text{imap } (\lambda s. \text{inth } xs \ s)$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k)))$
 by (metis *assms* *ilen-imap nfilter-ilen-n-zero*)
 have 2: $\bigwedge i. \text{inth } (\text{imap } (\lambda s. \text{inth } xs \ (s+(\text{inth } (\text{nfilter } P \ xs \ 0) \ k)))$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0)) \ i =$
 $\text{inth } xs$
 $((\text{inth } (\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0) \ i) + (\text{inth } (\text{nfilter } P \ xs \ 0) \ k))$

using *inth-imap* by blast

have 3: $\bigwedge i.$

$$\text{inth } (\text{imap } (\lambda s. \text{inth } xs \ s) \\
(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \\
(\text{inth } (\text{nfilter } P \ xs \ 0) \ k))) \ i = \\
\text{inth } xs \ (\text{inth } (\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \\
(\text{inth } (\text{nfilter } P \ xs \ 0) \ k))) \ i)$$

using *inth-imap by blast*
have 4: $\bigwedge i.$ $(\text{inth } (\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0) \ i) \\
+ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k) = \\
(\text{inth } (\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k))) \ i)$
by (*metis assms inth-imap nfilter-n-zero*)
show *?thesis*
by (*metis 1 2 3 4 interval-eq-inth-eq*)
qed

lemma *ifilter-nfilter-suffix*:

assumes $P \ (\text{inth } xs \ (\ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k))) \)$
 $k \leq \text{ilen } (\text{ifilter } P \ xs)$
shows $(\text{suffix } k \ (\text{ifilter } P \ xs) \) = \\
(\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \)$

proof –

have 1: $\exists x \in \text{iset } xs . P \ x$
by (*metis assms(1) assms(2) diff-zero ifilter-nfilter-prefix-ilen-0*)
have 2: $(\text{ifilter } P \ xs) = \text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ 0)$
by (*simp add: 1 nfilter-imap-ifilter*)
have 3: $(\text{suffix } k \ (\text{ifilter } P \ xs) \) = \\
(\text{suffix } k \ (\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ 0)))$
by (*simp add: 2*)
have 4: $(\text{suffix } k \ (\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ 0))) = \\
\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{suffix } k \ (\text{nfilter } P \ xs \ 0))$
by (*simp add: 1 assms(2) imap-suffix nfilter-ilen*)
have 5: $\exists x \in \text{iset}(\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k)) \ xs). P \ x$
by (*metis 1 add-cancel-right-left assms(1) assms(2) ifirst-suffix \\
ilen-gr-zero inth-and-iset nfilter-ilen nfilter-upper-bound*)
have 6: $(\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \) = \\
\text{imap } (\lambda s. \text{inth } (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ s) \\
(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0))$

by (*simp add: 5 nfilter-imap-ifilter*)
have 7: $\text{imap } (\lambda s. \text{inth } (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ s) \\
(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0) = \\
\text{imap } (\lambda s. \text{inth } xs \ (s+(\text{inth } (\text{nfilter } P \ xs \ 0) \ k))) \\
(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0))$

using 1 5 **by** (*simp add: add commute*)
have 8: $\text{imap } (\lambda s. \text{inth } xs \ (s+(\text{inth } (\text{nfilter } P \ xs \ 0) \ k))) \\
(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0) = \\
\text{imap } (\lambda s. \text{inth } xs \ s) \\
(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k)))$

using 5 *nfilter-imap-shift-suffix* **by** *metis*
have 9: $\text{imap } (\lambda s. \text{inth } xs \ s)$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k)) =$
 $\text{imap } (\lambda s. \text{inth } xs \ s)$
 $(\text{suffix } k \ (\text{nfilter } P \ xs \ 0))$

by (*simp add: assms(1) assms(2) nfilter-nfilter-suffix*)
show *?thesis*
by (*simp add: 3 4 6 7 8 9*)
qed

lemma *ifilter-nfilter-suffix-1:*

assumes $\exists x \in \text{iset } xs. P \ x$
 $k \leq \text{ilen } (\text{ifilter } P \ xs)$
shows $(\text{suffix } k \ (\text{ifilter } P \ xs) \) =$
 $(\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \)$
proof –
have 1: $\exists x \in \text{iset } xs. P \ x$
using *assms* **by** *auto*
have 2: $(\text{ifilter } P \ xs) = \text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ 0)$
by (*simp add: 1 nfilter-imap-ifilter*)
have 3: $(\text{suffix } k \ (\text{ifilter } P \ xs) \) =$
 $(\text{suffix } k \ (\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ 0)))$
by (*simp add: 2*)
have 4: $(\text{suffix } k \ (\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{nfilter } P \ xs \ 0))) =$
 $\text{imap } (\lambda s. \text{inth } xs \ s) \ (\text{suffix } k \ (\text{nfilter } P \ xs \ 0))$
by (*simp add: 1 assms(2) imap-suffix nfilter-ilen*)
have 5: $\exists x \in \text{iset}(\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k)) \ xs). P \ x$
using *assms inth-iset 2 inth-suffix nfilter-holds[of xs P 0]*
by (*metis add.right-neutral diff-zero ilen-imap le0*)
have 6: $(\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \) =$
 $\text{imap } (\lambda s. \text{inth } (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ s)$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0)$

by (*simp add: 5 nfilter-imap-ifilter*)
have 7: $\text{imap } (\lambda s. \text{inth } (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ s)$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0) =$
 $\text{imap } (\lambda s. \text{inth } xs \ (s + (\text{inth } (\text{nfilter } P \ xs \ 0) \ k)))$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0)$

using 1 5
by (*metis add.right-neutral inth-suffix suffix-suffix le0*)
have 8: $\text{imap } (\lambda s. \text{inth } xs \ (s + (\text{inth } (\text{nfilter } P \ xs \ 0) \ k)))$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ 0) =$
 $\text{imap } (\lambda s. \text{inth } xs \ s)$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k))$

using 5 *nfilter-imap-shift-suffix* **by** *metis*

have 9: $\text{imap } (\lambda s. \text{inth } xs \ s)$
 $(\text{nfilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs) \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k)) =$
 $\text{imap } (\lambda s. \text{inth } xs \ s)$
 $(\text{suffix } k \ (\text{nfilter } P \ xs \ 0))$

by (*simp add: assms(1) assms(2) nfilter-nfilter-suffix-1*)
show ?thesis
by (*simp add: 3 4 6 7 8 9*)
qed

lemma *ifilter-nfilter-suffix-ilen:*
assumes $P \ (\text{inth } xs \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k))$
 $k \leq \text{ilen } (\text{ifilter } P \ xs)$
shows $\text{ilen}(\text{suffix } k \ (\text{ifilter } P \ xs)) =$
 $\text{ilen}(\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs))$
using *assms*
by (*simp add: ifilter-nfilter-suffix*)

lemma *ifilter-nfilter-suffix-ilen-1:*
assumes $\exists x \in \text{iset } xs. P \ x$
 $k \leq \text{ilen } (\text{ifilter } P \ xs)$
shows $\text{ilen}(\text{suffix } k \ (\text{ifilter } P \ xs)) =$
 $\text{ilen}(\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs))$
using *assms*
by (*simp add: ifilter-nfilter-suffix-1*)

lemma *ifilter-nfilter-suffix-inth:*
assumes $P \ (\text{inth } xs \ (\text{inth } (\text{nfilter } P \ xs \ 0) \ k))$
 $k \leq \text{ilen } (\text{ifilter } P \ xs)$
 $j \leq \text{ilen}(\text{suffix } k \ (\text{ifilter } P \ xs))$
shows $\text{inth } (\text{suffix } k \ (\text{ifilter } P \ xs)) \ j =$
 $\text{inth } (\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs)) \ j$
using *assms*
by (*simp add: ifilter-nfilter-suffix*)

lemma *ifilter-nfilter-suffix-inth-1:*
assumes $\exists x \in \text{iset } xs. P \ x$
 $k \leq \text{ilen } (\text{ifilter } P \ xs)$
 $j \leq \text{ilen}(\text{suffix } k \ (\text{ifilter } P \ xs))$
shows $\text{inth } (\text{suffix } k \ (\text{ifilter } P \ xs)) \ j =$
 $\text{inth } (\text{ifilter } P \ (\text{suffix } ((\text{inth } (\text{nfilter } P \ xs \ 0) \ k) \) \ xs)) \ j$
using *assms*
by (*simp add: ifilter-nfilter-suffix-1*)

lemma *nfilter-ilast:*
assumes $P \ (\text{ilast } (\text{prefix } k \ xs))$
 $k \leq \text{ilen } xs$
shows $(\text{inth } xs \ ((\text{ilast } (\text{nfilter } P \ (\text{prefix } k \ xs) \ n)) - n)) = (\text{ilast } (\text{prefix } k \ xs))$
using *assms*
proof (*induction xs arbitrary: n k*)

```

case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (auto split:nat.split)
    fix x2
    fix x
    assume a0: ( $\bigwedge k n. P (inth\ xs\ k) \implies$ 
       $k \leq ilen\ xs \implies$ 
       $inth\ xs\ (inth\ (nfilter\ P\ (prefix\ k\ xs)\ n)\ (ilen\ (nfilter\ P\ (prefix\ k\ xs)\ n)) - n) =$ 
       $inth\ xs\ k$ )
    assume a1:  $P (inth\ xs\ x2)$ 
    assume a2:  $x \in iset\ (prefix\ x2\ xs)$ 
    assume a3:  $P\ x$ 
    assume a4:  $k = Suc\ x2$ 
    assume a5:  $x2 \leq ilen\ xs$ 
    assume a6:  $inth\ (nfilter\ P\ (prefix\ x2\ xs)\ (Suc\ n))\ (ilen\ (nfilter\ P\ (prefix\ x2\ xs)\ (Suc\ n)))$ 
       $\leq n$ 
    show  $x1a = inth\ xs\ x2$ 
    using a0 a1 a2 a3 a6 by (metis le-refl nfilter-inth-n-zero-a not-less-eq-eq)
  next
    fix x2
    fix x
    fix x2a
    assume b0: ( $\bigwedge k n. P (inth\ xs\ k) \implies$ 
       $k \leq ilen\ xs \implies$ 
       $inth\ xs\ (inth\ (nfilter\ P\ (prefix\ k\ xs)\ n)\ (ilen\ (nfilter\ P\ (prefix\ k\ xs)\ n)) - n) =$ 
       $inth\ xs\ k$ )
    assume b1:  $P (inth\ xs\ x2)$ 
    assume b2:  $x \in iset\ (prefix\ x2\ xs)$ 
    assume b3:  $P\ x$ 
    assume b4:  $k = Suc\ x2$ 
    assume b5:  $x2 \leq ilen\ xs$ 
    assume b6:  $inth\ (nfilter\ P\ (prefix\ x2\ xs)\ (Suc\ n))\ (ilen\ (nfilter\ P\ (prefix\ x2\ xs)\ (Suc\ n))) - n = Suc\ x2a$ 
    show  $inth\ xs\ x2a = inth\ xs\ x2$ 
    using b0 b1 b2 b3 b4 b5 b6
    by (metis Suc-eq-plus1 add-diff-cancel-left' diff-diff-left plus-1-eq-Suc)
  next
    fix x2
    assume c0: ( $\bigwedge k n. P (inth\ xs\ k) \implies$ 
       $k \leq ilen\ xs \implies$ 
       $inth\ xs\ (inth\ (nfilter\ P\ (prefix\ k\ xs)\ n)\ (ilen\ (nfilter\ P\ (prefix\ k\ xs)\ n)) - n) =$ 
       $inth\ xs\ k$ )
    assume c1:  $P (inth\ xs\ x2)$ 
    assume c2:  $\forall x \in iset\ (prefix\ x2\ xs). \neg P\ x$ 
    assume c3:  $k = Suc\ x2$ 
    assume c4:  $x2 \leq ilen\ xs$ 
    show  $x1a = inth\ xs\ x2$ 

```

```

    using c0 c1 c2 c3 c4 ilast-prefix inth-iset by fastforce
qed
qed

```

```

lemma nfilter-ifirst:
  assumes P (ifirst (suffix k xs))
    k ≤ ilen xs
  shows ifirst (nfilter P (suffix k xs) n) = n
using assms
proof (induction xs arbitrary: k n)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  by (auto split:nat.split)
qed

```

```

lemma ifilter-ilast:
  assumes P (ilast (prefix n xs))
    n ≤ ilen xs
  shows ilast (ifilter P (prefix n xs)) = (ilast (prefix n xs))
using assms
proof (induction n arbitrary: xs)
case 0
then show ?case by simp
next
case (Suc n)
then show ?case
  proof (cases xs)
  case (INil x1)
  then show ?thesis
  by simp
  next
  case (ICons x21 x22)
  then show ?thesis using Suc inth-iset by (auto, force)
  qed
qed

```

```

lemma ifilter-ifirst:
  assumes P (ifirst (suffix n xs))
  shows ifirst (ifilter P (suffix n xs)) = (ifirst (suffix n xs))
using assms
proof (induction xs arbitrary: n)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case by (auto split:nat.split)
qed

```

lemma *ifilter-inth-aa*:
assumes $(\exists x \in \text{iset } xs. P x)$
 $n \leq \text{ilen } (\text{ifilter } P \text{ } xs)$
shows $P (\text{inth } (\text{ifilter } P \text{ } xs) n)$
using *assms iset-ifilter inth-iset*[of $n (\text{ifilter } P \text{ } xs)$] **by** *simp*

lemma *ifilter-ilen-zero-conv-a*:
assumes $(\exists x \in \text{iset } xs. P x)$
 $\text{ilen } (\text{ifilter } P \text{ } xs) = 0$
shows $(\exists k \leq \text{ilen } xs. P (\text{inth } xs \ k) \wedge$
 $(\forall j \leq \text{ilen } xs. j \neq k \longrightarrow \neg P (\text{inth } xs \ j)))$

proof –

have 1: $P (\text{inth } xs \ (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ 0))$
by (*metis nfilter-imap-ifilter assms(1) assms(2) ifilter-inth-aa inth-imap*
le-numeral-extra(3))
have 2: $(\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ 0) \leq \text{ilen } xs$
by (*metis assms(1) diff-zero ilen-gr-zero nfilter-upper-bound*
ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 3: $(\forall j \leq \text{ilen } xs. j \neq (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ 0) \longrightarrow \neg P (\text{inth } xs \ j))$
using *assms nfilter-not-after*[of $xs \ P$] *nfilter-not-before*[of $xs \ P$]
by (*metis linorder-neqE-nat nfilter-ilen*)
show ?thesis
using 1 2 3 **by** *blast*
qed

lemma *ifilter-ilen-zero-conv-c*:
 $(\exists k \leq \text{ilen } xs. P (\text{inth } xs \ k) \wedge$
 $(\forall j \leq \text{ilen } xs. j \neq k \longrightarrow \neg P (\text{inth } xs \ j))) =$
 $(\exists k \leq \text{ilen } xs. P (\text{inth } xs \ k) \wedge$
 $(\forall j \leq \text{ilen } xs. j < k \vee k < j \longrightarrow \neg P (\text{inth } xs \ j)))$

using *antisym-conv3* **by** *auto*

lemma *ifilter-ilen-zero-conv-d*:
 $(\exists k \leq \text{ilen } xs. P (\text{inth } xs \ k) \wedge$
 $(\forall j \leq \text{ilen } xs. j < k \vee k < j \longrightarrow \neg P (\text{inth } xs \ j))) =$
 $(\exists k \leq \text{ilen } xs. P (\text{inth } xs \ k) \wedge$
 $(\forall j. j < k \longrightarrow \neg P (\text{inth } xs \ j)) \wedge$
 $(\forall j \leq \text{ilen } xs. k < j \longrightarrow \neg P (\text{inth } xs \ j)))$
 $)$

by *auto*

lemma *ifilter-ilen-zero-conv-b*:
assumes $(\exists k \leq \text{ilen } xs. P (\text{inth } xs \ k) \wedge$
 $(\forall j \leq \text{ilen } xs. j \neq k \longrightarrow \neg P (\text{inth } xs \ j)))$
shows $(\exists x \in \text{iset } xs. P x) \wedge \text{ilen } (\text{ifilter } P \text{ } xs) = 0$

proof –
have 1: $(\exists x \in \text{iset } xs. P x)$
using *assms inth-iset* **by** *auto*
obtain k **where** 2: $k \leq \text{ilen } xs \wedge P (\text{inth } xs k) \wedge$
 $(\forall j \leq \text{ilen } xs. j \neq k \longrightarrow \neg P (\text{inth } xs j))$
using *assms* **by** *auto*
have 3: $\text{ilen } (\text{ifilter } P xs) = 0$
using 1 2
proof (*induct xs arbitrary: k*)
case (*INil x*)
then show ?*case* **by** *simp*
next
case (*ICons x1a xs*)
then show ?*case*
proof (*cases k*)
case 0
then show ?*thesis* **using** *ICons*
by (*metis Suc-le-mono ifilter-ilen-d inth-Suc inth-and-iset ilen.simps(2)*
le0 not-less-eq-eq order-refl plus-1-eq-Suc)
next
case (*Suc nat*)
then show ?*thesis* **using** *ICons.hyps ICons.prem*s
by (*metis Suc-le-mono Zero-not-Suc add-diff-cancel-left' ifilter-ilen-c*
ilen-gr-zero inth-Suc inth-zero iset-IConsD ilen.simps(2) plus-1-eq-Suc)
qed
qed
show ?*thesis*
using 1 3 **by** *blast*
qed

lemma *ifilter-ilen-zero-conv*:

$$\begin{aligned}
 & ((\exists x \in \text{iset } xs. P x) \wedge \text{ilen } (\text{ifilter } P xs) = 0) = \\
 & (\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge \\
 & (\forall j \leq \text{ilen } xs. j \neq k \longrightarrow \neg P (\text{inth } xs j)))
 \end{aligned}$$

using *ifilter-ilen-zero-conv-a*[*of xs P*] *ifilter-ilen-zero-conv-b*[*of xs P*]
by *blast*

lemma *ifilter-ilen-zero-conv-1*:

$$\begin{aligned}
 & ((\exists x \in \text{iset } xs. P x) \wedge \text{ilen } (\text{ifilter } P xs) = 0) = \\
 & (\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge \\
 & (\forall j \leq \text{ilen } xs. j < k \vee k < j \longrightarrow \neg P (\text{inth } xs j)))
 \end{aligned}$$

proof –

$$\begin{aligned}
 \text{have } 1: & ((\exists x \in \text{iset } xs. P x) \wedge \text{ilen } (\text{ifilter } P xs) = 0) = \\
 & (\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge \\
 & (\forall j \leq \text{ilen } xs. j \neq k \longrightarrow \neg P (\text{inth } xs j)))
 \end{aligned}$$

by (*simp add: ifilter-ilen-zero-conv*)

$$\begin{aligned}
 \text{have } 2: & (\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge \\
 & (\forall j \leq \text{ilen } xs. j \neq k \longrightarrow \neg P (\text{inth } xs j))) = \\
 & (\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge \\
 & (\forall j \leq \text{ilen } xs. j < k \vee k < j \longrightarrow \neg P (\text{inth } xs j)))
 \end{aligned}$$

by fastforce
 show ?thesis by (simp add: 1 2)
 qed

lemma ifilter-ilen-zero-conv-2:

$((\exists x \in \text{iset } xs. P x) \wedge \text{ilen } (\text{ifilter } P \text{ } xs) = 0) =$
 $(\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge$
 $(\forall j. j < k \longrightarrow \neg P (\text{inth } xs j)) \wedge$
 $(\forall j \leq \text{ilen } xs. k < j \longrightarrow \neg P (\text{inth } xs j))$
 $)$

proof –

have 1: $((\exists x \in \text{iset } xs. P x) \wedge \text{ilen } (\text{ifilter } P \text{ } xs) = 0) =$
 $(\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge$
 $(\forall j \leq \text{ilen } xs. j < k \vee k < j \longrightarrow \neg P (\text{inth } xs j)))$

by (simp add: ifilter-ilen-zero-conv-1)

have 2: $(\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge$
 $(\forall j \leq \text{ilen } xs. j < k \vee k < j \longrightarrow \neg P (\text{inth } xs j))) =$
 $(\exists k \leq \text{ilen } xs. P (\text{inth } xs k) \wedge$
 $(\forall j. j < k \longrightarrow \neg P (\text{inth } xs j)) \wedge$
 $(\forall j \leq \text{ilen } xs. k < j \longrightarrow \neg P (\text{inth } xs j))$
 $)$

using dual-order.strict-trans1 by auto

from 1 2 show ?thesis by auto

qed

lemma ifilter-suffixes-imap-help-0:

assumes $j \leq \text{inth}(\text{nfilter } P \text{ } xs \ 0) \ 0$

$\exists x \in \text{iset } xs. P x$

shows $(\text{ifilter } P (\text{suffix } (\text{inth}(\text{nfilter } P \text{ } xs \ 0) \ 0) \ xs)) = (\text{ifilter } P (\text{suffix } j \ xs))$

using assms

proof (induct xs arbitrary: j)

case (INil x)

then show ?case by simp

next

case (ICons x1a xs)

then show ?case

proof (cases $\exists a \in \text{iset } xs. P a$)

case True

then show ?thesis

proof (cases P x1a)

case True

then show ?thesis using ICons.prem

by (metis le-zero-eq nfilter-inth-ICons)

next

case False

then show ?thesis

proof (cases j)

case 0

then show ?thesis using ICons False $\langle \exists a \in \text{iset } xs. P a \rangle$

by (metis ifilter-nfilter-suffix-1 suffix-zero le0)

```

next
case (Suc nat)
then show ?thesis using ICons False < $\exists a \in \text{iset } xs. P a$ > by (auto split: nat.split)
    (metis One-nat-def add-diff-cancel-left' le0 nfilter-inth-n-zero plus-1-eq-Suc)
qed
qed
next
case False
then show ?thesis
using ICons by auto
qed
qed

```

```

lemma ifilter-suffixes-imap-help-0-a:
assumes  $j \leq \text{inth}(\text{nfilter } P (\text{suffixes } xs) 0) 0$ 
     $\exists x \in \text{iset } (\text{suffixes } xs). P x$ 
shows  $(\text{ifilter } P (\text{suffixes } (\text{suffix } (\text{inth}(\text{nfilter } P (\text{suffixes } xs) 0) 0) xs))) =$ 
     $(\text{ifilter } P (\text{suffixes } (\text{suffix } j xs)))$ 
proof -
have 1:  $(\text{suffix } (\text{inth}(\text{nfilter } P (\text{suffixes } xs) 0) 0) (\text{suffixes } xs)) =$ 
     $(\text{suffixes } (\text{suffix } (\text{inth}(\text{nfilter } P (\text{suffixes } xs) 0) 0) xs))$ 
by (metis assms(2) diff-zero ilen-gr-zero nfilter-upper-bound
    ordered-cancel-comm-monoid-diff-class.add-diff-inverse suffix-suffixes)
have 2:  $\text{inth}(\text{nfilter } P (\text{suffixes } xs) 0) 0 \leq \text{ilen } (\text{suffixes } xs)$ 
by (metis add.left-neutral assms(2) ilen-gr-zero nfilter-upper-bound)
have 3:  $(\text{suffix } j (\text{suffixes } xs)) = (\text{suffixes } (\text{suffix } j xs))$ 
using 2 suffix-suffixes assms le-trans by blast
show ?thesis
using 1 3 assms ifilter-suffixes-imap-help-0 by fastforce
qed

```

```

lemma ifilter-suffixes-imap-help-1:
assumes  $j \leq \text{inth}(\text{nfilter } P xs 0) 1$ 
     $0 < \text{ilen}(\text{ifilter } P xs)$ 
     $\text{inth}(\text{nfilter } P xs 0) 0 < j$ 
     $\exists x \in \text{iset } xs. P x$ 
shows  $(\text{ifilter } P (\text{suffix } (\text{inth}(\text{nfilter } P xs 0) 1) xs)) = (\text{ifilter } P (\text{suffix } j xs))$ 
using assms
proof (induct xs arbitrary:j)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
proof (cases  $\exists a \in \text{iset } xs. P a$ )
case True
then show ?thesis
proof (cases  $P x1a$ )
case True
then show ?thesis

```



```

proof (cases j)
case 0
then show ?thesis
using ICons.premis by blast
next
case (Suc nat)
then show ?thesis using ICons True  $\langle \exists a \in \text{iset } xs. P a \rangle$  ifilter-suffixes-imap-help-0[of - P xs]
  nfilter-inth-n-zero[of xs P]
  proof auto
  fix x :: 'a
  assume j = Suc nat
  assume a1:  $\text{Suc nat} \leq \text{inth } (nfilter P xs) (\text{Suc } 0)$  0
  assume a2:  $\bigwedge k n. k \leq \text{ilen } (nfilter P xs n) \implies$ 
     $\text{inth } (nfilter P xs n) k - n = \text{inth } (nfilter P xs 0) k$ 
  assume a3:  $\bigwedge j. j \leq \text{inth } (nfilter P xs 0) 0 \implies$ 
     $\text{ifilter } P (\text{suffix } (\text{inth } (nfilter P xs 0) 0) xs) = \text{ifilter } P (\text{suffix } j xs)$ 
  have Suc ( $\text{inth } (nfilter P xs 0) 0$ ) =  $\text{inth } (nfilter P xs 1) 0$ 
  using a2 a1 by (metis (no-types) One-nat-def Suc-le-D diff-Suc-Suc diff-zero le0)
  then show ifilter P (case  $\text{inth } (nfilter P xs) (\text{Suc } 0)$  0 of 0  $\Rightarrow x1a \odot xs$  |
     $\text{Suc } n \Rightarrow \text{suffix } n xs$ ) = ifilter P (suffix nat xs)
  using a3 a2 a1 by (simp add: Nitpick.case-nat-unfold)
qed
qed
next
case False
then show ?thesis using False ICons  $\langle \exists a \in \text{iset } xs. P a \rangle$ 
  proof (auto split: nat.split)
  fix x2::nat
  fix x
  fix x2a
  assume a0:  $\neg P x1a$ 
  assume a1:  $(\bigwedge j. j \leq \text{inth } (nfilter P xs 0) (\text{Suc } 0) \implies$ 
     $\text{inth } (nfilter P xs 0) 0 < j \implies$ 
     $\text{ifilter } P (\text{suffix } (\text{inth } (nfilter P xs 0) (\text{Suc } 0)) xs) = \text{ifilter } P (\text{suffix } j xs))$ 
  assume a2:  $x2a \leq x2$ 
  assume a3:  $0 < \text{ilen } (\text{ifilter } P xs)$ 
  assume a4:  $\text{inth } (nfilter P xs (\text{Suc } 0)) 0 < \text{Suc } x2a$ 
  assume a5:  $x \in \text{iset } xs$ 
  assume a6:  $P x$ 
  assume a7:  $\text{inth } (nfilter P xs (\text{Suc } 0)) (\text{Suc } 0) = \text{Suc } x2$ 
  assume a8:  $j = \text{Suc } x2a$ 
  show ifilter P (suffix x2 xs) = ifilter P (suffix x2a xs)
  proof -
  have 1:  $j > 0$ 
  using a8 by auto
  have 2:  $x2a = j - 1$ 
  by (simp add: a8)
  have 3:  $x2 = \text{inth } (nfilter P xs (\text{Suc } 0)) (\text{Suc } 0) - (\text{Suc } 0)$ 
  by (simp add: a7)
  have 4:  $(\text{inth } (nfilter P xs (\text{Suc } 0)) (\text{Suc } 0) - (\text{Suc } 0)) = (\text{inth } (nfilter P xs 0) (\text{Suc } 0))$ 

```

```

  by (metis Suc-leI a3 a5 a6 nfilter-ilen nfilter-inth-n-zero)
have 5:  $0 < \text{ilen} (\text{ifilter } P \text{ } xs)$ 
  by (simp add: a3)
have 6:  $\text{ifilter } P (\text{suffix } x2 \text{ } xs) = \text{ifilter } P (\text{suffix } (\text{inth } (\text{nfilter } P \text{ } xs \ 0) (\text{Suc } 0)) \text{ } xs)$ 
  by (simp add: 3 4)
have 7:  $\text{ifilter } P (\text{suffix } x2a \text{ } xs) = \text{ifilter } P (\text{suffix } (j-1) \text{ } xs)$ 
  using 2 by blast
have 8:  $j-1 \leq \text{inth } (\text{nfilter } P \text{ } xs \ 0) \ 1$ 
  by (metis 2 3 4 One-nat-def a2)
have 9:  $(\text{inth } (\text{nfilter } P \text{ } xs \ (\text{Suc } 0)) \ 0) - (\text{Suc } 0) = (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ 0)$ 
  using a5 a6 nfilter-inth-n-zero by fastforce
have 10:  $(\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ 0) < j - (\text{Suc } 0)$ 
  using nfilter-inth-ICons[of  $P \ x1a \ xs \ -$ ]
    ifilter-inth-aa[of  $xs \ P \ j-1$ ]
    nfilter-inth-n-zero[of  $xs \ P$ ] nfilter-holds[of  $x1a \odot xs \ P$ ]
  using 9 2 True a0 a3 a4 a5 a6 a8
  by (metis 1 One-nat-def Suc-n-not-le-n diff-Suc-less less-Suc-eq less-imp-diff-less
    less-imp-le-nat nfilter-ilen nfilter-lower-bound)
have 11:  $\text{ifilter } P (\text{suffix } (\text{inth } (\text{nfilter } P \text{ } xs \ 0) \ 1) \text{ } xs) = \text{ifilter } P (\text{suffix } (j-1) \text{ } xs)$ 
  by (metis 10 8 One-nat-def a1)
show ?thesis by (metis 11 3 4 7 One-nat-def)
qed
qed
qed
next
case False
then show ?thesis
using ICons.premis(2) by auto
qed
qed

```

```

lemma ifilter-suffixes-imap-help-j:
assumes  $j \leq \text{inth}(\text{nfilter } P \text{ } xs \ 0) (\text{Suc } i)$ 
   $i < \text{ilen}(\text{ifilter } P \text{ } xs)$ 
   $\text{inth}(\text{nfilter } P \text{ } xs \ 0) \ i < j$ 
   $\exists x \in \text{iset } xs. P \ x$ 
shows  $(\text{ifilter } P (\text{suffix } (\text{inth}(\text{nfilter } P \text{ } xs \ 0) (\text{Suc } i)) \text{ } xs)) = (\text{ifilter } P (\text{suffix } j \text{ } xs))$ 
using assms
proof (induct xs arbitrary: j i)
case (INil x)
then show ?case by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases  $(\exists a \in \text{iset } xs. P \ a)$ )
  case True
  then show ?thesis
    proof (cases  $P \ x1a$ )
    case True
    then show ?thesis using ICons True  $\langle (\exists a \in \text{iset } xs. P \ a) \rangle$ 

```

```

proof (auto split: nat.split)
  fix x
  fix x2::nat
  fix x2a
  assume a0: ( $\bigwedge j$  i.  $j \leq \text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) (\text{Suc } i) \implies$ 
     $i < \text{ilen} (\text{ifilter } P \text{ } xs) \implies$ 
     $\text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) i < j \implies$ 
     $\text{ifilter } P (\text{suffix} (\text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) (\text{Suc } i)) xs) = \text{ifilter } P (\text{suffix } j \text{ } xs))$ 
  assume a1:  $x2a \leq x2$ 
  assume a2:  $i < \text{Suc} (\text{ilen} (\text{ifilter } P \text{ } xs))$ 
  assume a3: ( $\text{case } i \text{ of } 0 \Rightarrow 0 \mid \text{Suc } x \Rightarrow \text{inth} (\text{nfilter } P \text{ } xs (\text{Suc } 0)) x < \text{Suc } x2a$ )
  assume a4:  $P \text{ } x1a$ 
  assume a5:  $x \in \text{iset } xs$ 
  assume a6:  $P \text{ } x$ 
  assume a7:  $\text{inth} (\text{nfilter } P \text{ } xs (\text{Suc } 0)) i = \text{Suc } x2$ 
  assume a8:  $j = \text{Suc } x2a$ 
  show  $\text{ifilter } P (\text{suffix } x2 \text{ } xs) = \text{ifilter } P (\text{suffix } x2a \text{ } xs)$ 
proof –
  have 1:  $i=0 \longrightarrow \text{ifilter } P (\text{suffix } x2 \text{ } xs) = \text{ifilter } P (\text{suffix } x2a \text{ } xs)$ 
    using a0 a1 a2 a3 a4 a5 a6 a7 a8
    by (metis One-nat-def add-diff-cancel-left' ilen-gr-zero nfilter-inth-n-zero
      plus-1-eq-Suc ifilter-suffixes-imap-help-0)
  have 2:  $i>0 \longrightarrow (\text{inth} (\text{nfilter } P \text{ } xs (\text{Suc } 0)) (i-1)) - (\text{Suc } 0) = \text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) (i-1)$ 
    using a0 a1 a2 a3 a4 a5 a6 a7 a8
    by (metis One-nat-def Suc-leI add-leD1 le-add-diff-inverse2
      less-Suc-eq-le nfilter-ilen nfilter-inth-n-zero)
  have 3:  $i>0 \longrightarrow \text{inth} (\text{nfilter } P \text{ } xs (\text{Suc } 0)) (i-1) <_{\text{Suc}} x2a$ 
    using a0 a1 a2 a3 a4 a5 a6 a7 a8
    by (metis Nitpick.case-nat-unfold less-Suc0 not-less-eq)
  have 4:  $i>0 \longrightarrow \text{inth} (\text{nfilter } P \text{ } xs (\text{Suc } 0)) (i-1) > 0$ 
    using a0 a1 a2 a3 a4 a5 a6 a7 a8 nfilter-lower-bound[of xs P i-1 Suc 0]
    by (metis One-nat-def Suc-le-lessD Suc-le-mono Suc-pred less-imp-le-nat nfilter-ilen)
  have 5:  $i>0 \longrightarrow \text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) (i-1) < x2a$ 
    using 2 3 4 by linarith
  have 6:  $i>0 \longrightarrow x2a \leq \text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) i$ 
    using a0 a1 a2 a3 a4 a5 a6 a7 a8
    by (metis One-nat-def add-diff-cancel-left'
      less-Suc-eq-le nfilter-ilen nfilter-inth-n-zero plus-1-eq-Suc)
  have 7:  $i>0 \longrightarrow i - 1 < \text{ilen} (\text{ifilter } P \text{ } xs)$ 
    using a2 by linarith
  have 8:  $i>0 \longrightarrow \text{ifilter } P (\text{suffix} (\text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) i) xs) = \text{ifilter } P (\text{suffix } x2a \text{ } xs)$ 
    using a0 5 6 7 by (metis Suc-pred')
  have 9:  $i>0 \longrightarrow (\text{inth} (\text{nfilter } P \text{ } xs \text{ } 0) i) = x2$ 
    using 7 a0 a1 a2 a3 a4 a5 a6 a7 a8
    by (metis One-nat-def Suc-leI Suc-pred add-diff-cancel-left'
      nfilter-ilen nfilter-inth-n-zero plus-1-eq-Suc)
  from 1 2 8 9 show ?thesis by auto
qed
qed
next

```

```

case False
then show ?thesis using ICons False  $\langle (\exists a \in \text{iset } xs. P a) \rangle$ 
proof (auto split: nat.split)
  fix x
  fix x2::nat
  fix x2a
  assume b0:  $(\bigwedge j. j \leq \text{inth } (nfilter\ P\ xs\ 0)\ (Suc\ i) \implies$ 
     $i < \text{ilen } (ifilter\ P\ xs) \implies$ 
     $\text{inth } (nfilter\ P\ xs\ 0)\ i < j \implies$ 
     $ifilter\ P\ (\text{suffix } (\text{inth } (nfilter\ P\ xs\ 0)\ (Suc\ i))\ xs) = ifilter\ P\ (\text{suffix } j\ xs))$ 
  assume b1:  $x2a \leq x2$ 
  assume b2:  $i < \text{ilen } (ifilter\ P\ xs)$ 
  assume b3:  $\text{inth } (nfilter\ P\ xs\ (Suc\ 0))\ i < Suc\ x2a$ 
  assume b4:  $\neg P\ x1a$ 
  assume b5:  $x \in \text{iset } xs$ 
  assume b6:  $P\ x$ 
  assume b7:  $\text{inth } (nfilter\ P\ xs\ (Suc\ 0))\ (Suc\ i) = Suc\ x2$ 
  assume b8:  $j = Suc\ x2a$ 
  show  $ifilter\ P\ (\text{suffix } x2\ xs) = ifilter\ P\ (\text{suffix } x2a\ xs)$ 
  proof –
    have 1:  $i=0 \longrightarrow (\text{inth } (nfilter\ P\ xs\ (Suc\ 0))\ (Suc\ i)) - (Suc\ 0) =$ 
       $\text{inth } (nfilter\ P\ xs\ 0)\ (Suc\ i)$ 
    using b0 b1 b2 b3 b4 b5 b6 b7 b8
    by (metis Suc-leI nfilter-ilen nfilter-inth-n-zero)
    have 2:  $i=0 \longrightarrow (\text{inth } (nfilter\ P\ xs\ 0)\ (Suc\ i)) = x2$ 
    using 1 b7 by linarith
    have 3:  $i=0 \longrightarrow \text{inth } (nfilter\ P\ xs\ 0)\ i < x2a$ 
    using b0 b1 b2 b3 b4 b5 b6 b7 b8 nfilter-lower-bound[of xs P i Suc 0]
      nfilter-inth-n-zero[of xs P i Suc 0]
    by (metis One-nat-def Suc-le-lessD le0 less-Suc-eq-le
      ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc)
    have 4:  $i=0 \longrightarrow i < \text{ilen } (ifilter\ P\ xs)$ 
    using b2 by blast
    have 5:  $i=0 \longrightarrow x2a \leq \text{inth } (nfilter\ P\ xs\ 0)\ (Suc\ i)$ 
    using 1 2 b1 by blast
    have 6:  $i=0 \longrightarrow ifilter\ P\ (\text{suffix } x2\ xs) = ifilter\ P\ (\text{suffix } x2a\ xs)$ 
    using 2 3 5 b0 b2 by blast
    have 7:  $i>0 \longrightarrow (\text{inth } (nfilter\ P\ xs\ (Suc\ 0))\ (i)) - (Suc\ 0) = \text{inth } (nfilter\ P\ xs\ 0)\ (i)$ 
    using b0 b1 b2 b3 b4 b5 b6 b7 b8
    by (metis less-imp-le-nat nfilter-ilen nfilter-inth-n-zero)
    have 8:  $i>0 \longrightarrow \text{inth } (nfilter\ P\ xs\ (Suc\ 0))\ i < Suc\ x2a$ 
    using b3 by auto
    have 9:  $i>0 \longrightarrow \text{inth } (nfilter\ P\ xs\ (Suc\ 0))\ (i) > 0$ 
    using b0 b1 b2 b3 b4 b5 b6 b7 b8
    by (metis gr-zeroI idx-nfilter-gr less-imp-le-nat nfilter-ilen not-less0)
    have 10:  $i>0 \longrightarrow \text{inth } (nfilter\ P\ xs\ 0)\ (i) < x2a$ 
    using 7 9 b3 by linarith
    have 11:  $i>0 \longrightarrow x2a \leq \text{inth } (nfilter\ P\ xs\ 0)\ (Suc\ i)$ 
    using b0 b1 b2 b3 b4 b5 b6 b7 b8
    by (metis One-nat-def Suc-leI add-diff-cancel-left' nfilter-ilen

```

```

      nfilter-inth-n-zero plus-1-eq-Suc)
have 12:  $i > 0 \longrightarrow i < \text{ilen} (\text{ifilter } P \text{ } xs)$ 
  using b2 by simp
have 13:  $i > 0 \longrightarrow \text{ifilter } P (\text{suffix} (\text{inth} (\text{nfilter } P \text{ } xs \ 0) (\text{Suc } i)) \text{ } xs) =$ 
   $\text{ifilter } P (\text{suffix } x2a \text{ } xs)$ 
  using 10 11 b0 b1 b2 b3 b4 b5 b6 b7 b8 by blast
have 14:  $i > 0 \longrightarrow (\text{inth} (\text{nfilter } P \text{ } xs \ 0) (\text{Suc } i)) = x2$ 
  using b0 b1 b2 b3 b4 b5 b6 b7 b8
  by (metis One-nat-def Suc-leI add-diff-cancel-left' nfilter-ilen
    nfilter-inth-n-zero plus-1-eq-Suc)
show ?thesis using 13 14 6 by blast
qed
qed
qed
next
case False
then show ?thesis using ICons False by auto
qed
qed

lemma ifilter-suffixes-imap-help-j-aa:
assumes  $j \leq \text{inth}(\text{nfilter } P (\text{suffixes } xs) \ 0) (\text{Suc } i)$ 
   $i < \text{ilen}(\text{ifilter } P (\text{suffixes } xs))$ 
   $\text{inth}(\text{nfilter } P (\text{suffixes } xs) \ 0) \ i < j$ 
   $\exists x \in \text{iset} (\text{suffixes } xs). P \ x$ 
shows  $(\text{ifilter } P (\text{suffixes} (\text{suffix} (\text{inth}(\text{nfilter } P (\text{suffixes } xs) \ 0) (\text{Suc } i)) \text{ } xs))) =$ 
   $(\text{ifilter } P (\text{suffixes} (\text{suffix } j \text{ } xs)))$ 
proof -
have 1:  $\text{inth}(\text{nfilter } P (\text{suffixes } xs) \ 0) (\text{Suc } i) \leq \text{ilen} (\text{suffixes } xs)$ 
  by (metis Suc-leI add-cancel-right-left assms(2) assms(4) nfilter-ilen nfilter-upper-bound)
have 2:  $(\text{suffixes} (\text{suffix} (\text{inth}(\text{nfilter } P (\text{suffixes } xs) \ 0) (\text{Suc } i)) \text{ } xs)) =$ 
   $(\text{suffix} (\text{inth}(\text{nfilter } P (\text{suffixes } xs) \ 0) (\text{Suc } i)) (\text{suffixes } xs))$ 
  using 1 suffix-suffixes by fastforce
have 3:  $(\text{suffixes} (\text{suffix } j \text{ } xs)) = (\text{suffix } j (\text{suffixes } xs))$ 
  using 1 assms le-trans suffix-suffixes[of j xs] by fastforce
show ?thesis
by (simp add: 2 3 assms(1) assms(2) assms(3) assms(4) ifilter-suffixes-imap-help-j)
qed

lemma ifilter-suffixes-imap:
assumes  $(\text{Suc } i) \leq \text{ilen} (\text{ifilter } f (\text{suffixes } \sigma))$ 
   $\exists x \in \text{iset}(\text{suffixes } \sigma). f \ x$ 
shows  $(\text{suffix} (\text{Suc } i) (\text{imap} (\lambda s. \text{inth } s \ 0) (\text{ifilter } f (\text{suffixes } \sigma)))) =$ 
   $(\text{imap} (\lambda s. \text{inth } s \ 0)$ 
     $(\text{ifilter } f (\text{suffixes} (\text{suffix} (\text{Suc} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ i)) \ \sigma))))$ 
proof -
have 1:  $(\text{suffix} (\text{Suc } i) (\text{imap} (\lambda s. \text{inth } s \ 0) (\text{ifilter } f (\text{suffixes } \sigma)))) =$ 
   $(\text{imap} (\lambda s. \text{inth } s \ 0) (\text{suffix} (\text{Suc } i) (\text{ifilter } f (\text{suffixes } \sigma))))$ 
  by (simp add: assms(1) imap-suffix)
have 2:  $(\text{Suc} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ i)) \leq \text{ilen}(\text{suffixes } \sigma)$ 

```

```

    by (metis ilen-ifilter-le Suc-leD assms(1) assms(2) diff-is-0-eq diff-zero
        ifilter-nfilter-suffix-1 suffix-ilen not-less-eq-eq)
  have 3: (ifilter f (suffixes (suffix (Suc (inth (nfilter f (suffixes  $\sigma$ ) 0) i))  $\sigma$ ))) =
    (ifilter f (suffix (Suc (inth (nfilter f (suffixes  $\sigma$ ) 0) i)) (suffixes  $\sigma$ )))

    using 2 suffix-suffixes by fastforce
  have 4: (suffix (Suc i) (ifilter f (suffixes  $\sigma$ ))) =
    (ifilter f (suffix (inth (nfilter f (suffixes  $\sigma$ ) 0) (Suc i)) (suffixes  $\sigma$ )))
    by (simp add: assms(1) assms(2) ifilter-nfilter-suffix-1)
  have 5: (Suc (inth (nfilter f (suffixes  $\sigma$ ) 0) i))  $\leq$  inth(nfilter f (suffixes  $\sigma$ ) 0) (Suc i)
    by (simp add: Suc-leI Suc-le-lessD assms(1) assms(2) idx-nfilter-mono nfilter-ilen)
  have 6: i < ilen(ifilter f (suffixes  $\sigma$ ))
    using Suc-le-eq assms(1) by blast
  have 7: inth (nfilter f (suffixes  $\sigma$ ) 0) i < (Suc (inth (nfilter f (suffixes  $\sigma$ ) 0) i))
    by simp
  have 8: (ifilter f (suffix (inth (nfilter f (suffixes  $\sigma$ ) 0) (Suc i)) (suffixes  $\sigma$ ))) =
    (ifilter f (suffix (Suc (inth (nfilter f (suffixes  $\sigma$ ) 0) i)) (suffixes  $\sigma$ )))
    using 5 6 7
    ifilter-suffixes-imap-help-j[of (Suc (inth (nfilter f (suffixes  $\sigma$ ) 0) i)) f suffixes  $\sigma$  i ]
    assms(2) by blast
  show ?thesis
    by (simp add: 1 3 4 8)
qed

```

lemma *sfxfilter-suffix-ilen:*

```

  assumes k ≤ ilen  $\sigma$ 
     $\exists x \in \text{iset } (\text{suffixes}(\text{suffix } k \ \sigma)). f \ x$ 
  shows ilen (ifilter f (suffixes (suffix k  $\sigma$ ))) ≤ ilen (ifilter f (suffixes  $\sigma$ ))
  using assms
  proof (induct k arbitrary:  $\sigma$ )
  case 0
  then show ?case by simp
  next
  case (Suc k)
  then show ?case
    proof (cases  $\sigma$ )
    case (INil x1)
    then show ?thesis
      by simp
    next
    case (ICons x21 x22)
    then show ?thesis
      proof auto
      fix y
      assume a0:  $\sigma = x21 \odot x22$ 
      assume a1:  $f(x21 \odot x22)$ 
      assume a2:  $\text{osfx } y \ x22$ 
      assume a3:  $f \ y$ 
      show ilen (ifilter f (suffixes (suffix k  $x22$ ))) ≤ Suc (ilen (ifilter f (suffixes  $x22$ )))
        using Suc a0

```

```

    by (metis ilen-ifilter-le Suc.hyps Suc.premis(2) suffix-ilen-code
        suffix-suc ilen-suffixes le-0-eq le-SucI less-imp-le-nat zero-less-Suc)
next
  assume b0:  $\sigma = x21 \odot x22$ 
  assume b1:  $f (x21 \odot x22)$ 
  assume b2:  $\forall x \in \text{iset} (\text{suffixes } x22). \neg f x$ 
  show ilen (ifilter f (suffixes (suffix k x22))) = 0
    using b0 b1 b2 Suc.premis osfx-suffix
    by (simp add: osfx-order.order.trans)
next
  fix y
  assume c0:  $\sigma = x21 \odot x22$ 
  assume c1:  $\neg f (x21 \odot x22)$ 
  assume c2: osfx y x22
  assume c3: f y
  show ilen (ifilter f (suffixes (suffix k x22)))  $\leq$  ilen (ifilter f (suffixes x22))
  using c0 c1
    by (metis ilen-ifilter-le Suc.hyps Suc.premis(2) ilen-gr-zero
        suffix-ilen-code suffix-suc ilen-suffixes le-0-eq)
next
  assume d0:  $\sigma = x21 \odot x22$ 
  assume d1:  $\neg f (x21 \odot x22)$ 
  assume d2:  $\forall x \in \text{iset} (\text{suffixes } x22). \neg f x$ 
  show ilen (ifilter f (suffixes (suffix k x22))) = 0
  using d0 d1 d2 Suc.premis osfx-suffix
  by (simp add: osfx-order.order.trans)
qed
qed
qed

lemma sfxfilter-suffix-inth:
  assumes  $k \leq \text{ilen } \sigma$ 
     $\exists x \in \text{iset} (\text{suffixes}(\text{suffix } k \ \sigma)). f x$ 
     $j \leq \text{ilen} (\text{ifilter } f (\text{suffixes} (\text{suffix } k \ \sigma)))$ 
  shows (inth (ifilter f (suffixes (suffix k  $\sigma$ ))) j) =
    (inth (suffix (ilen(ifilter f (suffixes  $\sigma$ )) - ilen((ifilter f (suffixes (suffix k  $\sigma$ ))))))
      (ifilter f (suffixes  $\sigma$ ))) j)
  using assms
proof (induct k arbitrary:  $\sigma$ )
case 0
then show ?case by simp
next
case (Suc k)
then show ?case
  proof (cases  $\sigma$ )
  case (INil x1)
  then show ?thesis
  by simp
  next
  case (ICons x21 x22)

```

```

then show ?thesis
proof auto
  fix y
  assume a0:  $\sigma = x21 \odot x22$ 
  assume a1:  $f (x21 \odot x22)$ 
  assume a2:  $osfx\ y\ x22$ 
  assume a3:  $f\ y$ 
  show
    inth (ifilter f (suffixes (suffix k x22))) j =
    inth
      (case Suc (ilen (ifilter f (suffixes x22))) - ilen (ifilter f (suffixes (suffix k x22)))) of
        0  $\Rightarrow$   $(x21 \odot x22) \odot$  ifilter f (suffixes x22) | Suc m  $\Rightarrow$  suffix m (ifilter f (suffixes x22)))
      j
  using a0 Suc
  by (simp add: Suc-diff-le sfxfilter-suffix-ilen)
next
  assume b0:  $\sigma = x21 \odot x22$ 
  assume b1:  $f (x21 \odot x22)$ 
  assume b2:  $\forall x \in iset\ (suffixes\ x22). \neg f\ x$ 
  show inth (ifilter f (suffixes (suffix k x22))) j =  $x21 \odot x22$ 
  using Suc b0 b1 b2 using osfx-suffix
  using osfx-order.order-trans by fastforce
next
  fix y
  assume c0:  $\sigma = x21 \odot x22$ 
  assume c1:  $\neg f (x21 \odot x22)$ 
  assume c2:  $osfx\ y\ x22$ 
  assume c4:  $f\ y$ 
  show inth (ifilter f (suffixes (suffix k x22))) j =
    inth
      (suffix (ilen (ifilter f (suffixes x22)) - ilen (ifilter f (suffixes (suffix k x22))))
        (ifilter f (suffixes x22)))
      j
  using Suc c0 by simp
next
  assume d0:  $\sigma = x21 \odot x22$ 
  assume d1:  $\neg f (x21 \odot x22)$ 
  assume d2:  $\forall x \in iset\ (suffixes\ x22). \neg f\ x$ 
  show inth (ifilter f (suffixes (suffix k x22))) j =  $x21 \odot x22$ 
  using Suc d0 d1 d2 osfx-suffix
  using osfx-order.order-trans by fastforce
qed
qed
qed

```

lemma sfxfilter-suffix-suffix:

assumes $k \leq ilen\ \sigma$

$\exists x \in iset\ (suffixes\ (suffix\ k\ \sigma)).\ f\ x$

shows $(ifilter\ f\ (suffixes\ (suffix\ k\ \sigma))) =$
 $(suffix\ (ilen(ifilter\ f\ (suffixes\ \sigma)) - ilen((ifilter\ f\ (suffixes\ (suffix\ k\ \sigma)))))$

$(\text{ifilter } f \text{ (suffixes } \sigma)))$

by (simp add: assms(1) assms(2) interval-eq-inth-eq sfxfilter-suffix-ilen sfxfilter-suffix-inth)

lemma sfxfilter-suffix-suffix-a:

assumes $jj \leq \text{ilen} (\text{ifilter } f \text{ (suffixes } xs))$

$\exists x \in \text{iset} (\text{suffixes } xs). f \ x$

shows $(\text{suffix } jj (\text{ifilter } f \text{ (suffixes } xs))) =$

$\text{ifilter } f \text{ (suffixes (suffix ((ilen } xs) - \text{ilen (inth (ifilter } f \text{ (suffixes } xs)) } jj)) \text{ } xs))$

proof –

have 1: $(\text{suffix } jj (\text{ifilter } f \text{ (suffixes } xs))) =$

$\text{ifilter } f \text{ (suffix (inth (nfilter } f \text{ (suffixes } xs) 0) } jj) \text{ (suffixes } xs))$

using ifilter-nfilter-suffix-1

by (simp add: ifilter-nfilter-suffix-1 assms(1) assms(2))

have 2: $(\text{inth (nfilter } f \text{ (suffixes } xs) 0) } jj) \leq \text{ilen} (\text{suffixes } xs)$

by (metis add-cancel-right-left assms(1) assms(2) nfilter-ilen nfilter-upper-bound)

have 3: $(\text{suffix (inth (nfilter } f \text{ (suffixes } xs) 0) } jj) \text{ (suffixes } xs)) =$

$(\text{suffixes (suffix (inth (nfilter } f \text{ (suffixes } xs) 0) } jj) \text{ } xs))$

using suffix-suffixes[of (inth (nfilter f (suffixes xs) 0) jj) xs] 2 **by** blast

have 4: $(\text{inth (ifilter } f \text{ (suffixes } xs)) } jj) = \text{inth} (\text{suffixes } xs) (\text{inth (nfilter } f \text{ (suffixes } xs) 0) } jj)$

by (metis nfilter-imap-ifilter assms(2) inth-imap)

have 5: $\text{ilen}(\text{suffixes } xs) \leq \text{ilen} (xs)$

by simp

have 6: $(\text{inth (nfilter } f \text{ (suffixes } xs) 0) } jj) \leq \text{ilen} (xs)$

using 2 5 le-trans **by** blast

have 7: $\text{inth} (\text{suffixes } xs) (\text{inth (nfilter } f \text{ (suffixes } xs) 0) } jj) =$

$(\text{suffix (inth (nfilter } f \text{ (suffixes } xs) 0) } jj) \text{ } xs)$

by (simp add: 6 inth-suffixes)

have 8: $(\text{inth (nfilter } f \text{ (suffixes } xs) 0) } jj) =$

$((\text{ilen } xs) - \text{ilen} (\text{inth (ifilter } f \text{ (suffixes } xs)) } jj))$

by (simp add: 4 6 7)

show ?thesis

using 1 3 8 **by** auto

qed

lemma sfx-suffix-upperbound:

$(\forall jj < (\text{ilen}(\text{ifilter } f \text{ (suffixes } \sigma)) - \text{ilen} (\text{ifilter } f \text{ (suffixes (suffix } k \text{ } \sigma))))).$

$((\text{ilen } \sigma) - \text{ilen} (\text{inth (ifilter } f \text{ (suffixes } \sigma)) } jj)) < k$

)

proof

fix jj

show $jj < \text{ilen} (\text{ifilter } f \text{ (suffixes } \sigma)) - \text{ilen} (\text{ifilter } f \text{ (suffixes (suffix } k \text{ } \sigma))) \longrightarrow$

$\text{ilen } \sigma - \text{ilen} (\text{inth (ifilter } f \text{ (suffixes } \sigma)) } jj) < k$

proof (induct k arbitrary: σ jj)

case 0

then show ?case **by** simp

next

case (Suc k)

then show ?case

proof (cases σ)

```

case (INil x1)
then show ?thesis using Suc by simp
next
case (ICons x21 x22)
then show ?thesis
  proof (cases jj)
    case 0
    then show ?thesis using Suc ICons using Suc-diff-le less-Suc-eq-0-disj by auto
    next
    case (Suc nat)
    then show ?thesis
      using Suc.hyps ICons Suc-diff-le less-Suc-eq-0-disj diff-is-0-eq gr-implies-not-zero by auto
    qed
  qed
qed
qed

end

```

14 Until and Since operator

```

theory UntilSince
imports Semantics Fuse Theorems TimeReversal
begin

```

This theory introduces the weak and strong versions of the until and since operators. The theorems from [11] are proven in a mostly deductive style.

14.1 Definitions

```

definition until-d :: ('a :: world) formula  $\Rightarrow$  'a formula  $\Rightarrow$  'a formula
where until-d F G  $\equiv \lambda s. ( (\exists k \leq \text{ilen } s. ((\text{suffix } k \ s) \models G) \wedge$ 
   $(\forall j. j < k \longrightarrow ((\text{suffix } j \ s) \models F) )) )$ 

```

```

definition since-d :: ('a :: world) formula  $\Rightarrow$  'a formula  $\Rightarrow$  'a formula
where since-d F G  $\equiv \lambda s. ( (\exists k \leq \text{ilen } s. ( (\text{prefix } k \ s) \models G) \wedge$ 
   $(\forall j. k < j \wedge j \leq \text{ilen } s \longrightarrow ((\text{prefix } j \ s) \models F) )) )$ 

```

syntax

```

-until-d :: [lift,lift]  $\Rightarrow$  lift      ((- U -) [84,84] 83)
-since-d :: [lift,lift]  $\Rightarrow$  lift      ((- S -) [84,84] 83)

```

syntax (*ASCII*)

```

-until-d    :: [lift,lift]  $\Rightarrow$  lift      ((- until -) [84,84] 83)
-since-d    :: [lift,lift]  $\Rightarrow$  lift      ((- since -) [84,84] 83)

```

translations

```

-until-d     $\equiv$  CONST until-d

```

$-since-d \quad \Rightarrow \text{CONST } since-d$

definition $suntil-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $suntil-d F G \equiv LIFT(\odot(F \mathcal{U} G))$

definition $ssince-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $ssince-d F G \equiv LIFT(prev(F \mathcal{S} G))$

definition $wait-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $wait-d F G \equiv LIFT(\Box F \vee F \mathcal{U} G)$

definition $pwait-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $pwait-d F G \equiv LIFT(bi F \vee F \mathcal{S} G)$

definition $release-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $release-d F G \equiv LIFT(\neg((\neg F) \mathcal{U} (\neg G)))$

definition $prelease-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $prelease-d F G \equiv LIFT(\neg((\neg F) \mathcal{S} (\neg G)))$

syntax

$-wait-d \quad :: [lift, lift] \Rightarrow lift \quad ((- \mathcal{W} -) [84, 84] 83)$
 $-pwait-d \quad :: [lift, lift] \Rightarrow lift \quad ((- \mathcal{PW} -) [84, 84] 83)$
 $-release-d \quad :: [lift, lift] \Rightarrow lift \quad ((- \mathcal{R} -) [84, 84] 83)$
 $-prelease-d \quad :: [lift, lift] \Rightarrow lift \quad ((- \mathcal{PR} -) [84, 84] 83)$
 $-suntil-d \quad :: [lift, lift] \Rightarrow lift \quad ((- \mathcal{U}^s -) [84, 84] 83)$
 $-ssince-d \quad :: [lift, lift] \Rightarrow lift \quad ((- \mathcal{S}^s -) [84, 84] 83)$

syntax (ASCII)

$-wait-d \quad :: [lift, lift] \Rightarrow lift \quad ((- wait -) [84, 84] 83)$
 $-pwait-d \quad :: [lift, lift] \Rightarrow lift \quad ((- pwait -) [84, 84] 83)$
 $-release-d \quad :: [lift, lift] \Rightarrow lift \quad ((- release -) [84, 84] 83)$
 $-prelease-d \quad :: [lift, lift] \Rightarrow lift \quad ((- prelease -) [84, 84] 83)$
 $-suntil-d \quad :: [lift, lift] \Rightarrow lift \quad ((- until -) [84, 84] 83)$
 $-ssince-d \quad :: [lift, lift] \Rightarrow lift \quad ((- since -) [84, 84] 83)$

translations

$-wait-d \quad \Rightarrow \text{CONST } wait-d$
 $-pwait-d \quad \Rightarrow \text{CONST } pwait-d$
 $-release-d \quad \Rightarrow \text{CONST } release-d$
 $-prelease-d \quad \Rightarrow \text{CONST } prelease-d$
 $-suntil-d \quad \Rightarrow \text{CONST } until-d$
 $-ssince-d \quad \Rightarrow \text{CONST } since-d$

definition $srelease-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $srelease-d F G \equiv LIFT(\neg((\neg F) \mathcal{W} (\neg G)))$

definition $psrelease-d :: ('a :: world) formula \Rightarrow 'a formula \Rightarrow 'a formula$
where $psrelease-d F G \equiv LIFT(\neg((\neg F) \mathcal{PW} (\neg G)))$

syntax

$\text{-srelease-d} :: [\text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((- \mathcal{M} -) [84, 84] \ 83)$
 $\text{-psrelease-d} :: [\text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((- \mathcal{PM} -) [84, 84] \ 83)$

syntax (ASCII)

$\text{-srelease-d} :: [\text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((- \text{ srelease } -) [84, 84] \ 83)$
 $\text{-psrelease-d} :: [\text{lift}, \text{lift}] \Rightarrow \text{lift} \quad ((- \text{ psrelease } -) [84, 84] \ 83)$

translations

$\text{-srelease-d} \quad \Rightarrow \text{CONST srelease-d}$
 $\text{-psrelease-d} \quad \Rightarrow \text{CONST psrelease-d}$

14.2 Axioms

14.2.1 NextUntil

lemma *NextUntilsema*:

assumes $(\sigma \models \bigcirc(f \ \mathcal{U} \ g))$

shows $(\sigma \models (\bigcirc f) \ \mathcal{U} \ (\bigcirc g))$

proof –

have 0: $0 < \text{ilen } \sigma \wedge$

$(\exists k \leq \text{ilen } (\text{suffix } (\text{Suc } 0) \ \sigma). \ g \ (\text{suffix } (\text{Suc } k) \ \sigma) \wedge (\forall j < k. \ f \ (\text{suffix } (\text{Suc } j) \ \sigma)))$

using *assms* **by** (*auto simp add: itl-defs until-d-def*)

have 1: $0 < \text{ilen } \sigma$

using 0 **by** *auto*

have 2: $(\exists k \leq \text{ilen } (\text{suffix } (\text{Suc } 0) \ \sigma). \ g \ (\text{suffix } (\text{Suc } k) \ \sigma) \wedge (\forall j < k. \ f \ (\text{suffix } (\text{Suc } j) \ \sigma)))$

using 0 **by** *auto*

obtain *k* **where** 3: $k \leq \text{ilen } (\text{suffix } (\text{Suc } 0) \ \sigma) \wedge g \ (\text{suffix } (\text{Suc } k) \ \sigma) \wedge$

$(\forall j < k. \ f \ (\text{suffix } (\text{Suc } j) \ \sigma))$

using 2 **by** *auto*

have 4: $g \ (\text{suffix } (\text{Suc } k) \ \sigma)$

using 3 **by** *auto*

have 5: $k \leq \text{ilen } \sigma$

using 3 0 **by** *auto*

have 6: $0 < \text{ilen } (\text{suffix } k \ \sigma)$

using 1 3 **by** *auto*

have 7: $(\forall j < k. \ 0 < \text{ilen } (\text{suffix } j \ \sigma) \wedge f \ (\text{suffix } (\text{Suc } j) \ \sigma))$

using 3 5 **by** *force*

have 8: $\exists k \leq \text{ilen } \sigma.$

$0 < \text{ilen } (\text{suffix } k \ \sigma) \wedge$

$g \ (\text{suffix } (\text{Suc } k) \ \sigma) \wedge (\forall j < k. \ 0 < \text{ilen } (\text{suffix } j \ \sigma) \wedge f \ (\text{suffix } (\text{Suc } j) \ \sigma))$

using 3 5 6 7 **by** *blast*

from 8 **show** *?thesis* **by** (*simp add: itl-defs until-d-def*)

qed

lemma *NextUntilsemb*:

assumes $(\sigma \models (\bigcirc f) \ \mathcal{U} \ (\bigcirc g))$

shows $(\sigma \models \bigcirc(f \ \mathcal{U} \ g))$

proof –

have 1: $\exists k \leq \text{ilen } \sigma$.
 $0 < \text{ilen } (\text{suffix } k \ \sigma) \wedge$
 $g (\text{suffix } (\text{Suc } k) \ \sigma) \wedge (\forall j < k. 0 < \text{ilen } (\text{suffix } j \ \sigma) \wedge f (\text{suffix } (\text{Suc } j) \ \sigma))$
using *assms* **by** (*auto simp add: itl-defs until-d-def*)
obtain k **where** 2: $k \leq \text{ilen } \sigma \wedge 0 < \text{ilen } (\text{suffix } k \ \sigma) \wedge$
 $g (\text{suffix } (\text{Suc } k) \ \sigma) \wedge (\forall j < k. 0 < \text{ilen } (\text{suffix } j \ \sigma) \wedge f (\text{suffix } (\text{Suc } j) \ \sigma))$
using 1 **by** *auto*
have 3: $0 < \text{ilen } \sigma$
using 2 **by** *auto*
have 4: $k \leq \text{ilen } (\text{suffix } (\text{Suc } 0) \ \sigma)$
using 2 *suffix-ilen-good* **by** *auto*
have 5: $g (\text{suffix } (\text{Suc } k) \ \sigma)$
using 2 **by** *auto*
have 6: $(\forall j < k. f (\text{suffix } (\text{Suc } j) \ \sigma))$
using 2 **by** *blast*
have 7: $0 < \text{ilen } \sigma \wedge$
 $(\exists k \leq \text{ilen } (\text{suffix } (\text{Suc } 0) \ \sigma). g (\text{suffix } (\text{Suc } k) \ \sigma) \wedge (\forall j < k. f (\text{suffix } (\text{Suc } j) \ \sigma)))$
using 2 3 4 **by** *blast*
from 7 **show** ?thesis **by** (*auto simp: itl-defs until-d-def*)
qed

lemma *NextUntilsem*:

$\sigma \models \bigcirc(f \ \mathcal{U} \ g) = (\bigcirc f) \ \mathcal{U} \ (\bigcirc g)$
using *NextUntilsema NextUntilsemb* **using** *unl-lift2* **by** *blast*

lemma *NextUntil*:

$\vdash \bigcirc(f \ \mathcal{U} \ g) = (\bigcirc f) \ \mathcal{U} \ (\bigcirc g)$
using *NextUntilsem Valid-def* **by** *blast*

14.2.2 UntilNextUntil

lemma *UntilNextUntilsema*:

assumes $0 < \text{ilen } \sigma \wedge$
 $(\exists k > 0. k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \ \sigma) \wedge (\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j \ \sigma)))$
shows $\sigma \models \bigcirc (f \ \mathcal{U} \ g)$

proof –

have 1: $0 < \text{ilen } \sigma \wedge$
 $(\exists k > 0. k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \ \sigma) \wedge (\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j \ \sigma)))$
using *assms* **by** *auto*
have 3: $(\exists k > 0. k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \ \sigma) \wedge (\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j \ \sigma)))$
using 1 **by** *auto*
obtain k **where** 4: $(\text{Suc } k) > 0 \wedge (\text{Suc } k) \leq \text{ilen } \sigma \wedge g (\text{suffix } (\text{Suc } k) \ \sigma) \wedge$
 $(\forall j. 0 < j \wedge j < (\text{Suc } k) \longrightarrow f (\text{suffix } j \ \sigma))$
using 3 **by** (*metis Suc-pred*)
have 5: $k \leq \text{ilen } (\text{suffix } (\text{Suc } 0) \ \sigma)$
using 4 **by** *auto*
have 6: $g (\text{suffix } (\text{Suc } k) \ \sigma)$
using 4 **by** *auto*
have 7: $(\forall j < k. f (\text{suffix } (\text{Suc } j) \ \sigma))$

using 4 by blast
 have 8: $(\exists k \leq \text{ilen} (\text{suffix} (\text{Suc } 0) \sigma). g (\text{suffix} (\text{Suc } k) \sigma) \wedge (\forall j < k. f (\text{suffix} (\text{Suc } j) \sigma)))$
 using 4 5 by blast
 have 9: $0 < \text{ilen } \sigma \wedge (\exists k \leq \text{ilen} (\text{suffix} (\text{Suc } 0) \sigma). g (\text{suffix} (\text{Suc } k) \sigma) \wedge (\forall j < k. f (\text{suffix} (\text{Suc } j) \sigma)))$
 using 1 8 by blast
 from 9 show ?thesis by (auto simp add: itl-defs until-d-def)
 qed

lemma *UntilNextUntilsemb:*

assumes $\sigma \models \bigcirc (f \mathcal{U} g)$

shows $0 < \text{ilen } \sigma \wedge$

$(\exists k > 0. k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \sigma) \wedge (\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j \sigma)))$

proof –

have 1: $0 < \text{ilen } \sigma \wedge$

$(\exists k \leq \text{ilen} (\text{suffix} (\text{Suc } 0) \sigma). g (\text{suffix} (\text{Suc } k) \sigma) \wedge (\forall j < k. f (\text{suffix} (\text{Suc } j) \sigma)))$

using *assms* **by** (auto simp add: itl-defs until-d-def)

have 2: $(\exists k \leq \text{ilen} (\text{suffix} (\text{Suc } 0) \sigma). g (\text{suffix} (\text{Suc } k) \sigma) \wedge (\forall j < k. f (\text{suffix} (\text{Suc } j) \sigma)))$

using 1 **by** auto

obtain *k* **where** 3: $k \leq \text{ilen} (\text{suffix} (\text{Suc } 0) \sigma) \wedge g (\text{suffix} (\text{Suc } k) \sigma) \wedge$

$(\forall j < k. f (\text{suffix} (\text{Suc } j) \sigma))$

using 2 **by** auto

have 4: $(\text{Suc } k) > 0$

by *simp*

have 5: $g (\text{suffix} (\text{Suc } k) \sigma)$

using 3 **by** auto

have 6: $(\text{Suc } k) \leq \text{ilen } \sigma$

using 1 3 **by** auto

have 7: $(\forall j. 0 < j \wedge j < (\text{Suc } k) \longrightarrow f (\text{suffix } j \sigma))$

using 3 *less-Suc-eq-0-disj* **by** auto

have 8: $(\exists k > 0. k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \sigma) \wedge (\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j \sigma)))$

using 3 6 7 **by** blast

show ?thesis **using** 1 8 **by** blast

qed

lemma *UntilNextUntilsem:*

$(\sigma \models \bigcirc (f \mathcal{U} g)) = (0 < \text{ilen } \sigma \wedge$

$(\exists k > 0. k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \sigma) \wedge (\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j \sigma))))$

using *UntilNextUntilsema*[of σ g f] *UntilNextUntilsemb*[of f g σ] **by** *meson*

lemma *UntilNextUntilsem1:*

$(\sigma \models f \mathcal{U} g) = (\sigma \models (g \vee (f \wedge \bigcirc(f \mathcal{U} g))))$

unfolding *UntilNextUntilsem*

proof

assume *a*: $(\sigma \models f \mathcal{U} g)$

show $\sigma \models g \vee f \wedge (\lambda s. 0 < \text{ilen } s \wedge (\exists k > 0. k \leq \text{ilen } s \wedge g (\text{suffix } k s) \wedge$

$(\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j s))))$

using *a* **unfolding** *until-d-def* **by** *simp force*
next
assume $b: \sigma \models g \vee f \wedge (\lambda s. 0 < \text{ilen } s \wedge (\exists k > 0. k \leq \text{ilen } s \wedge g (\text{suffix } k \ s) \wedge (\forall j. 0 < j \wedge j < k \longrightarrow f (\text{suffix } j \ s))))$
show $(\sigma \models f \ \mathcal{U} \ g)$
using *b* **unfolding** *until-d-def* **by** *simp*
(metis gr-implies-not-zero ilen-gr-zero suffix-zero linorder-cases)
qed

lemma *UntilNextUntil*:
 $\vdash f \ \mathcal{U} \ g = (g \vee (f \wedge \bigcirc(f \ \mathcal{U} \ g)))$
by (*simp add: UntilNextUntilsem1 Valid-def*)

14.2.3 NotUntilFalse

lemma *NotUntilFalse*:
 $\vdash \neg (f \ \mathcal{U} \ \#False)$
by (*simp add: intI until-d-def*)

14.2.4 UntilOrDist

lemma *UntilOrDistsem*:
 $\sigma \models f \ \mathcal{U} \ (g \vee h) = (f \ \mathcal{U} \ g \vee f \ \mathcal{U} \ h)$
by (*auto simp add: until-d-def*)

lemma *UntilOrDist*:
 $\vdash f \ \mathcal{U} \ (g \vee h) = (f \ \mathcal{U} \ g \vee f \ \mathcal{U} \ h)$
using *UntilOrDistsem Valid-def* **by** *blast*

14.2.5 UntilRightDistOr

lemma *UntilRightDistOr*:
 $\vdash f \ \mathcal{U} \ h \vee g \ \mathcal{U} \ h \longrightarrow (f \vee g) \ \mathcal{U} \ h$
by (*auto simp add: Valid-def until-d-def*)

14.2.6 UntilLeftDistAnd

lemma *UntilLeftDistAnd*:
 $\vdash f \ \mathcal{U} \ (g \wedge h) \longrightarrow f \ \mathcal{U} \ g \wedge f \ \mathcal{U} \ h$
by (*auto simp add: Valid-def until-d-def*)

14.2.7 UntilAndDist

lemma *UntilAndDistsem*:
 $\sigma \models (f \wedge g) \ \mathcal{U} \ h = ((f \ \mathcal{U} \ h) \wedge (g \ \mathcal{U} \ h))$
by (*auto simp add: until-d-def*)
(metis dual-order.strict-trans linorder-cases)

lemma *UntilAndDist*:
 $\vdash (f \wedge g) \ \mathcal{U} \ h = ((f \ \mathcal{U} \ h) \wedge (g \ \mathcal{U} \ h))$
using *UntilAndDistsem Valid-def* **by** *blast*

14.2.8 untilNotImp

lemma *UntilNotImp*:

$\vdash f \mathcal{U} g \wedge (\neg g) \mathcal{U} h \longrightarrow f \mathcal{U} h$

by (*auto simp add: Valid-def until-d-def*)
(metis less-trans linorder-cases)

14.2.9 UntilUntil

lemma *UntilUntilsem*:

$(\sigma \models f \mathcal{U} g) = (\sigma \models f \mathcal{U} (f \mathcal{U} g))$

unfolding *until-d-def*

proof

assume *a*: $\exists k \leq \text{ilen } \sigma. g (\text{suffix } k \sigma) \wedge (\forall j < k. f (\text{suffix } j \sigma))$

show $\exists k \leq \text{ilen } \sigma. (\exists ka \leq \text{ilen } (\text{suffix } k \sigma). g (\text{suffix } ka (\text{suffix } k \sigma)) \wedge$
 $(\forall j < ka. f (\text{suffix } j (\text{suffix } k \sigma)))) \wedge (\forall j < k. f (\text{suffix } j \sigma))$

using *a* **by** *simp-all force*

next

assume *b*: $\exists k \leq \text{ilen } \sigma. (\exists ka \leq \text{ilen } (\text{suffix } k \sigma). g (\text{suffix } ka (\text{suffix } k \sigma)) \wedge$
 $(\forall j < ka. f (\text{suffix } j (\text{suffix } k \sigma)))) \wedge (\forall j < k. f (\text{suffix } j \sigma))$

show $\exists k \leq \text{ilen } \sigma. g (\text{suffix } k \sigma) \wedge (\forall j < k. f (\text{suffix } j \sigma))$

proof –

obtain *k* **where** *1*: $k \leq \text{ilen } \sigma \wedge (\exists ka \leq \text{ilen } (\text{suffix } k \sigma). g (\text{suffix } ka (\text{suffix } k \sigma)) \wedge$
 $(\forall j < ka. f (\text{suffix } j (\text{suffix } k \sigma)))) \wedge (\forall j < k. f (\text{suffix } j \sigma))$

using *b* **by** *auto*

have *2*: $(\exists ka \leq \text{ilen } (\text{suffix } k \sigma). g (\text{suffix } ka (\text{suffix } k \sigma)) \wedge$
 $(\forall j < ka. f (\text{suffix } j (\text{suffix } k \sigma))))$

using *1* **by** *auto*

obtain *ka* **where** *3*: $ka \leq \text{ilen } (\text{suffix } k \sigma) \wedge g (\text{suffix } ka (\text{suffix } k \sigma)) \wedge$
 $(\forall j < ka. f (\text{suffix } j (\text{suffix } k \sigma)))$

using *2* **by** *auto*

have *4*: $ka + k \leq \text{ilen } \sigma$

using *1 3* **by** *auto*

have *5*: $g (\text{suffix } (ka+k) \sigma)$

using *3* **by** *auto*

have *6*: $(\forall j < ka+k. f (\text{suffix } j \sigma))$

by (*metis 1 3 add commute add-diff-inverse-nat suffix-suffix nat-add-left-cancel-less*)

show *?thesis*

using *4 5 6* **by** *blast*

qed

qed

lemma *UntilUntil*:

$\vdash f \mathcal{U} g = f \mathcal{U} (f \mathcal{U} g)$

using *UntilUntilsem* **by** *fastforce*

14.2.10 UntilRightor

lemma *UntilRightOr*:

$\vdash f \mathcal{U} (g \mathcal{U} h) \longrightarrow (f \vee g) \mathcal{U} h$

by (*auto simp add: Valid-def until-d-def*)

(metis Nat.le-diff-conv2 leI le-add-diff-inverse2 less-diff-conv2)

14.2.11 UntilRightAnd

lemma *UntilRightAndsem*:

assumes $(\sigma \models f \mathcal{U} (g \wedge h))$

shows $(\sigma \models (f \mathcal{U} g) \mathcal{U} h)$

proof –

have 1: $\exists k \leq \text{ilen } \sigma. g (\text{suffix } k \sigma) \wedge h (\text{suffix } k \sigma) \wedge (\forall j < k. f (\text{suffix } j \sigma))$

using *assms* **by** (*simp add: until-d-def*)

obtain *k* **where** 2: $k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \sigma) \wedge h (\text{suffix } k \sigma) \wedge (\forall j < k. f (\text{suffix } j \sigma))$

using 1 **by** *auto*

have 3: $h (\text{suffix } k \sigma)$

using 2 **by** *auto*

have 4: $k \leq \text{ilen } \sigma$

using 2 **by** *auto*

have 5: $(\forall j < k.$

$\exists ka \leq \text{ilen } (\text{suffix } j \sigma). g (\text{suffix } (ka + j) \sigma) \wedge (\forall ja < ka. f (\text{suffix } (ja + j) \sigma)))$

proof

fix *j*

show $j < k \longrightarrow$

$(\exists ka \leq \text{ilen } (\text{suffix } j \sigma). g (\text{suffix } (ka + j) \sigma) \wedge (\forall ja < ka. f (\text{suffix } (ja + j) \sigma)))$

proof

assume *a0*: $j < k$

show $(\exists ka \leq \text{ilen } (\text{suffix } j \sigma). g (\text{suffix } (ka + j) \sigma) \wedge (\forall ja < ka. f (\text{suffix } (ja + j) \sigma)))$

proof –

have 51: $k - j \leq \text{ilen } (\text{suffix } j \sigma)$

using 4 *a0* **by** *auto*

have 52: $g (\text{suffix } ((k - j) + j) \sigma)$

by (*simp add: 2 a0 less-imp-le-nat*)

have 53: $(\forall ja < (k - j). f (\text{suffix } (ja + j) \sigma))$

using 2 *less-diff-conv* **by** *blast*

show *?thesis*

using 51 52 53 **by** *blast*

qed

qed

qed

have 6: $\exists k \leq \text{ilen } \sigma.$

$h (\text{suffix } k \sigma) \wedge$

$(\forall j < k. \exists k' \leq \text{ilen } (\text{suffix } j \sigma). g (\text{suffix } (k' + j) \sigma) \wedge (\forall ja < k'. f (\text{suffix } (ja + j) \sigma)))$

using 2 5 **by** *blast*

from 6 **show** *?thesis* **by** (*simp add: until-d-def*)

qed

lemma *UntilRightAnd*:

$\vdash f \mathcal{U} (g \wedge h) \longrightarrow (f \mathcal{U} g) \mathcal{U} h$

using *UntilRightAndsem Valid-def* **by** *auto*

14.2.12 SincePrevSince

lemma *SincePrevSincesema*:

assumes $0 < \text{ilen } \sigma \wedge$
 $(\exists k < \text{ilen } \sigma. g (\text{prefix } k \sigma) \wedge (\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \sigma)))$
shows $\sigma \models \text{prev } (f \mathcal{S} g)$
proof –
have 1: $0 < \text{ilen } \sigma \wedge$
 $(\exists k < \text{ilen } \sigma. g (\text{prefix } k \sigma) \wedge (\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \sigma)))$
using *assms* **by** *auto*
have 2: $(\exists k < \text{ilen } \sigma. g (\text{prefix } k \sigma) \wedge (\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \sigma)))$
using 1 **by** *auto*
obtain k **where** 3: $k < \text{ilen } \sigma \wedge g (\text{prefix } k \sigma) \wedge (\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \sigma))$
using 2 **by** *auto*
have 4: $k \leq \text{ilen } \sigma - \text{Suc } 0$
using 3 **by** *linarith*
have 5: $g (\text{prefix } k (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma))$
by (*simp add: 1 3 pref-pref-help*)
have 6: $(\forall j. k < j \wedge j \leq \text{ilen } \sigma - \text{Suc } 0 \longrightarrow f (\text{prefix } j (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma)))$
using 3 *pref-pref-help* **by** *fastforce*
have 7: $(\exists k \leq \text{ilen } \sigma - \text{Suc } 0.$
 $g (\text{prefix } k (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq \text{ilen } \sigma - \text{Suc } 0 \longrightarrow f (\text{prefix } j (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma))))$
using 4 5 6 **by** *blast*
have 8: $0 < \text{ilen } \sigma \wedge$
 $(\exists k \leq \text{ilen } \sigma - \text{Suc } 0.$
 $g (\text{prefix } k (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq \text{ilen } \sigma - \text{Suc } 0 \longrightarrow f (\text{prefix } j (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma))))$
using 1 7 **by** *blast*
from 8 **show** *?thesis* **by** (*simp add: itl-defs since-d-def*)
qed

lemma *SincePrevSincesemb:*

assumes $\sigma \models \text{prev } (f \mathcal{S} g)$
shows $0 < \text{ilen } \sigma \wedge$
 $(\exists k < \text{ilen } \sigma. g (\text{prefix } k \sigma) \wedge (\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \sigma)))$

proof –

have 1: $0 < \text{ilen } \sigma \wedge$
 $(\exists k \leq \text{ilen } \sigma - \text{Suc } 0.$
 $g (\text{prefix } k (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq \text{ilen } \sigma - \text{Suc } 0 \longrightarrow f (\text{prefix } j (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma))))$
using *assms* **by** (*simp add: itl-defs since-d-def*)
have 2: $(\exists k \leq \text{ilen } \sigma - \text{Suc } 0.$
 $g (\text{prefix } k (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq \text{ilen } \sigma - \text{Suc } 0 \longrightarrow f (\text{prefix } j (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma))))$
using 1 **by** *auto*
obtain k **where** 3: $k \leq \text{ilen } \sigma - \text{Suc } 0 \wedge g (\text{prefix } k (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq \text{ilen } \sigma - \text{Suc } 0 \longrightarrow f (\text{prefix } j (\text{prefix } (\text{ilen } \sigma - \text{Suc } 0) \sigma)))$
using 2 **by** *auto*
have 4: $k < \text{ilen } \sigma$
using 1 3 **by** *linarith*
have 5: $g (\text{prefix } k \sigma)$
using 3 4 *pref-pref-help*[*of* σ] **by** *force*

have 6: $(\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \ \sigma))$
using 3 *pref-pref-help*[of σ] **by** *fastforce*
have 7: $(\exists k < \text{ilen } \sigma. g (\text{prefix } k \ \sigma) \wedge (\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \ \sigma)))$
using 4 5 6 **by** *blast*
show ?thesis **using** 1 7 **by** *blast*
qed

lemma *SincePrevSincesem*:

$(\sigma \models \text{prev } (f \ \mathcal{S} \ g)) =$
 $(0 < \text{ilen } \sigma \wedge$
 $(\exists k < \text{ilen } \sigma. g (\text{prefix } k \ \sigma) \wedge (\forall j. k < j \wedge j < \text{ilen } \sigma \longrightarrow f (\text{prefix } j \ \sigma))))$
using *SincePrevSincesema*[of $\sigma \ g \ f$] *SincePrevSincesemb*[of $f \ g \ \sigma$] *unl-lift2* **by** *blast*

lemma *SincePrevSincesem1*:

$(\sigma \models f \ \mathcal{S} \ g) = (\sigma \models (g \vee (f \wedge \text{prev}(f \ \mathcal{S} \ g))))$

unfolding *SincePrevSincesem*

proof

assume a: $(\sigma \models f \ \mathcal{S} \ g)$
show $\sigma \models g \vee f \wedge (\lambda s. 0 < \text{ilen } s \wedge (\exists k < \text{ilen } s. g (\text{prefix } k \ s) \wedge$
 $(\forall j. k < j \wedge j < \text{ilen } s \longrightarrow f (\text{prefix } j \ s))))$
using a **unfolding** *since-d-def* **by** *simp*
 $(\text{metis prefix-ilen-bound prefix-ilen le-neq-implies-less le-zero-eq less-imp-le-nat neq0-conv})$

next

assume b: $\sigma \models g \vee f \wedge (\lambda s. 0 < \text{ilen } s \wedge (\exists k < \text{ilen } s. g (\text{prefix } k \ s) \wedge$
 $(\forall j. k < j \wedge j < \text{ilen } s \longrightarrow f (\text{prefix } j \ s))))$
show $(\sigma \models f \ \mathcal{S} \ g)$
using b **unfolding** *since-d-def* **by** *simp*
 $(\text{metis prefix-ilen leD order-le-less})$

qed

lemma *SincePrevSince*:

$\vdash f \ \mathcal{S} \ g = (g \vee (f \wedge \text{prev}(f \ \mathcal{S} \ g)))$
by (*simp add: SincePrevSincesem1 Valid-def*)

14.2.13 RevUntil

lemma *RevUntilsema*:

assumes $\sigma \models (f \ \mathcal{U} \ g)^r$

shows $\sigma \models ((f^r) \ \mathcal{S} \ (g^r))$

proof –

have 1: $\sigma \models (f \ \mathcal{U} \ g)^r$
using *assms* **by** *auto*
have 2: $\exists k \leq \text{ilen } \sigma.$
 $g (\text{suffix } k \ (\text{irev } \sigma)) \wedge (\forall j < k. f (\text{suffix } j \ (\text{irev } \sigma)))$
using 1 **by** (*simp add: until-d-def reverse-d-def*)
obtain k **where** 3: $k \leq \text{ilen } \sigma \wedge g (\text{suffix } k \ (\text{irev } \sigma)) \wedge$
 $(\forall j < k. f (\text{suffix } j \ (\text{irev } \sigma)))$
using 2 **by** *auto*
have 4: $g (\text{irev } (\text{prefix } (\text{ilen } \sigma - k) \ \sigma))$
by (*simp add: 3 irev-prefix*)

have 5: $(\forall j < k. f (irev (prefix (ilen \sigma - j) \sigma)))$
by (*simp add: 3 irev-prefix*)
have 6: $(ilen \sigma - k) \leq ilen \sigma$
using *diff-le-self* **by** *blast*
have 7: $(\forall j. (ilen \sigma - k) < j \wedge j \leq ilen \sigma \longrightarrow f (irev (prefix j \sigma)))$
by (*simp add: 3 irev-prefix less-diff-conv2*)
have 8: $\exists k \leq ilen \sigma. g (irev (prefix k \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq ilen \sigma \longrightarrow f (irev (prefix j \sigma)))$
using 4 6 7 **by** *blast*
have 10: $\sigma \models (f^r) \mathcal{S} (g^r)$
using 8 **by** (*simp add: since-d-def reverse-d-def*)
show *?thesis* **by** (*simp add: 10*)
qed

lemma *RevUntilsemb*:

assumes $\sigma \models (f^r) \mathcal{S} (g^r)$

shows $\sigma \models (f \mathcal{U} g)^r$

proof –

have 1: $\sigma \models (f^r) \mathcal{S} (g^r)$
using *assms* **by** *auto*
have 2: $\exists k \leq ilen \sigma. g (irev (prefix k \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq ilen \sigma \longrightarrow f (irev (prefix j \sigma)))$
using 1 **by** (*simp add: since-d-def reverse-d-def*)
obtain *k* **where** 3: $k \leq ilen \sigma \wedge g (irev (prefix k \sigma)) \wedge$
 $(\forall j. k < j \wedge j \leq ilen \sigma \longrightarrow f (irev (prefix j \sigma)))$
using 2 **by** *auto*
have 4: $g (suffix (ilen \sigma - k) (irev \sigma))$
using 3 *irev-prefix* **by** *fastforce*
have 5: $(\forall j. j < ilen \sigma - k \longrightarrow f (suffix j (irev \sigma)))$
using 3 *irev-prefix[of - σ]*
by (*metis (mono-tags, lifting) diff-diff-cancel diff-le-self diff-less-mono2 less-imp-le-nat less-le-trans*)
have 6: $\exists k \leq ilen \sigma. g (suffix k (irev \sigma)) \wedge$
 $(\forall j < k. f (suffix j (irev \sigma)))$
using 4 5 *diff-le-self* **by** *blast*
have 7: $\sigma \models (f \mathcal{U} g)^r$
using 6 **by** (*simp add: until-d-def reverse-d-def*)
show *?thesis*
using 7 **by** *blast*
qed

lemma *RevUntilsem*:

$\sigma \models (f \mathcal{U} g)^r = (f^r) \mathcal{S} (g^r)$

using *RevUntilsema RevUntilsemb* **using** *unl-lift2* **by** *blast*

lemma *RevUntil*:

$\vdash (f \mathcal{U} g)^r = (f^r) \mathcal{S} (g^r)$

using *RevUntilsem Valid-def* **by** *blast*

14.2.14 DiamondEqvTrueUntil

lemma *DiamondEqvTrueUntil*:

$\vdash \Diamond f = \# \text{True } \mathcal{U} f$

by (*simp add: Valid-def itl-defs until-d-def*)

14.2.15 TrueUntillImpNotUntil

lemma *interval-suf-first*:

assumes $(\exists i \leq \text{ilen } xs. f (\text{suffix } i \text{ } xs))$

shows $(\exists i \leq \text{ilen } xs. f (\text{suffix } i \text{ } xs) \wedge$
 $(\forall j. j < i \longrightarrow \neg (f (\text{suffix } j \text{ } xs))))$

using *assms suf-first-upto[of ilen xs + 1 f xs]*

by (*simp add: less-eq-iff-succ-less*)

lemma *NotSuffixFirst*:

assumes $(\exists n \leq \text{ilen } xs. \neg f (\text{suffix } n \text{ } xs))$

shows $(\exists n \leq \text{ilen } xs. \neg f (\text{suffix } n \text{ } xs) \wedge (\forall k. k < n \longrightarrow f (\text{suffix } k \text{ } xs)))$

using *assms interval-suf-first[of xs LIFT($\neg f$)]* **by** *auto*

lemma *NotSuffixFirst-upto*:

assumes $(\exists i < k. \neg f (\text{suffix } i \text{ } xs))$

$k \leq \text{ilen } xs + 1$

shows $(\exists i < k. \neg f (\text{suffix } i \text{ } xs) \wedge$
 $(\forall j < i. (f (\text{suffix } j \text{ } xs))))$

using *assms suf-first-upto[of k LIFT($\neg f$) xs]* **by** *auto*

lemma *TrueUntilImpNotUntil*:

$\vdash \# \text{True } \mathcal{U} g \longrightarrow (\neg g) \mathcal{U} g$

by (*simp add: intI interval-suf-first until-d-def*)

14.2.16 WaitNotDistUntil

lemma *WaitNotDistUntilsem1*:

assumes $(\sigma \models \neg(f \mathcal{W} g))$

shows $(\sigma \models ((\neg g) \mathcal{U} ((\neg f) \wedge (\neg g))))$

proof –

have 1: $(\forall k. g (\text{suffix } k \text{ } \sigma) \longrightarrow k \leq \text{ilen } \sigma \longrightarrow (\exists j < k. \neg f (\text{suffix } j \text{ } \sigma))) \wedge$
 $(\exists n \leq \text{ilen } \sigma. \neg f (\text{suffix } n \text{ } \sigma))$

using *assms* **by** (*simp add: wait-d-def until-d-def itl-defs*)

have 2: $(\forall k. k \leq \text{ilen } \sigma \longrightarrow \neg g (\text{suffix } k \text{ } \sigma) \vee (\exists j < k. \neg f (\text{suffix } j \text{ } \sigma)))$

using 1 **by** *auto*

have 3: $(\exists n \leq \text{ilen } \sigma. \neg f (\text{suffix } n \text{ } \sigma))$

using 1 **by** *auto*

obtain *n* **where** 4: $n \leq \text{ilen } \sigma \wedge \neg f (\text{suffix } n \text{ } \sigma) \wedge$
 $(\forall k < n. f (\text{suffix } k \text{ } \sigma))$

using 3 **using** *NotSuffixFirst* **by** *blast*

have 16: $n \leq \text{ilen } \sigma$

by (*simp add: 4*)

have 17: $\neg g (\text{suffix } n \text{ } \sigma)$

```

  using 1 4 by blast
have 18:  $(\forall j < n. \neg g (\text{suffix } j \ \sigma))$ 
  by (meson 2 4 le-eq-less-or-eq less-le-trans)
have 19:  $\exists k \leq \text{ilen } \sigma. \neg f (\text{suffix } k \ \sigma) \wedge \neg g (\text{suffix } k \ \sigma) \wedge (\forall j < k. \neg g (\text{suffix } j \ \sigma))$ 
  using 16 17 18 4 by blast
have 20:  $(\sigma \models ((\neg g) \mathcal{U} (\neg f \wedge \neg g)))$ 
  using 19 by (simp add: until-d-def)
show ?thesis using 20 by auto
qed

```

```

lemma WaitNotDistUntilsem2:
assumes  $(\sigma \models ((\neg g) \mathcal{U} ((\neg f) \wedge (\neg g))))$ 
shows  $(\sigma \models \neg(f \mathcal{W} g))$ 
using assms not-less-iff-gr-or-eq by (auto simp add: itl-defs wait-d-def until-d-def)

```

```

lemma WaitNotDistUntilsem:
 $(\sigma \models (\neg(f \mathcal{W} g)) = ((\neg g) \mathcal{U} ((\neg f) \wedge (\neg g))))$ 
using WaitNotDistUntilsem1 WaitNotDistUntilsem2 unl-lift2 by blast

```

```

lemma WaitNotDistUntil:
 $\vdash (\neg(f \mathcal{W} g)) = ((\neg g) \mathcal{U} (\neg f \wedge \neg g))$ 
using WaitNotDistUntilsem Valid-def by blast

```

14.2.17 UntilInduction

```

lemma LFPUntilsem1:
assumes  $\forall n \leq \text{ilen } \sigma.$ 
   $(g (\text{suffix } n \ \sigma) \longrightarrow h (\text{suffix } n \ \sigma)) \wedge$ 
   $(f (\text{suffix } n \ \sigma) \wedge n < \text{ilen } \sigma \wedge h (\text{suffix } (\text{Suc } n) \ \sigma) \longrightarrow$ 
     $h (\text{suffix } n \ \sigma))$ 
   $k \leq \text{ilen } \sigma$ 
   $g (\text{suffix } k \ \sigma)$ 
   $\forall j < k. f (\text{suffix } j \ \sigma)$ 
shows  $h \ \sigma$ 
using assms
proof (induct k arbitrary:  $\sigma$ )
case 0
then show ?case by auto
next
case (Suc k)
then show ?case
  proof (cases  $\sigma$ )
  case (INil x1)
  then show ?thesis using Suc.prem1 Suc.hyps by (metis suffix.simps(1))
  next
  case (ICons x21 x22)
  then show ?thesis
    using Suc.prem1 Suc.hyps
    by (metis Suc-less-eq suffix-suc suffix-zero
      ilen.simps(2) le-eq-less-or-eq not-less-eq-eq plus-1-eq-Suc zero-less-Suc)
  end

```

qed
qed

lemma *LFPUntilsem*:

$\sigma \models \Box((g \vee (f \wedge \bigcirc h)) \longrightarrow h) \longrightarrow (f \mathcal{U} g \longrightarrow h)$
using *LFPUntilsem1* **by** (*simp add: itl-defs until-d-def, blast*)

lemma *LFPUntil*:

$\vdash \Box((g \vee (f \wedge \bigcirc h)) \longrightarrow h) \longrightarrow (f \mathcal{U} g \longrightarrow h)$
using *LFPUntilsem Valid-def* **by** *blast*

lemma *UntilInduction-a*:

$\vdash \Box(f \longrightarrow ((\bigcirc f) \wedge g) \vee h) \longrightarrow (f \longrightarrow \Box g \vee g \mathcal{U} h)$

proof –

have 1: $\vdash (\Box g \vee g \mathcal{U} h) = g \mathcal{W} h$

by (*auto simp add: wait-d-def*)

have 2: $\vdash (f \longrightarrow \Box g \vee g \mathcal{U} h) = ((\neg h) \mathcal{U} (\neg g \wedge \neg h) \longrightarrow \neg f)$

using 1 *WaitNotDistUntil* **by** *fastforce*

have 3: $\vdash \Box(((\neg g \wedge \neg h) \vee (\neg h \wedge \bigcirc(\neg f))) \longrightarrow \neg f) \longrightarrow ((\neg h) \mathcal{U} (\neg g \wedge \neg h) \longrightarrow \neg f)$

using *LFPUntil* **by** *blast*

have 4: $\vdash (f \longrightarrow ((\bigcirc f) \wedge g) \vee h) \longrightarrow (((\neg g \wedge \neg h) \vee (\neg h \wedge \bigcirc(\neg f))) \longrightarrow \neg f)$

using *NextImpNotNextNot[of f]* **by** *auto*

have 5: $\vdash \Box(f \longrightarrow ((\bigcirc f) \wedge g) \vee h) \longrightarrow \Box(((\neg g \wedge \neg h) \vee (\neg h \wedge \bigcirc(\neg f))) \longrightarrow \neg f)$

using 4 **by** (*rule ImpBoxRule*)

show *?thesis*

using 2 3 5 **by** *fastforce*

qed

lemma *UntilInduction-b*:

$\vdash \Box(f \longrightarrow (\bigcirc f) \vee g) \longrightarrow (f \longrightarrow \Box f \vee f \mathcal{U} g)$

proof –

have 1: $\vdash (\Box f \vee f \mathcal{U} g) = f \mathcal{W} g$

by (*auto simp add: wait-d-def*)

have 2: $\vdash (f \longrightarrow \Box f \vee f \mathcal{U} g) = ((\neg g) \mathcal{U} (\neg f \wedge \neg g) \longrightarrow \neg f)$

using 1 *WaitNotDistUntil* **by** *fastforce*

have 3: $\vdash \Box(((\neg f \wedge \neg g) \vee (\neg g \wedge \bigcirc(\neg f))) \longrightarrow \neg f) \longrightarrow ((\neg g) \mathcal{U} (\neg f \wedge \neg g) \longrightarrow \neg f)$

using *LFPUntil* **by** *blast*

have 4: $\vdash (f \longrightarrow (\bigcirc f) \vee g) \longrightarrow (((\neg f \wedge \neg g) \vee (\neg g \wedge \bigcirc(\neg f))) \longrightarrow \neg f)$

using *NextImpNotNextNot[of f]* **by** *auto*

have 5: $\vdash \Box(f \longrightarrow (\bigcirc f) \vee g) \longrightarrow \Box(((\neg f \wedge \neg g) \vee (\neg g \wedge \bigcirc(\neg f))) \longrightarrow \neg f)$

using 4 *BoxImpBoxRule* **by** *blast*

show *?thesis*

using 2 3 5 **by** *fastforce*

qed

14.3 Theorems

lemma *NextFalseSUntil*:

$\vdash \bigcirc g = \#False \mathcal{U}^s g$

proof –

have 1: $\vdash \#False \mathcal{U} g = g$
using *UntilNextUntil*[of *LIFT*($\#False$) *g*] **by** *auto*
show ?thesis **unfolding** *suntil-d-def* **using** 1 *inteq-reflection* **by** *force*
qed

lemma *PrevFalseSSince*:
 $\vdash prev\ g = \#False\ \mathcal{S}^s\ g$
proof –
have 1: $\vdash \#False\ \mathcal{S}\ g = g$
using *SincePrevSince*[of *LIFT*($\#False$) *g*] **by** *auto*
show ?thesis **unfolding** *ssince-d-def* **using** 1 *inteq-reflection* **by** *force*
qed

lemma *WNextUntil*:
 $\vdash wnext(f\ \mathcal{U}\ g) = (empty \vee (\bigcirc f)\ \mathcal{U}\ (\bigcirc g))$
by (*meson* *NextUntil* *Prop06* *WnextEqvEmptyOrNext*)

lemma *UntilRelease*:
 $\vdash f\ \mathcal{R}\ g = (\neg (\neg f)\ \mathcal{U}\ (\neg g))$
by (*simp* *add*: *release-d-def*)

lemma *SReleaseWait*:
 $\vdash f\ \mathcal{M}\ g = (\neg (\neg f)\ \mathcal{W}\ (\neg g))$
by (*simp* *add*: *srelease-d-def*)

lemma *ReleaseUntil*:
 $\vdash f\ \mathcal{U}\ g = (\neg (\neg f)\ \mathcal{R}\ (\neg g))$
by (*simp* *add*: *release-d-def*)

lemma *WaitSRelease*:
 $\vdash f\ \mathcal{W}\ g = (\neg (\neg f)\ \mathcal{M}\ (\neg g))$
by (*simp* *add*: *srelease-d-def*)

lemma *NotUntilRelease*:
 $\vdash \neg(f\ \mathcal{U}\ g) = (\neg f)\ \mathcal{R}\ (\neg g)$
by (*simp* *add*: *ReleaseUntil*)

lemma *NotWaitSRelease*:
 $\vdash \neg(f\ \mathcal{W}\ g) = (\neg f)\ \mathcal{M}\ (\neg g)$
by (*simp* *add*: *WaitSRelease*)

lemma *NotReleaseUntil*:
 $\vdash \neg(f\ \mathcal{R}\ g) = (\neg f)\ \mathcal{U}\ (\neg g)$
by (*simp* *add*: *UntilRelease*)

lemma *NotSReleaseWait*:
 $\vdash \neg(f\ \mathcal{M}\ g) = (\neg f)\ \mathcal{W}\ (\neg g)$
by (*simp* *add*: *SReleaseWait*)

lemma *BoxEqvFalseRelease*:

$\vdash \Box f = \#False \mathcal{R} f$
by (*metis DiamondEqvTrueUntil EqvReverseReverse always-d-def int-simps(3) inteq-reflection release-d-def*)

lemma *RevSince*:

$\vdash (f \mathcal{S} g)^r = (f^r) \mathcal{U} (g^r)$

proof –

have 1: $\vdash (f^r \mathcal{U} g^r)^r = (f^r)^r \mathcal{S} (g^r)^r$

by (*simp add: RevUntil*)

show *?thesis*

by (*metis 1 EqvReverseReverse inteq-reflection*)

qed

lemma *LFPSince*:

$\vdash bi((g \vee (f \wedge prev\ h)) \longrightarrow h) \longrightarrow (f \mathcal{S} g \longrightarrow h)$

by (*metis (no-types, lifting) LFPUntil RBiEqvBox RPrevEqvNext RevSince ReverseEqv all-rev-eq(3) inteq-reflection*)

lemma *UntilTrue*:

$\vdash f \mathcal{U} \#True$

using *UntilNextUntil* **by** *fastforce*

lemma *UntilIdempotent*:

$\vdash f \mathcal{U} f = f$

using *UntilNextUntil* **by** *fastforce*

lemma *UntilImpUntil*:

assumes $\vdash f0 \longrightarrow f1$

$\vdash g0 \longrightarrow g1$

shows $\vdash f0 \mathcal{U} g0 \longrightarrow f1 \mathcal{U} g1$

using *assms*

by (*metis Prop10 Prop12 UntilAndDist UntilLeftDistAnd int-eq*)

lemma *UntilEqvUntil*:

assumes $\vdash f0 = f1$

$\vdash g0 = g1$

shows $\vdash f0 \mathcal{U} g0 = f1 \mathcal{U} g1$

proof –

have 1: $\vdash f0 \longrightarrow f1$

using *assms* **by** *auto*

have 2: $\vdash g0 \longrightarrow g1$

using *assms* **by** *auto*

have 3: $\vdash f0 \mathcal{U} g0 \longrightarrow f1 \mathcal{U} g1$

using 1 2 *UntilImpUntil*[*of f0 f1 g0 g1*] **by** *auto*

have 4: $\vdash f1 \longrightarrow f0$

using *assms* **by** *auto*

have 5: $\vdash g1 \longrightarrow g0$

using *assms* **by** *auto*

have 6: $\vdash f1 \mathcal{U} g1 \longrightarrow f0 \mathcal{U} g0$

using 4 5 *UntilImpUntil*[*of f1 f0 g1 g0*] **by** *auto*

from 3 6 show ?thesis by fastforce
qed

lemma *SinceImpSince*:

assumes $\vdash f0 \longrightarrow f1$

$\vdash g0 \longrightarrow g1$

shows $\vdash f0 \mathcal{S} g0 \longrightarrow f1 \mathcal{S} g1$

using *assms*

by (*metis RevSince ReverseEqv UntilImpUntil all-rev-eq(3) inteq-reflection*)

lemma *SinceEqvSince*:

assumes $\vdash f0 = f1$

$\vdash g0 = g1$

shows $\vdash f0 \mathcal{S} g0 = f1 \mathcal{S} g1$

proof –

have 1: $\vdash f0 \longrightarrow f1$

using *assms* by auto

have 2: $\vdash g0 \longrightarrow g1$

using *assms* by auto

have 3: $\vdash f0 \mathcal{S} g0 \longrightarrow f1 \mathcal{S} g1$

using 1 2 *SinceImpSince*[of *f0 f1 g0 g1*] by auto

have 4: $\vdash f1 \longrightarrow f0$

using *assms* by auto

have 5: $\vdash g1 \longrightarrow g0$

using *assms* by auto

have 6: $\vdash f1 \mathcal{S} g1 \longrightarrow f0 \mathcal{S} g0$

using 4 5 *SinceImpSince*[of *f1 f0 g1 g0*] by auto

from 3 6 show ?thesis by fastforce

qed

lemma *UntilRightDistImp*:

$\vdash (f \longrightarrow g) \mathcal{U} h \longrightarrow (f \mathcal{U} h \longrightarrow g \mathcal{U} h)$

proof –

have 1: $\vdash (f \longrightarrow g) \mathcal{U} h \longrightarrow (f \mathcal{U} h \longrightarrow g \mathcal{U} h) =$

$((f \longrightarrow g) \mathcal{U} h \wedge f \mathcal{U} h \longrightarrow g \mathcal{U} h)$

by auto

have 2: $\vdash ((f \longrightarrow g) \mathcal{U} h \wedge f \mathcal{U} h) = ((f \longrightarrow g) \wedge f) \mathcal{U} h$

by (*simp add: UntilAndDist int-iffD1 int-iffD2 int-iffI*)

have 3: $\vdash ((f \longrightarrow g) \wedge f) = (f \wedge g)$

by auto

have 4: $\vdash h = h$

by auto

have 5: $\vdash ((f \longrightarrow g) \wedge f) \mathcal{U} h = (f \wedge g) \mathcal{U} h$

using 3 4 using *UntilEqvUntil* by blast

have 6: $\vdash (f \wedge g) \mathcal{U} h = (f \mathcal{U} h \wedge g \mathcal{U} h)$

by (*simp add: UntilAndDist*)

show ?thesis

using 2 5 6 by fastforce

qed

lemma *FalseUntil*:

$\vdash \#False \mathcal{U} g = g$

by (*metis Prop10 Prop12 TrueW UntilNextUntil int-simps(14) int-simps(21) int-simps(25) int-simps(3) inteq-reflection*)

lemma *UntilExclMid*:

$\vdash f \mathcal{U} g \vee f \mathcal{U} (\neg g)$

using *UntilOrDist UntilTrue* **by** *fastforce*

lemma *NotUntilImp*:

$\vdash (\neg f) \mathcal{U} (g \mathcal{U} h) \wedge f \mathcal{U} h \longrightarrow g \mathcal{U} h$

proof –

have 1: $\vdash (\neg f) \mathcal{U} (g \mathcal{U} h) \longrightarrow (\neg f \vee g) \mathcal{U} h$

by (*simp add: UntilRightOr*)

have 2: $\vdash (\neg f \vee g) = (f \longrightarrow g)$

by *auto*

have 3: $\vdash h = h$

by *auto*

have 4: $\vdash (\neg f \vee g) \mathcal{U} h = (f \longrightarrow g) \mathcal{U} h$

by (*simp add: 2 UntilEqvUntil*)

have 5: $\vdash (f \longrightarrow g) \mathcal{U} h \longrightarrow (f \mathcal{U} h \longrightarrow g \mathcal{U} h)$

by (*simp add: UntilRightDistImp*)

have 6: $\vdash (\neg f) \mathcal{U} (g \mathcal{U} h) \longrightarrow (f \mathcal{U} h \longrightarrow g \mathcal{U} h)$

using 1 4 5 **by** *fastforce*

from 6 **show** *?thesis* **by** *auto*

qed

lemma *UntilNotImpa*:

$\vdash f \mathcal{U} ((\neg g) \mathcal{U} h) \wedge g \mathcal{U} h \longrightarrow f \mathcal{U} h$

proof –

have 1: $\vdash f \mathcal{U} ((\neg g) \mathcal{U} h) \longrightarrow (f \vee (\neg g)) \mathcal{U} h$

by (*simp add: UntilRightOr*)

have 2: $\vdash (f \vee (\neg g)) = (g \longrightarrow f)$

by *auto*

have 3: $\vdash h = h$

by *auto*

have 4: $\vdash (f \vee (\neg g)) \mathcal{U} h = (g \longrightarrow f) \mathcal{U} h$

by (*simp add: 2 UntilEqvUntil*)

have 5: $\vdash (g \longrightarrow f) \mathcal{U} h \longrightarrow (g \mathcal{U} h \longrightarrow f \mathcal{U} h)$

by (*simp add: UntilRightDistImp*)

have 6: $\vdash f \mathcal{U} ((\neg g) \mathcal{U} h) \longrightarrow (g \mathcal{U} h \longrightarrow f \mathcal{U} h)$

using 1 4 5 **by** *fastforce*

from 6 **show** *?thesis* **by** *auto*

qed

lemma *UntilNotUntilImp*:

$\vdash f \mathcal{U} g \wedge (\neg g) \mathcal{U} f \longrightarrow f$

proof –

have 1: $\vdash f \mathcal{U} g \wedge (\neg g) \mathcal{U} f \longrightarrow f \mathcal{U} f$

using *UntilNotImp* **by** *auto*

have 2: $\vdash f \mathcal{U} f = f$
using *UntilIdempotent* **by** *auto*
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *AndNotUntilImp*:
 $\vdash f \wedge (\neg f) \mathcal{U} g \longrightarrow g$
proof –
have 1: $\vdash f = f \mathcal{U} f$
by (*simp add: UntilIdempotent int-iffD1 int-iffD2 int-iffI*)
have 2: $\vdash g = \#False \mathcal{U} g$
by (*meson FalseUntil Prop11*)
have 3: $\vdash f \mathcal{U} f \wedge (\neg f) \mathcal{U} g \longrightarrow \#False \mathcal{U} g$
by (*metis 1 FalseUntil UntilNotImp inteq-reflection*)
from 1 2 3 **show** *?thesis* **by** *fastforce*
qed

lemma *UntilImpOr*:
 $\vdash f \mathcal{U} g \longrightarrow f \vee g$

proof –
have $\vdash f \wedge \bigcirc(f \mathcal{U} g) \longrightarrow f \vee g$
by *force*
then show *?thesis*
using *UntilNextUntil[of f g]* **by** *auto*
qed

lemma *UntilIntro*:
 $\vdash g \longrightarrow f \mathcal{U} g$
proof –
have 1: $\vdash g = \#False \mathcal{U} g$
by (*meson FalseUntil Prop11*)
have 2: $\vdash \#False \longrightarrow f$
by *auto*
have 3: $\vdash g \longrightarrow g$
by *auto*
have 4: $\vdash \#False \mathcal{U} g \longrightarrow f \mathcal{U} g$
by (*simp add: UntilImpUntil*)
from 1 4 **show** *?thesis* **by** *fastforce*
qed

lemma *OrImpUntil*:
 $\vdash f \wedge g \longrightarrow f \mathcal{U} g$
by (*simp add: Prop01 Prop05 UntilIntro*)

lemma *UntilAbsorp-a*:
 $\vdash (f \vee f \mathcal{U} g) = (f \vee g)$
proof –
have 1: $\vdash (f \vee f \mathcal{U} g) \longrightarrow f \vee g$
using *UntilImpOr* **by** *fastforce*

have 2: $\vdash f \vee g \longrightarrow (f \vee f \mathcal{U} g)$
using *UntilIntro* **by** *fastforce*
from 1 2 **show** *?thesis* **by** *fastforce*
qed

lemma *UntilAbsorp-b*:
 $\vdash (f \mathcal{U} g \vee g) = f \mathcal{U} g$
using *UntilNextUntil* **by** *fastforce*

lemma *UntilAbsorp-c*:
 $\vdash (f \mathcal{U} g \wedge g) = g$
using *UntilIntro* **by** *fastforce*

lemma *UntilAbsorp-d*:
 $\vdash (f \mathcal{U} g \vee (f \wedge g)) = f \mathcal{U} g$
using *UntilNextUntil* **by** *fastforce*

lemma *UntilAbsorp-e*:
 $\vdash (f \mathcal{U} g \wedge (f \vee g)) = f \mathcal{U} g$
by (*meson Prop10 Prop11 UntilImpOr*)

lemma *LeftUntilAbsorp*:
 $\vdash f \mathcal{U} (f \mathcal{U} g) = f \mathcal{U} g$
by (*meson Prop11 UntilUntil*)

lemma *RightUntilAbsorp*:
 $\vdash (f \mathcal{U} g) \mathcal{U} g = f \mathcal{U} g$
by (*metis Prop11 UntilAbsorp-b UntilAbsorp-c UntilImpOr UntilRightAnd UntilUntil inteq-reflection*)

lemma *UntilAbsorpAndDiamond*:
 $\vdash (f \mathcal{U} g \wedge \Diamond g) = f \mathcal{U} g$
by (*metis DiamondEqvTrueUntil Prop11 Prop12 UntilAbsorp-c UntilRightAnd int-simps(17) inteq-reflection*)

lemma *UntilAbsorpOrDiamond*:
 $\vdash (f \mathcal{U} g \vee \Diamond g) = \Diamond g$
using *UntilAbsorpAndDiamond* **by** *fastforce*

lemma *UntilAbsorpDiamond*:
 $\vdash f \mathcal{U} (\Diamond g) = \Diamond g$
using *DiamondDiamondEqvDiamond UntilAbsorpOrDiamond UntilAbsorp-b* **by** *fastforce*

lemma *UntilImpDiamond*:
 $\vdash f \mathcal{U} g \longrightarrow \Diamond g$
using *UntilAbsorpAndDiamond* **by** *fastforce*

lemma *AlwaysImpNotUntilNot*:
 $\vdash \Box f \longrightarrow \neg(g \mathcal{U} (\neg f))$
by (*simp add: UntilImpDiamond always-d-def*)

lemma *UntilAlwaysAndDist*:

$\vdash \Box f \wedge g \mathcal{U} h \longrightarrow (f \wedge g) \mathcal{U} (f \wedge h)$

proof –

have 1: $\vdash \Box(h \vee g \wedge \bigcirc((f \wedge g) \mathcal{U} (f \wedge h))) \longrightarrow (f \wedge g) \mathcal{U} (f \wedge h) \longrightarrow$
 $g \mathcal{U} h \longrightarrow (f \wedge g) \mathcal{U} (f \wedge h)$

using *LFPUntil* **by** *blast*

have 2: $\vdash f \longrightarrow (h \vee g \wedge \bigcirc((f \wedge g) \mathcal{U} (f \wedge h))) \longrightarrow (f \wedge g) \mathcal{U} (f \wedge h)$

using *UntilNextUntil* **by** *fastforce*

have 3: $\vdash \Box f \longrightarrow \Box(h \vee g \wedge \bigcirc((f \wedge g) \mathcal{U} (f \wedge h))) \longrightarrow (f \wedge g) \mathcal{U} (f \wedge h)$

using 2 *BoxImpBoxRule* **by** *blast*

have 4: $\vdash \Box f \longrightarrow (g \mathcal{U} h \longrightarrow (f \wedge g) \mathcal{U} (f \wedge h))$

using 1 3 *lift-imp-trans* **by** *blast*

show *?thesis* **using** 4 **by** *fastforce*

qed

lemma *UntilAndImp*:

$\vdash \Box f \wedge \Diamond g \longrightarrow f \mathcal{U} g$

proof –

have 1: $\vdash \Diamond g = \#True \mathcal{U} g$

by (*simp add: DiamondEqvTrueUntil*)

have 2: $\vdash \Box f \wedge \#True \mathcal{U} g \longrightarrow (f \wedge \#True) \mathcal{U} (f \wedge g)$

using *UntilAlwaysAndDist* **by** *blast*

have 3: $\vdash (f \wedge \#True) \mathcal{U} (f \wedge g) = f \mathcal{U} (f \wedge g)$

by *simp*

have 4: $\vdash f \mathcal{U} (f \wedge g) \longrightarrow (f \mathcal{U} f) \mathcal{U} g$

by (*simp add: UntilRightAnd*)

have 5: $\vdash (f \mathcal{U} f) = f$

by (*simp add: UntilIdempotent*)

have 6: $\vdash (f \mathcal{U} f) \mathcal{U} g = f \mathcal{U} g$

by (*simp add: 5 UntilEqvUntil*)

show *?thesis*

by (*metis 1 2 3 4 5 inteq-reflection lift-imp-trans*)

qed

lemma *UntilRightMono*:

$\vdash \Box(f \longrightarrow g) \longrightarrow (h \mathcal{U} f \longrightarrow h \mathcal{U} g)$

proof –

have 1: $\vdash \Box(f \longrightarrow g) \wedge h \mathcal{U} f \longrightarrow ((f \longrightarrow g) \wedge h) \mathcal{U} ((f \longrightarrow g) \wedge f)$

using *UntilAlwaysAndDist* **by** *blast*

have 2: $\vdash ((f \longrightarrow g) \wedge h) \mathcal{U} ((f \longrightarrow g) \wedge f) \longrightarrow h \mathcal{U} ((f \longrightarrow g) \wedge f)$

by (*meson Prop12 UntilImpUntil int-iffD2 lift-and-com*)

have 3: $\vdash ((f \longrightarrow g) \wedge f) \longrightarrow g$

by *auto*

have 4: $\vdash h \mathcal{U} ((f \longrightarrow g) \wedge f) \longrightarrow h \mathcal{U} g$

by (*simp add: 3 UntilImpUntil*)

show *?thesis*

by (*meson 1 2 4 Prop09 lift-imp-trans*)

qed

lemma *UntilLeftMono*:

$\vdash \Box (f \longrightarrow g) \longrightarrow (f \mathcal{U} h \longrightarrow g \mathcal{U} h)$
proof –
have 1: $\vdash \Box (f \longrightarrow g) \wedge f \mathcal{U} h \longrightarrow ((f \longrightarrow g) \wedge f) \mathcal{U} ((f \longrightarrow g) \wedge h)$
by (*simp add: UntilAlwaysAndDist*)
have 2: $\vdash ((f \longrightarrow g) \wedge f) \mathcal{U} ((f \longrightarrow g) \wedge h) \longrightarrow ((f \longrightarrow g) \wedge f) \mathcal{U} h$
by (*meson Prop12 UntilLeftDistAnd*)
have 3: $\vdash ((f \longrightarrow g) \wedge f) \longrightarrow g$
by *auto*
have 4: $\vdash ((f \longrightarrow g) \wedge f) \mathcal{U} h \longrightarrow g \mathcal{U} h$
by (*simp add: 3 UntilImpUntil*)
show *?thesis*
by (*meson 1 2 4 Prop09 lift-imp-trans*)
qed

lemma *UntilCatRule*:

$\vdash \Box ((f \longrightarrow g \mathcal{U} h) \wedge (h \longrightarrow g \mathcal{U} i)) \longrightarrow (f \longrightarrow (g \mathcal{U} i))$
proof –
have 1: $\vdash \Box ((f \longrightarrow g \mathcal{U} h) \wedge (h \longrightarrow g \mathcal{U} i)) \longrightarrow \Box (f \longrightarrow g \mathcal{U} h)$
by (*metis BoxElim BoxEqvBoxBox BoxImpBoxRule Prop12 inteq-reflection*)
have 2: $\vdash \Box ((f \longrightarrow g \mathcal{U} h) \wedge (h \longrightarrow g \mathcal{U} i)) \longrightarrow \Box (h \longrightarrow g \mathcal{U} i)$
by (*metis BoxElim BoxEqvBoxBox BoxImpBoxRule Prop12 inteq-reflection*)
have 3: $\vdash \Box (h \longrightarrow g \mathcal{U} i) \longrightarrow \Box (g \mathcal{U} h \longrightarrow g \mathcal{U} (g \mathcal{U} i))$
by (*metis BoxEqvBoxBox BoxImpBoxRule UntilRightMono inteq-reflection*)
have 4: $\vdash \Box (g \mathcal{U} h \longrightarrow g \mathcal{U} (g \mathcal{U} i)) \longrightarrow \Box (g \mathcal{U} h \longrightarrow g \mathcal{U} i)$
by (*metis BoxEqvBoxBox UntilUntil int-iffD1 inteq-reflection*)
have 5: $\vdash \Box (f \longrightarrow g \mathcal{U} h) \longrightarrow (f \longrightarrow g \mathcal{U} h)$
by (*simp add: BoxElim*)
have 6: $\vdash \Box (g \mathcal{U} h \longrightarrow g \mathcal{U} i) \longrightarrow (g \mathcal{U} h \longrightarrow g \mathcal{U} i)$
by (*simp add: BoxElim*)
have 7: $\vdash (f \longrightarrow g \mathcal{U} h) \wedge (g \mathcal{U} h \longrightarrow g \mathcal{U} i) \longrightarrow (f \longrightarrow g \mathcal{U} i)$
by *auto*
have 8: $\vdash \Box ((f \longrightarrow g \mathcal{U} h) \wedge (h \longrightarrow g \mathcal{U} i)) \longrightarrow$
 $(f \longrightarrow g \mathcal{U} h) \wedge (g \mathcal{U} h \longrightarrow g \mathcal{U} i)$
using 1 2 3 4 5 6 **by** *fastforce*
from 7 8 **show** *?thesis* **by** *auto*
qed

lemma *UntilStrengthen*:

$\vdash \Box ((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (f \mathcal{U} g \longrightarrow h \mathcal{U} i)$
proof –
have 11: $\vdash \Box ((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow \Box (f \longrightarrow h)$
by (*meson BoxImpBoxRule Prop12 int-iffD2 lift-and-com*)
have 1: $\vdash \Box ((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (f \mathcal{U} g \longrightarrow h \mathcal{U} g)$
using 11 *UntilLeftMono*[*of f h g*] **by** *fastforce*
have 21: $\vdash \Box ((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow \Box (g \longrightarrow i)$
by (*simp add: BoxImpBoxRule Prop01 Prop05*)
have 2: $\vdash \Box ((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (h \mathcal{U} g \longrightarrow h \mathcal{U} i)$
using 21 *UntilRightMono*[*of g i h*] **by** *fastforce*
have 3: $\vdash \Box ((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (f \mathcal{U} g \longrightarrow h \mathcal{U} g) \wedge (h \mathcal{U} g \longrightarrow h \mathcal{U} i)$

using 1 2 by fastforce
 have 4: $\vdash (f \mathcal{U} g \longrightarrow h \mathcal{U} g) \wedge (h \mathcal{U} g \longrightarrow h \mathcal{U} i) \longrightarrow (f \mathcal{U} g \longrightarrow h \mathcal{U} i)$
 by auto
 from 3 4 show ?thesis by auto
 qed

lemma *UntilInduction*:

$\vdash \Box (f \longrightarrow \neg g \wedge \bigcirc f) \longrightarrow (f \longrightarrow \neg(h \mathcal{U} g))$
proof –
 have 1: $\vdash \Box (\neg g) \longrightarrow \neg(h \mathcal{U} g)$
 by (simp add: UntilImpDiamond always-d-def)
 have 15: $\vdash (f \longrightarrow \neg g \wedge \bigcirc f) \longrightarrow (g \vee \bigcirc (\neg f) \longrightarrow \neg f)$
 using NextImpNotNextNot[of f] by fastforce
 have 16: $\vdash (f \longrightarrow \neg g \wedge \bigcirc f) \longrightarrow (g \vee \#True \wedge \bigcirc (\neg f) \longrightarrow \neg f)$
 using 15 by auto
 have 2: $\vdash \Box (f \longrightarrow \neg g \wedge \bigcirc f) \longrightarrow \Box (g \vee \#True \wedge \bigcirc (\neg f) \longrightarrow \neg f)$
 using 16 BoxImpBoxRule by blast
 have 3: $\vdash \Box (f \longrightarrow \neg g \wedge \bigcirc f) \longrightarrow (\#True \mathcal{U} g \longrightarrow \neg f)$
 using 2 LFPUntil[of g LIFT(#True) LIFT($\neg f$)]
 by fastforce
 have 4: $\vdash (\#True \mathcal{U} g \longrightarrow \neg f) \longrightarrow (f \longrightarrow \neg(\#True \mathcal{U} g))$
 by auto
 have 5: $\vdash \neg(\#True \mathcal{U} g) = \Box (\neg g)$
 using BoxEqFalseRelease NotUntilRelease integ-reflection by fastforce
 from 5 4 3 1 show ?thesis by fastforce
 qed

lemma *UntilBoxImp*:

$\vdash f \mathcal{U} (\Box g) \longrightarrow \Box (f \mathcal{U} g)$
proof –
 have 1: $\vdash f \mathcal{U} \Box g \longrightarrow f \mathcal{U} g$
 by (meson BoxElim BoxGen MP UntilRightMono)
 have 2: $\vdash wnext (f \mathcal{U} \Box g) = (empty \vee \bigcirc (f \mathcal{U} \Box g))$
 by (meson WnextEqvEmptyOrNext)
 have 3: $\vdash \Box g = (g \wedge wnext (\Box g))$
 by (metis (no-types) BoxEqvAndWnextBox)
 have 4: $\vdash f \mathcal{U} \Box g = (\Box g \vee f \wedge \bigcirc (f \mathcal{U} \Box g))$
 by (meson UntilNextUntil)
 have 5: $\vdash g \wedge wnext (\Box g) \longrightarrow empty \vee \bigcirc (f \mathcal{U} \Box g)$
 by (metis NextUntil Prop01 Prop05 Prop08 UntilIntro WnextEqvEmptyOrNext int-iffD1 integ-reflection)
 have 6: $\vdash more \wedge f \mathcal{U} \Box g \longrightarrow empty \vee \bigcirc (f \mathcal{U} \Box g)$
 using 5 3 4 by fastforce
 have 7: $\vdash more \wedge f \mathcal{U} \Box g \longrightarrow \bigcirc (f \mathcal{U} \Box g)$
 using 2 6 WnextAndMoreEqvNext by fastforce
 from 1 7 show ?thesis using BoxIntro[of LIFT($f \mathcal{U} (\Box g)$) LIFT($f \mathcal{U} g$)]
 by auto
 qed

lemma *UntilBoxEqvBox*:


```

  ⊢ f U (□ f) = □ f
proof –
  have 1: ⊢ f U (□ f) → □(f U f)
    using UntilBoxImp[of f f] by auto
  have 2: ⊢ □(f U f) = □ f
    by (simp add: BoxEqvBox UntilIdempotent)
  have 3: ⊢ □ f → f U (□ f)
    by (simp add: UntilIntro)
  from 1 2 3 show ?thesis by fastforce
qed

lemma UntilRightStrengthen:
  ⊢ f U (g ∧ h) → f U (g U h)
by (meson BoxGen MP OrImpUntil UntilRightMono)

lemma UntilLeftStrengthen:
  ⊢ (f ∧ g) U h → (f U g) U h
  by (simp add: OrImpUntil UntilImpUntil)

lemma UntilLeftAndOrder:
  ⊢ (f ∧ g) U h → f U (g U h)
  by (metis Prop12 UntilIdempotent UntilImpUntil UntilIntro inteq-reflection)

lemma UntilFrameNext:
  ⊢ □ f → (○ g → ○ (f U g))
by (simp add: NextImpNext Prop01 Prop05 Prop09 UntilIntro)

lemma UntilFrameDiamond:
  ⊢ □ f → (◇ g → ◇ (f U g))
by (meson NowImpDiamond Prop09 UntilAndImp lift-imp-trans)

lemma UntilFrameBox:
  ⊢ □ f → (□ g → □ (f U g))
by (simp add: BoxAndBoxImpBoxRule OrImpUntil Prop09)

lemma UntilImpNot:
  ⊢ f U g → (f ∧ ¬g) U g
proof –
  have 1: ⊢ f U g → ◇ g
    by (simp add: UntilImpDiamond)
  have 2: ⊢ ◇ g = #True U g
    by (simp add: DiamondEqvTrueUntil)
  have 3: ⊢ #True U g → (¬g) U g
    by (simp add: TrueUntilImpNotUntil)
  have 4: ⊢ (f U g ∧ (¬g) U g) = (f ∧ ¬g) U g
    using UntilAndDist by fastforce
  show ?thesis
  by (meson 1 2 3 4 Prop10 int-iffD1 lift-imp-trans)
qed

```

lemma *UntilAndRule*:

$\vdash f \mathcal{U} g = (f \wedge \neg g) \mathcal{U} g$

proof –

have 1: $\vdash (f \wedge \neg g) \mathcal{U} g \longrightarrow f \mathcal{U} g$

using *UntilAndDist* **by** *fastforce*

show *?thesis* **by** (*simp add: 1 UntilImpNot int-iffI*)

qed

lemma *UntilWait*:

$\vdash f \mathcal{U} g = (f \mathcal{W} g \wedge \Diamond g)$

proof –

have 1: $\vdash f \mathcal{U} g \longrightarrow f \mathcal{W} g \wedge \Diamond g$

by (*simp add: Prop05 Prop12 UntilImpDiamond wait-d-def*)

have 2: $\vdash (f \mathcal{W} g \wedge \Diamond g) = ((\Box f \vee f \mathcal{U} g) \wedge \Diamond g)$

by (*auto simp add: wait-d-def*)

have 3: $\vdash ((\Box f \vee f \mathcal{U} g) \wedge \Diamond g) = ((\Box f \wedge \Diamond g) \vee (f \mathcal{U} g \wedge \Diamond g))$

by *auto*

have 4: $\vdash (\Box f \wedge \Diamond g) \longrightarrow f \mathcal{U} g$

by (*simp add: UntilAndImp*)

have 5: $\vdash (f \mathcal{U} g \wedge \Diamond g) \longrightarrow f \mathcal{U} g$

by *auto*

show *?thesis*

using 1 2 4 **by** *fastforce*

qed

lemma *WaitLeftDistAnd*:

$\vdash f \mathcal{W} (g \wedge h) \longrightarrow f \mathcal{W} g \wedge f \mathcal{W} h$

proof –

have 1: $\vdash f \mathcal{W} (g \wedge h) \longrightarrow f \mathcal{W} g$

unfolding *wait-d-def*

by (*metis Prop08 Prop12 UntilAbsorp-a UntilLeftDistAnd int-iffD1 inteq-reflection*)

have 2: $\vdash f \mathcal{W} (g \wedge h) \longrightarrow f \mathcal{W} h$

unfolding *wait-d-def*

by (*metis Prop08 Prop12 UntilAbsorp-a UntilLeftDistAnd int-iffD1 inteq-reflection*)

show *?thesis* **by** (*simp add: 1 2 Prop12*)

qed

lemma *WaitRightDistAnd*:

$\vdash (f \wedge g) \mathcal{W} h = (f \mathcal{W} h \wedge g \mathcal{W} h)$

proof –

have 1: $\vdash \Box(f \wedge g) = (\Box f \wedge \Box g)$

by (*metis BoxAndBoxEqvBoxRule inteq-reflection lift-and-com*)

have 2: $\vdash (f \wedge g) \mathcal{U} h = (f \mathcal{U} h \wedge g \mathcal{U} h)$

by (*simp add: UntilAndDist*)

have 3: $\vdash ((\Box f \wedge \Box g) \longrightarrow ((\Box f \vee f \mathcal{U} h) \wedge (\Box g \vee g \mathcal{U} h)))$

by (*simp add: intI*)

have 4: $\vdash (f \mathcal{U} h \wedge g \mathcal{U} h) \longrightarrow ((\Box f \vee f \mathcal{U} h) \wedge (\Box g \vee g \mathcal{U} h))$

by *auto*

have 5: $\vdash ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h)) \longrightarrow ((\Box f \vee f \mathcal{U} h) \wedge (\Box g \vee g \mathcal{U} h))$

using 3 4 **by** *fastforce*

have 6: $\vdash \Box f \wedge \Box g \longrightarrow ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h))$
by *auto*
have 7: $\vdash \Box f \wedge g \mathcal{U} h \longrightarrow ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h))$
by (*metis 2 Prop05 Prop12 UntilAlwaysAndDist UntilLeftDistAnd inteq-reflection lift-imp-trans*)
have 8: $\vdash f \mathcal{U} h \wedge \Box g \longrightarrow ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h))$
by (*metis Prop05 Prop08 Prop12 UntilAlwaysAndDist UntilAndDist UntilLeftDistAnd inteq-reflection lift-and-com*)
have 9: $\vdash f \mathcal{U} h \wedge g \mathcal{U} h \longrightarrow ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h))$
by *auto*
have 10: $\vdash ((\Box f \vee f \mathcal{U} h) \wedge (\Box g \vee g \mathcal{U} h)) \longrightarrow ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h))$
using 6 7 8 9 **by** *fastforce*
have 11: $\vdash ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h)) = ((\Box f \vee f \mathcal{U} h) \wedge (\Box g \vee g \mathcal{U} h))$
using 5 10 **by** *auto*
have 12: $\vdash (\Box(f \wedge g) \vee (f \wedge g) \mathcal{U} h) = ((\Box f \wedge \Box g) \vee (f \mathcal{U} h \wedge g \mathcal{U} h))$
using 1 2 **by** *fastforce*
show *?thesis unfolding wait-d-def using 11 12*
by (*meson Prop04 UntilIdempotent*)
qed

lemma *WaitAndRule:*

$\vdash f \mathcal{W} g = (f \wedge \neg g) \mathcal{W} g$

proof –

have 1: $\vdash (f \mathcal{W} g \wedge (\neg g) \mathcal{W} g) = (f \wedge \neg g) \mathcal{W} g$

by (*meson Prop11 WaitRightDistAnd*)

have 2: $\vdash (\neg g) \mathcal{W} g$

by (*metis NotUntilFalse WaitNotDistUntil int-eq int-simps(21) int-simps(4)*)

show *?thesis*

using 1 2 **by** *fastforce*

qed

lemma *WaitUntilb:*

$\vdash f \mathcal{W} g = (\Box (f \wedge \neg g) \vee f \mathcal{U} g)$

proof –

have 1: $\vdash f \mathcal{W} g = (f \wedge \neg g) \mathcal{W} g$

by (*simp add: WaitAndRule*)

have 2: $\vdash (f \wedge \neg g) \mathcal{W} g = (\Box (f \wedge \neg g) \vee (f \wedge \neg g) \mathcal{U} g)$

by (*auto simp add: wait-d-def*)

have 3: $\vdash (f \wedge \neg g) \mathcal{U} g = f \mathcal{U} g$

by (*meson Prop11 UntilAndRule*)

show *?thesis*

using 1 2 3 **by** *fastforce*

qed

lemma *UntilNotDistWait:*

$\vdash (\neg (f \mathcal{U} g)) = (\neg g) \mathcal{W} (\neg f \wedge \neg g)$

proof –

have 1: $\vdash (\neg ((\neg g) \mathcal{W} (\neg f \wedge \neg g))) = (\neg(\neg f \wedge \neg g)) \mathcal{U} (\neg(\neg g) \wedge \neg(\neg f \wedge \neg g))$

using *WaitNotDistUntil* **by** *blast*

have 2: $\vdash (\neg(\neg f \wedge \neg g)) = (f \vee g)$

```

  by auto
have 3:  $\vdash (\neg (\neg g) \wedge \neg (\neg f \wedge \neg g)) = g$ 
  by auto
have 4:  $\vdash (\neg (\neg f \wedge \neg g)) \mathcal{U} (\neg (\neg g) \wedge \neg (\neg f \wedge \neg g)) =$ 
       $(f \vee g) \mathcal{U} g$ 
  using 2 3 UntilEqvUntil by blast
have 5:  $\vdash (f \vee g) \mathcal{U} g = ((f \vee g) \wedge \neg g) \mathcal{U} g$ 
  by (simp add: UntilAndRule)
have 6:  $\vdash ((f \vee g) \wedge \neg g) = (f \wedge \neg g)$ 
  by auto
have 7:  $\vdash ((f \vee g) \wedge \neg g) \mathcal{U} g = (f \wedge \neg g) \mathcal{U} g$ 
  using 6 inteq-reflection by fastforce
have 8:  $\vdash (f \wedge \neg g) \mathcal{U} g = f \mathcal{U} g$ 
  by (meson Prop11 UntilAndRule)
have 9:  $\vdash f \mathcal{U} g = (\neg ((\neg g) \mathcal{W} (\neg f \wedge \neg g)))$ 
  using 1 4 5 7 8 by fastforce
show ?thesis using 9 by auto
qed

```

lemma *UntilImpWait*:

```

 $\vdash f \mathcal{U} g \longrightarrow f \mathcal{W} g$ 
by (meson Prop03 WaitUntilb)

```

lemma *WaitAndDist*:

```

 $\vdash (\Box f \wedge g \mathcal{W} h) \longrightarrow (f \wedge g) \mathcal{W} (f \wedge h)$ 
proof -
  have 1:  $\vdash (\Box f \wedge g \mathcal{W} h) = (\Box f \wedge (\Box g \vee g \mathcal{U} h))$ 
    by (auto simp add: wait-d-def)
  have 2:  $\vdash (\Box f \wedge (\Box g \vee g \mathcal{U} h)) = ((\Box f \wedge \Box g) \vee (\Box f \wedge g \mathcal{U} h))$ 
    by auto
  have 3:  $\vdash (\Box f \wedge \Box g) = \Box(f \wedge g)$ 
    by (simp add: BoxAndBoxEqvBoxRule)
  have 4:  $\vdash (\Box f \wedge g \mathcal{U} h) \longrightarrow (f \wedge g) \mathcal{U} (f \wedge h)$ 
    by (simp add: UntilAlwaysAndDist)
  have 5:  $\vdash ((\Box f \wedge \Box g) \vee (\Box f \wedge g \mathcal{U} h)) \longrightarrow \Box(f \wedge g) \vee (f \wedge g) \mathcal{U} (f \wedge h)$ 
    using 3 4 by fastforce
  have 6:  $\vdash (\Box(f \wedge g) \vee (f \wedge g) \mathcal{U} (f \wedge h)) = (f \wedge g) \mathcal{W} (f \wedge h)$ 
    by (auto simp add: wait-d-def)
  show ?thesis
  using 1 5 6 by fastforce
qed

```

lemma *WaitDiamondOr*:

```

 $\vdash f \mathcal{W} (\Diamond g) = (\Box f \vee \Diamond g)$ 
proof -
  have 1:  $\vdash f \mathcal{W} (\Diamond g) = (\Box f \vee f \mathcal{U} (\Diamond g))$ 
    by (auto simp add: wait-d-def)
  have 2:  $\vdash f \mathcal{U} (\Diamond g) = \Diamond g$ 
    by (simp add: UntilAbsorpDiamond)
  show ?thesis using 1 2 Prop06 by blast

```

qed

lemma *WaitBoxImp:*

$\vdash f \mathcal{W} (\Box g) \longrightarrow \Box (f \mathcal{W} g)$

proof —

have 1: $\vdash f \mathcal{W} (\Box g) = (\Box f \vee f \mathcal{U} (\Box g))$

by (*auto simp add: wait-d-def*)

have 2: $\vdash \Box f = \Box (\Box f)$

by (*simp add: BoxEqvBoxBox*)

have 3: $\vdash f \mathcal{U} (\Box g) \longrightarrow \Box (f \mathcal{U} g)$

by (*simp add: UntilBoxImp*)

have 4: $\vdash (\Box f \vee f \mathcal{U} (\Box g)) \longrightarrow (\Box (\Box f) \vee \Box (f \mathcal{U} g))$

using 2 3 **by** *fastforce*

have 5: $\vdash \Box (\Box f) \longrightarrow \Box (\Box f \vee f \mathcal{U} g)$

by (*metis BoxImpBoxRule Prop08 UntilIdempotent UntilIntro int-simps(11) int-simps(25) inteq-reflection*)

have 6: $\vdash \Box (f \mathcal{U} g) \longrightarrow \Box (\Box f \vee f \mathcal{U} g)$

by (*metis BoxImpBoxRule UntilImpWait wait-d-def*)

have 7: $\vdash (\Box (\Box f) \vee \Box (f \mathcal{U} g)) \longrightarrow \Box (\Box f \vee f \mathcal{U} g)$

using 5 6 **by** *fastforce*

have 6: $\vdash \Box (\Box f \vee f \mathcal{U} g) = \Box (f \mathcal{W} g)$

by (*simp add: wait-d-def*)

show *?thesis*

by (*metis 4 7 lift-imp-trans wait-d-def*)

qed

lemma *WaitAbsorptionBox:*

$\vdash f \mathcal{W} (\Box f) = \Box f$

by (*metis Prop02 Prop11 UntilBoxEqvBox UntilImpWait inteq-reflection wait-d-def*)

lemma *BoxImpWait:*

$\vdash \Box f \longrightarrow f \mathcal{W} g$

by (*auto simp add: wait-d-def*)

lemma *WaitDistNext:*

$\vdash \bigcirc (f \mathcal{W} g) = (\bigcirc f) \mathcal{W} (\bigcirc g)$

— nitpick finds counterexample, does not hold because of finite intervals

oops

lemma *WnextAlwaysEqvAlwaysWnext:*

$\vdash \text{wnext} (\Box f) = \Box (\text{wnext } f)$

by (*metis (no-types, lifting) NextDiamondEqvDiamondNext always-d-def int-eq int-simps(4) wnext-d-def*)

lemma *WaitExpand:*

$\vdash f \mathcal{W} g = (g \vee (f \wedge \bigcirc (f \mathcal{W} g)))$

— nitpick finds counterexample, does not hold because of finite intervals

oops

lemma *WaitExpand*:

$\vdash f \mathcal{W} g = (g \vee (f \wedge \text{wnext}(f \mathcal{W} g)))$

proof –

have 1: $\vdash f \mathcal{W} g = (\Box f \vee f \mathcal{U} g)$

by (*simp add: wait-d-def*)

have 2: $\vdash \Box f = (f \wedge \text{wnext}(\Box f))$

by (*simp add: BoxEqvAndWnextBox*)

have 3: $\vdash f \mathcal{U} g = (g \vee (f \wedge \bigcirc (f \mathcal{U} g)))$

using *UntilNextUntil* **by** *blast*

have 4: $\vdash (f \wedge \text{wnext}(\Box f)) = (f \wedge (\text{empty} \vee \bigcirc (\Box f)))$

using 2 *BoxEqvAndEmptyOrNextBox* **by** *fastforce*

have 5: $\vdash \text{wnext}(f \mathcal{W} g) = (\text{empty} \vee \bigcirc (f \mathcal{W} g))$

using *WnextEqvEmptyOrNext* **by** *blast*

have 6: $\vdash (f \wedge (\text{empty} \vee \bigcirc (\Box f))) = ((f \wedge \text{empty}) \vee (f \wedge \bigcirc (\Box f)))$

by *auto*

have 7: $\vdash (((f \wedge \text{empty}) \vee (f \wedge \bigcirc (\Box f))) \vee$

$(g \vee (f \wedge \bigcirc (f \mathcal{U} g)))) =$

$(g \vee (f \wedge (\text{empty} \vee \bigcirc (\Box f) \vee \bigcirc (f \mathcal{U} g))))$

by *auto*

have 8: $\vdash (\bigcirc (\Box f) \vee \bigcirc (f \mathcal{U} g)) = \bigcirc (\Box f \vee f \mathcal{U} g)$

by (*metis ChopOrEqv Prop11 next-d-def*)

show *?thesis*

by (*metis 1 2 3 4 5 6 7 8 inteq-reflection*)

qed

lemma *WaitExclMid*:

$\vdash f \mathcal{W} g \vee f \mathcal{W} (\neg g)$

using *WaitExpand*

proof –

have 1: $\vdash f \mathcal{W} g = (g \vee f \wedge \text{wnext} (f \mathcal{W} g))$

by (*simp add: WaitExpand*)

have 2: $\vdash f \mathcal{W} (\neg g) = ((\neg g) \vee f \wedge \text{wnext} (f \mathcal{W} (\neg g)))$

by (*simp add: WaitExpand*)

have 3: $\vdash (f \mathcal{W} g \vee f \mathcal{W} (\neg g)) =$

$((g \vee f \wedge \text{wnext} (f \mathcal{W} g)) \vee ((\neg g) \vee f \wedge \text{wnext} (f \mathcal{W} (\neg g))))$

using 1 2 **by** *fastforce*

from 3 **show** *?thesis* **by** *fastforce*

qed

lemma *WaitleftZero*:

$\vdash \# \text{True} \mathcal{W} g$

by (*meson BoxGen BoxImpWait MP TrueW*)

lemma *WaitLeftDistOr*:

$\vdash f \mathcal{W} (g \vee h) = (f \mathcal{W} g \vee f \mathcal{W} h)$

proof –

have 1: $\vdash f \mathcal{W} (g \vee h) = (\Box f \vee f \mathcal{U} (g \vee h))$

by (*simp add: wait-d-def*)

have 2: $\vdash (f \mathcal{W} g \vee f \mathcal{W} h) = ((\Box f \vee f \mathcal{U} g) \vee (\Box f \vee f \mathcal{U} h))$

by (simp add: wait-d-def)
 have 3: $\vdash f \mathcal{U} (g \vee h) = (f \mathcal{U} g \vee f \mathcal{U} h)$
 by (simp add: UntilOrDist)
 from 1 2 3 show ?thesis by fastforce
 qed

lemma *WaitRightDistOr*:

$\vdash f \mathcal{W} h \vee g \mathcal{W} h \longrightarrow (f \vee g) \mathcal{W} h$
proof –
 have 0: $\vdash \Box g \longrightarrow \Box (f \vee g)$
 by (simp add: BoxImpBoxRule intI)
 have 1: $\vdash \Box f \longrightarrow \Box (f \vee g)$
 by (simp add: BoxImpBoxRule intI)
 have 11: $\vdash \Box f \vee \Box g \longrightarrow \Box (f \vee g)$
 using 0 1 Prop02 by blast
 have 2: $\vdash f \mathcal{W} h = (\Box f \vee f \mathcal{U} h)$
 by (simp add: wait-d-def)
 have 3: $\vdash (f \vee g) \mathcal{W} h = (\Box (f \vee g) \vee (f \vee g) \mathcal{U} h)$
 by (simp add: wait-d-def)
 have 4: $\vdash g \mathcal{W} h = (\Box g \vee g \mathcal{U} h)$
 by (simp add: wait-d-def)
 have 5: $\vdash f \mathcal{U} h \vee g \mathcal{U} h \longrightarrow (f \vee g) \mathcal{U} h$
 using UntilRightDistOr by simp
 have 6: $\vdash (f \mathcal{W} h \vee g \mathcal{W} h) = ((\Box f \vee \Box g) \vee (f \mathcal{U} h \vee g \mathcal{U} h))$
 using 2 4 by fastforce
 from 11 5 6 3 show ?thesis
 by (meson BoxImpWait Prop02 Prop11 UntilImpWait lift-imp-trans)
 qed

lemma *WaitOrRule*:

$\vdash f \mathcal{W} g = (f \vee g) \mathcal{W} g$
proof –
 have 1: $\vdash f \mathcal{W} g \longrightarrow (f \vee g) \mathcal{W} g$
 by (metis (no-types, lifting) Prop03 Prop10 UntilAbsorp-a WaitNotDistUntil int-iffD1 int-simps(14) int-simps(32) int-simps(33) inteq-reflection)
 have 2: $\vdash (f \vee g) \mathcal{W} g \longrightarrow f \mathcal{W} g$
 by (metis (no-types, lifting) Prop03 Prop10 WaitNotDistUntil int-iffD2 int-simps(14) int-simps(32) int-simps(33) inteq-reflection)
 from 1 2 show ?thesis by fastforce
 qed

lemma *UntilOrRule*:

$\vdash f \mathcal{U} g = (f \vee g) \mathcal{U} g$
 by (metis UntilWait WaitOrRule inteq-reflection)

lemma *WaitRule*:

$\vdash (\neg f) \mathcal{W} f$
 by (metis BoxGen BoxImpWait MP WaitOrRule int-eq-true int-simps(29) inteq-reflection)

lemma *UntilRule*:

$\vdash (\neg f) \mathcal{U} f = \Diamond f$

using *DiamondEqvTrueUntil UntilOrRule inteq-reflection* **by** *fastforce*

lemma *WaitImpRule*:

$\vdash (f \longrightarrow g) \mathcal{W} f$

proof –

have 1: $\vdash (f \longrightarrow g) \mathcal{W} f = ((f \longrightarrow g) \vee f) \mathcal{W} f$

by (*simp add: WaitOrRule*)

have 2: $\vdash (f \longrightarrow g) \vee f$

by *auto*

have 3: $\vdash ((f \longrightarrow g) \vee f) \mathcal{W} f = \#True \mathcal{W} f$

by (*metis 1 2 int-eq-true inteq-reflection*)

show *?thesis*

using 1 3 *WaitleftZero* **by** *fastforce*

qed

lemma *DiamondUntilImpRule*:

$\vdash \Diamond f \longrightarrow (f \longrightarrow g) \mathcal{U} f$

using *UntilWait WaitImpRule* **by** *fastforce*

lemma *WaitNotDist*:

$\vdash (\neg (f \mathcal{W} g)) = (f \wedge \neg g) \mathcal{U} (\neg f \wedge \neg g)$

proof –

have 1: $\vdash (\neg (f \mathcal{W} g)) = (\neg g) \mathcal{U} (\neg f \wedge \neg g)$

using *WaitNotDistUntil* **by** *blast*

have 2: $\vdash (\neg g) \mathcal{U} (\neg f \wedge \neg g) = (\neg g \wedge \neg(\neg f \wedge \neg g)) \mathcal{U} (\neg f \wedge \neg g)$

using *UntilAndRule* **by** *blast*

have 3: $\vdash (\neg g \wedge \neg(\neg f \wedge \neg g)) = (f \wedge \neg g)$

by *auto*

have 4: $\vdash (\neg g \wedge \neg(\neg f \wedge \neg g)) \mathcal{U} (\neg f \wedge \neg g) = (f \wedge \neg g) \mathcal{U} (\neg f \wedge \neg g)$

using 3 *inteq-reflection* **by** *force*

show *?thesis* **using** 1 2 4 **by** *fastforce*

qed

lemma *UntilNotDist*:

$\vdash (\neg (f \mathcal{U} g)) = (f \wedge \neg g) \mathcal{W} (\neg f \wedge \neg g)$

proof –

have 1: $\vdash (\neg (f \mathcal{U} g)) = (\neg g) \mathcal{W} (\neg f \wedge \neg g)$

using *UntilNotDistWait* **by** *blast*

have 2: $\vdash (\neg g) \mathcal{W} (\neg f \wedge \neg g) = (\neg g \wedge \neg(\neg f \wedge \neg g)) \mathcal{W} (\neg f \wedge \neg g)$

by (*simp add: WaitAndRule*)

have 3: $\vdash (\neg g \wedge \neg(\neg f \wedge \neg g)) = (f \wedge \neg g)$

by *auto*

have 4: $\vdash (\neg g \wedge \neg(\neg f \wedge \neg g)) \mathcal{W} (\neg f \wedge \neg g) = (f \wedge \neg g) \mathcal{W} (\neg f \wedge \neg g)$

using 3 *inteq-reflection* **by** *force*

show *?thesis* **using** 1 2 4 **by** *fastforce*

qed

lemma *UntilDuala*:

$\vdash (\neg (\neg f) \mathcal{U} (\neg g))) = g \mathcal{W} (f \wedge g)$

proof –

have 1: $\vdash (\neg (\neg f) \mathcal{U} (\neg g))) = (\neg f \wedge \neg (\neg g)) \mathcal{W} (\neg(\neg f) \wedge \neg (\neg g))$

using *UntilNotDist* **by** *blast*

have 2: $\vdash (\neg f \wedge g) \mathcal{W} (f \wedge g) = g \mathcal{W} (f \wedge g)$

using 1 *UntilNotDistWait int-eq* **by** *fastforce*

show *?thesis*

using 1 2 **by** *fastforce*

qed

lemma *UntilDualb*:

$\vdash (\neg (\neg f) \mathcal{U} (\neg g))) = (\neg f \wedge g) \mathcal{W} (f \wedge g)$

proof –

have 1: $\vdash (\neg (\neg f) \mathcal{U} (\neg g))) = (\neg f \wedge \neg (\neg g)) \mathcal{W} (\neg(\neg f) \wedge \neg (\neg g))$

using *UntilNotDist* **by** *blast*

show *?thesis*

using 1 **by** *auto*

qed

lemma *WaitDuala*:

$\vdash (\neg (\neg f) \mathcal{W} (\neg g))) = g \mathcal{U} (f \wedge g)$

proof –

have 1: $\vdash (\neg (\neg f) \mathcal{W} (\neg g))) = (\neg f \wedge \neg (\neg g)) \mathcal{U} (\neg(\neg f) \wedge \neg (\neg g))$

using *WaitNotDist* **by** *blast*

have 2: $\vdash (\neg f \wedge g) \mathcal{U} (f \wedge g) = g \mathcal{U} (f \wedge g)$

using 1 *WaitNotDistUntil int-eq* **by** *fastforce*

show *?thesis*

using 1 2 **by** *fastforce*

qed

lemma *WaitDualb*:

$\vdash (\neg (\neg f) \mathcal{W} (\neg g))) = (\neg f \wedge g) \mathcal{U} (f \wedge g)$

proof –

have 1: $\vdash (\neg (\neg f) \mathcal{W} (\neg g))) = (\neg f \wedge \neg (\neg g)) \mathcal{U} (\neg(\neg f) \wedge \neg (\neg g))$

using *WaitNotDist* **by** *blast*

show *?thesis* **using** 1 **by** *auto*

qed

lemma *WaitIdempotent*:

$\vdash f \mathcal{W} f = f$

by (*meson BoxElim Prop02 Prop12 UntilIdempotent UntilImpWait UntilIntro WaitUntilb int-iffD1 int-iffI lift-imp-trans*)

lemma *WaitRightZero*:

$\vdash f \mathcal{W} \# \text{True}$

by (*meson MP TrueW UntilImpWait UntilIntro*)

lemma *WaitLeftIdentity*:

$\vdash \#False \mathcal{W} g = g$
by (*metis* (*no-types*, *lifting*) *UntilAbsorp-c* *UntilNotDistWait* *WaitDuala* *WaitIdempotent* *WaitSRelease* *int-eq* *int-simps*(17) *int-simps*(3) *srelease-d-def*)

lemma *WaitImpOr*:

$\vdash f \mathcal{W} g \longrightarrow f \vee g$

by (*metis* *Prop03* *WaitIdempotent* *WaitLeftDistOr* *WaitOrRule* *inteq-reflection*)

lemma *BoxOrImpWait*:

$\vdash \Box(f \vee g) \longrightarrow f \mathcal{W} g$

using *BoxImpWait* *WaitOrRule* **by** *fastforce*

lemma *BoxImpImpWait*:

$\vdash \Box(\neg g \longrightarrow f) \longrightarrow f \mathcal{W} g$

proof –

have 1: $\vdash (\neg g \longrightarrow f) = (f \vee g)$

by *auto*

have 2: $\vdash \Box(\neg g \longrightarrow f) = \Box(f \vee g)$

using 1 *BoxEqvBox* **by** *blast*

show *?thesis* **using** 2 *BoxOrImpWait* **by** *fastforce*

qed

lemma *WaitInsertion*:

$\vdash g \longrightarrow f \mathcal{W} g$

by (*simp* *add*: *Prop05* *UntilIntro* *wait-d-def*)

lemma *WaitFrameNext*:

$\vdash \Box f \longrightarrow (\bigcirc g \longrightarrow \bigcirc (f \mathcal{W} g))$

by (*simp* *add*: *NextImpNext* *Prop01* *Prop05* *Prop09* *WaitInsertion*)

lemma *WaitFrameDiamond*:

$\vdash \Box f \longrightarrow (\Diamond g \longrightarrow \Diamond (f \mathcal{W} g))$

by (*simp* *add*: *DiamondImpDiamond* *Prop01* *Prop05* *Prop09* *WaitInsertion*)

lemma *WaitFrameBox*:

$\vdash \Box f \longrightarrow (\Box g \longrightarrow \Box (f \mathcal{W} g))$

by (*meson* *BoxAndBoxImpBoxRule* *OrImpUntil* *Prop09* *UntilImpWait* *lift-imp-trans*)

lemma *WaitInductiona*:

$\vdash \Box (f \longrightarrow (\bigcirc f \wedge g) \vee h) \longrightarrow (f \longrightarrow g \mathcal{W} h)$

by (*simp* *add*: *UntilInduction-a* *wait-d-def*)

lemma *WaitInductionb*:

$\vdash \Box (f \longrightarrow \bigcirc f \vee g) \longrightarrow (f \longrightarrow f \mathcal{W} g)$

by (*simp* *add*: *UntilInduction-b* *wait-d-def*)

lemma *WaitInductionc*:

$\vdash \Box(f \longrightarrow \bigcirc f) \longrightarrow (f \longrightarrow f \mathcal{W} g)$

proof –

have 1: $\vdash (f \longrightarrow \bigcirc f) \longrightarrow (f \longrightarrow \text{wnext } f)$
unfolding *wnext-d-def* **using** *NextImpNotNextNot[of f]* **by** *auto*
have 2: $\vdash \Box(f \longrightarrow \bigcirc f) \longrightarrow \Box(f \longrightarrow \text{wnext } f)$
using 1 *BoxImpBoxRule* **by** *blast*
show ?thesis **by** (meson 2 *BoxImpWait BoxInduct Prop09 lift-imp-trans*)
qed

lemma *WaitInductiond*:
 $\vdash \Box(f \longrightarrow g \wedge \bigcirc f) \longrightarrow (f \longrightarrow f \mathcal{W} g)$
proof –
have 1: $\vdash (f \longrightarrow g \wedge \bigcirc f) \longrightarrow (f \longrightarrow \text{wnext } f)$
unfolding *wnext-d-def* **using** *NextImpNotNextNot[of f]* **by** *auto*
have 2: $\vdash \Box(f \longrightarrow g \wedge \bigcirc f) \longrightarrow \Box(f \longrightarrow \text{wnext } f)$
using 1 *BoxImpBoxRule* **by** *blast*
show ?thesis **by** (meson 2 *BoxImpWait BoxInduct Prop09 lift-imp-trans*)
qed

lemma *WaitAbsorptiona*:
 $\vdash (f \vee f \mathcal{W} g) = (f \vee g)$
proof –
have 1: $\vdash (f \vee f \mathcal{W} g) \longrightarrow (f \vee g)$
using *WaitImpOr* **by** *fastforce*
have 2: $\vdash f \vee g \longrightarrow f \vee f \mathcal{W} g$
using *WaitInsertion* **by** *fastforce*
show ?thesis **using** 1 2 *int-iffI* **by** *blast*
qed

lemma *WaitAbsorptionb*:
 $\vdash (f \mathcal{W} g \vee g) = f \mathcal{W} g$
using *WaitInsertion[of g f]* **by** *auto*

lemma *WaitAbsorptionc*:
 $\vdash (f \mathcal{W} g \wedge g) = g$
using *WaitInsertion* **by** *fastforce*

lemma *WaitAbsorptiond*:
 $\vdash (f \mathcal{W} g \wedge (f \vee g)) = f \mathcal{W} g$
by (meson *Prop10 Prop11 WaitImpOr*)

lemma *WaitAbsorptione*:
 $\vdash (f \mathcal{W} g \vee (f \wedge g)) = f \mathcal{W} g$
by (metis (no-types, lifting) *BoxEqvBoxBox UntilAbsorp-a UntilAbsorp-d WaitAbsorptiona WaitLeftDistOr WaitOrRule inteq-reflection wait-d-def*)

lemma *WaitLeftAbsorption*:
 $\vdash f \mathcal{W} (f \mathcal{W} g) = f \mathcal{W} g$
by (metis (no-types, lifting) *BoxEqvBoxBox UntilUntil WaitAbsorptionBox WaitAbsorptiona WaitLeftDistOr inteq-reflection wait-d-def*)

lemma *WaitRightAbsorption*:

$\vdash (f \mathcal{W} g) \mathcal{W} g = f \mathcal{W} g$
by (*metis* (*no-types*, *lifting*) *LeftUntilAbsorp Prop10 WaitInsertion WaitNotDistUntil int-iffD1*
int-iffI int-simps(32) inteq-reflection)

lemma *WaitBox*:

$\vdash \Box f = f \mathcal{W} \#False$
by (*metis* (*no-types*, *lifting*) *BoxGen DiamondNotEqvNotBox UntilAbsorpAndDiamond UntilAbsorp-c*
int-eq-true int-simps(2) int-simps(25) inteq-reflection wait-d-def)

lemma *WaitDiamond*:

$\vdash \Diamond f = (\neg(\neg f) \mathcal{W} \#False)$
using *DiamondNotEqvNotBox WaitBox* **by** *fastforce*

lemma *WaitImp*:

$\vdash f \mathcal{W} g \longrightarrow \Box f \vee \Diamond g$
by (*metis Prop08 UntilImpDiamond WaitAbsorptionb WaitImpOr WaitRightAbsorption int-eq wait-d-def*)

lemma *WaitRightUntilAbsorption*:

$\vdash f \mathcal{W} (f \mathcal{U} g) = f \mathcal{W} g$
by (*metis UntilUntil WaitOrRule inteq-reflection wait-d-def*)

lemma *WaitLeftUntilAbsorption*:

$\vdash (f \mathcal{U} g) \mathcal{W} g = f \mathcal{U} g$
by (*metis Prop11 RightUntilAbsorp UntilAbsorp-b UntilImpWait WaitImpOr inteq-reflection*)

lemma *UntilRightWaitAbsorption*:

$\vdash f \mathcal{U} (f \mathcal{W} g) = f \mathcal{W} g$
using *UntilImpWait UntilIntro WaitLeftAbsorption* **by** *fastforce*

lemma *UntilLeftWaitAbsorption*:

$\vdash (f \mathcal{W} g) \mathcal{U} g = f \mathcal{U} g$
by (*metis UntilWait WaitRightAbsorption inteq-reflection*)

lemma *WaitDiamondAbsorption*:

$\vdash (\Diamond g) \mathcal{W} g = \Diamond g$
by (*metis DiamondEqvTrueUntil WaitLeftUntilAbsorption inteq-reflection*)

lemma *WaitAndBoxAbsorption*:

$\vdash (\Box f \wedge f \mathcal{W} g) = \Box f$
by (*meson BoxImpWait NotDiamondNotEqvBox Prop04 Prop10*)

lemma *WaitOrBoxAbsorption*:

$\vdash (\Box f \vee f \mathcal{W} g) = f \mathcal{W} g$
by (*metis UntilRightWaitAbsorption WaitLeftAbsorption inteq-reflection wait-d-def*)

lemma *WaitAndBoxImpBox*:

$\vdash f \mathcal{W} g \wedge \Box(\neg g) \longrightarrow \Box f$
by (*metis* (*no-types*, *opaque-lifting*) *Prop02 Prop07 Prop08 UntilIdempotent UntilImpDiamond*
always-d-def int-simps(25) int-simps(4) inteq-reflection wait-d-def Prop03)

lemma *BoxImpUntilOrBox*:

$\vdash \Box f \longrightarrow f \mathcal{U} g \vee \Box (\neg g)$

proof –

have 1: $\vdash (\Box f \longrightarrow f \mathcal{U} g \vee \Box (\neg g)) =$
 $((\Box f \wedge \Diamond g) \longrightarrow f \mathcal{U} g)$

by (*auto simp add: always-d-def*)

have 2: $\vdash (\Box f \wedge \Diamond g) \longrightarrow f \mathcal{U} g$

using *UntilAndImp* **by** *blast*

show *?thesis*

using 1 2 **by** *fastforce*

qed

lemma *NotBoxAndWaitImpDiamond*:

$\vdash \neg(\Box f) \wedge f \mathcal{W} g \longrightarrow \Diamond g$

using *WaitImp* **by** *fastforce*

lemma *DiamondImpNotBoxOrUntil*:

$\vdash \Diamond g \longrightarrow \neg(\Box f) \vee f \mathcal{U} g$

proof –

have 1: $\vdash \Diamond g \wedge \Box f \longrightarrow f \mathcal{U} g$

using *UntilAndImp* **by** *fastforce*

show *?thesis* **using** 1 **by** *auto*

qed

lemma *WaitRightDistImp*:

$\vdash (f \longrightarrow g) \mathcal{W} h \longrightarrow (f \mathcal{W} h \longrightarrow g \mathcal{W} h)$

proof –

have 1: $\vdash \Box(f \longrightarrow g) \wedge \Box f \longrightarrow \Box g \vee g \mathcal{U} h$

by (*simp add: itl-defs intI*)

have 2: $\vdash \Box(f \longrightarrow g) \wedge f \mathcal{U} h \longrightarrow ((f \longrightarrow g) \wedge f) \mathcal{U} ((f \longrightarrow g) \wedge h)$

using *UntilAlwaysAndDist*[of *LIFT*(*f* \longrightarrow *g*) *f* *h*] **by** *auto*

have 3: $\vdash (f \longrightarrow g) \wedge f \longrightarrow g$

by *auto*

have 4: $\vdash (f \longrightarrow g) \wedge h \longrightarrow h$

by *auto*

have 5: $\vdash ((f \longrightarrow g) \wedge f) \mathcal{U} ((f \longrightarrow g) \wedge h) \longrightarrow \Box g \vee g \mathcal{U} h$

by (*simp add: 3 4 Prop05 UntilImpUntil*)

have 6: $\vdash \Box(f \longrightarrow g) \wedge f \mathcal{U} h \longrightarrow \Box g \vee g \mathcal{U} h$

using 2 5 *lift-imp-trans* **by** *blast*

have 7: $\vdash \Box(f \longrightarrow g) \longrightarrow \Box f \vee f \mathcal{U} h \longrightarrow \Box g \vee g \mathcal{U} h$

by (*simp add: 1 6 Prop09 Prop12*)

have 8: $\vdash \Box f \wedge (f \longrightarrow g) \mathcal{U} h \longrightarrow (f \wedge (f \longrightarrow g)) \mathcal{U} (f \wedge h)$

by (*simp add: UntilAlwaysAndDist*)

have 9: $\vdash f \wedge (f \longrightarrow g) \longrightarrow g$

by *auto*

have 10: $\vdash f \wedge h \longrightarrow h$

by *auto*

have 11: $\vdash (f \wedge (f \longrightarrow g)) \mathcal{U} (f \wedge h) \longrightarrow \Box g \vee g \mathcal{U} h$

by (*simp add: 10 9 Prop05 UntilImpUntil*)

have 12: $\vdash (f \longrightarrow g) \mathcal{U} h \wedge \Box f \longrightarrow \Box g \vee g \mathcal{U} h$

using 8 11 by fastforce
 have 13: $\vdash (f \longrightarrow g) \mathcal{U} h \wedge f \mathcal{U} h \longrightarrow \Box g \vee g \mathcal{U} h$
 by (metis 3 Prop05 UntilAndDist UntilImpUntil UntilIntro UntilUntil inteq-reflection)
 have 14: $\vdash (f \longrightarrow g) \mathcal{U} h \longrightarrow \Box f \vee f \mathcal{U} h \longrightarrow \Box g \vee g \mathcal{U} h$
 by (simp add: 12 13 Prop09 Prop12)
 show ?thesis unfolding wait-d-def using 7 14 Prop02 by blast
 qed

lemma WaitLeftMono:
 $\vdash \Box (f \longrightarrow g) \longrightarrow (f \mathcal{W} h \longrightarrow g \mathcal{W} h)$
 by (meson BoxImpWait WaitRightDistImp lift-imp-trans)

lemma WaitRightMono:
 $\vdash \Box (f \longrightarrow g) \longrightarrow (h \mathcal{W} f \longrightarrow h \mathcal{W} g)$
proof –
 have 1: $\vdash \Box (f \longrightarrow g) \longrightarrow (h \mathcal{U} f \longrightarrow h \mathcal{U} g)$
 by (simp add: UntilRightMono)
 have 2: $\vdash \Box (f \longrightarrow g) \longrightarrow (\Box h \longrightarrow \Box h \vee h \mathcal{U} g)$
 by auto
 have 3: $\vdash \Box (f \longrightarrow g) \longrightarrow (h \mathcal{U} f \longrightarrow \Box h \vee h \mathcal{U} g)$
 using 1 by auto
 have 4: $\vdash \Box (f \longrightarrow g) \longrightarrow (\Box h \vee h \mathcal{U} f \longrightarrow \Box h \vee h \mathcal{U} g)$
 using 2 3 by fastforce
 from 4 show ?thesis by (simp add: wait-d-def)
 qed

lemma WaitStrengthen:
 $\vdash \Box((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (f \mathcal{W} g \longrightarrow h \mathcal{W} i)$
proof –
 have 1: $\vdash \Box((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (f \mathcal{W} g \longrightarrow h \mathcal{W} g)$
 by (meson BoxAndBoxEqvBoxRule Prop01 Prop05 Prop11 WaitLeftMono lift-and-com lift-imp-trans)
 have 2: $\vdash \Box((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (h \mathcal{W} g \longrightarrow h \mathcal{W} i)$
 by (meson BoxElim BoxImpBoxBox BoxImpBoxRule Prop12 WaitRightMono lift-imp-trans)
 have 3: $\vdash \Box((f \longrightarrow h) \wedge (g \longrightarrow i)) \longrightarrow (f \mathcal{W} g \longrightarrow h \mathcal{W} g) \wedge (h \mathcal{W} g \longrightarrow h \mathcal{W} i)$
 using 1 2 by fastforce
 have 4: $\vdash (f \mathcal{W} g \longrightarrow h \mathcal{W} g) \wedge (h \mathcal{W} g \longrightarrow h \mathcal{W} i) \longrightarrow (f \mathcal{W} g \longrightarrow h \mathcal{W} i)$
 by auto
 from 3 4 show ?thesis by auto
 qed

lemma WaitCatRule:
 $\vdash \Box((f \longrightarrow g \mathcal{W} h) \wedge (h \longrightarrow g \mathcal{W} i)) \longrightarrow (f \longrightarrow g \mathcal{W} i)$
proof –
 have 1: $\vdash \Box((f \longrightarrow g \mathcal{W} h) \wedge (h \longrightarrow g \mathcal{W} i)) \longrightarrow \Box(f \longrightarrow g \mathcal{W} h)$
 by (metis BoxElim BoxEqvBoxBox BoxImpBoxRule Prop12 inteq-reflection)
 have 2: $\vdash \Box((f \longrightarrow g \mathcal{W} h) \wedge (h \longrightarrow g \mathcal{W} i)) \longrightarrow \Box(h \longrightarrow g \mathcal{W} i)$
 by (metis BoxElim BoxEqvBoxBox BoxImpBoxRule Prop12 inteq-reflection)
 have 3: $\vdash \Box(h \longrightarrow g \mathcal{W} i) \longrightarrow \Box(g \mathcal{W} h \longrightarrow g \mathcal{W} (g \mathcal{W} i))$
 by (metis BoxEqvBoxBox BoxImpBoxRule WaitRightMono inteq-reflection)

have 4: $\vdash \Box (g \mathcal{W} h \longrightarrow g \mathcal{W} (g \mathcal{W} i)) \longrightarrow \Box (g \mathcal{W} h \longrightarrow g \mathcal{W} i)$
by (*metis BoxEqvBoxBox WaitLeftAbsorption int-iffD1 inteq-reflection*)
have 5: $\vdash \Box (f \longrightarrow g \mathcal{W} h) \longrightarrow (f \longrightarrow g \mathcal{W} h)$
by (*simp add: BoxElim*)
have 6: $\vdash \Box (g \mathcal{W} h \longrightarrow g \mathcal{W} i) \longrightarrow (g \mathcal{W} h \longrightarrow g \mathcal{W} i)$
by (*simp add: BoxElim*)
have 7: $\vdash (f \longrightarrow g \mathcal{W} h) \wedge (g \mathcal{W} h \longrightarrow g \mathcal{W} i) \longrightarrow (f \longrightarrow g \mathcal{W} i)$
by *auto*
have 8: $\vdash \Box ((f \longrightarrow g \mathcal{W} h) \wedge (h \longrightarrow g \mathcal{W} i)) \longrightarrow$
 $(f \longrightarrow g \mathcal{W} h) \wedge (g \mathcal{W} h \longrightarrow g \mathcal{W} i)$
using 1 2 3 4 5 6 **by** *fastforce*
from 7 8 **show** ?thesis **by** *auto*
qed

lemma *LeftUntilWaitImp*:
 $\vdash (f \mathcal{U} g) \mathcal{W} h \longrightarrow (f \mathcal{W} g) \mathcal{W} h$
by (*meson BoxGen MP UntilImpWait WaitLeftMono*)

lemma *RightWaitUntilImp*:
 $\vdash f \mathcal{W} (g \mathcal{U} h) \longrightarrow f \mathcal{W} (g \mathcal{W} h)$
by (*meson BoxGen MP UntilImpWait WaitRightMono*)

lemma *RightUntilUntilImp*:
 $\vdash f \mathcal{U} (g \mathcal{U} h) \longrightarrow f \mathcal{U} (g \mathcal{W} h)$
by (*meson BoxGen MP UntilImpWait UntilRightMono*)

lemma *LeftUntilUntilImp*:
 $\vdash (f \mathcal{U} g) \mathcal{U} h \longrightarrow (f \mathcal{W} g) \mathcal{U} h$
by (*simp add: UntilImpUntil UntilImpWait*)

lemma *LeftUntilOrStrengthen*:
 $\vdash (f \mathcal{U} g) \mathcal{U} h \longrightarrow (f \vee g) \mathcal{U} h$
by (*simp add: UntilImpOr UntilImpUntil*)

lemma *LeftWaitOrStrengthen*:
 $\vdash (f \mathcal{W} g) \mathcal{W} h \longrightarrow (f \vee g) \mathcal{W} h$
by (*meson BoxGen MP WaitImpOr WaitLeftMono*)

lemma *RightWaitOrStrengthen*:
 $\vdash f \mathcal{W} (g \mathcal{W} h) \longrightarrow f \mathcal{W} (g \vee h)$
by (*meson BoxGen MP WaitImpOr WaitRightMono*)

lemma *BoxImpBoxOr*:
 $\vdash \Box f \longrightarrow \Box (f \vee g)$
by (*metis BoxEqvBoxBox BoxImpBoxRule BoxImpWait Prop12 WaitAbsorptiond inteq-reflection*)

lemma *RightWaitOrOrder*:
 $\vdash f \mathcal{W} (g \mathcal{W} h) \longrightarrow (f \vee g) \mathcal{W} h$

proof –

have 1: $\vdash f \mathcal{W} (g \mathcal{W} h) = (\Box f \vee f \mathcal{U} (\Box g \vee g \mathcal{U} h))$
by (*simp add: wait-d-def*)
have 2: $\vdash (f \vee g) \mathcal{W} h = (\Box (f \vee g) \vee (f \vee g) \mathcal{U} h)$
by (*simp add: wait-d-def*)
have 3: $\vdash \Box f \longrightarrow (\Box (f \vee g) \vee (f \vee g) \mathcal{U} h)$
using *BoxImpBoxOr* **by** *fastforce*
have 4: $\vdash f \mathcal{U} (\Box g \vee g \mathcal{U} h) = (f \mathcal{U} (\Box g) \vee f \mathcal{U} (g \mathcal{U} h))$
using *UntilOrDist* **by** *blast*
have 5: $\vdash f \mathcal{U} (g \mathcal{U} h) \longrightarrow (\Box (f \vee g) \vee (f \vee g) \mathcal{U} h)$
by (*simp add: Prop05 UntilRightOr*)
have 6: $\vdash f \mathcal{U} (\Box g) \longrightarrow (\Box (f \vee g) \vee (f \vee g) \mathcal{U} h)$
by (*metis BoxImpBoxRule BoxImpWait UntilBoxImp UntilImpOr lift-imp-trans wait-d-def*)
have 7: $\vdash (f \mathcal{U} (\Box g) \vee f \mathcal{U} (g \mathcal{U} h)) \longrightarrow (\Box (f \vee g) \vee (f \vee g) \mathcal{U} h)$
using 5 6 **by** *fastforce*
show *?thesis*
using 1 2 3 4 7 **by** *fastforce*
qed

lemma *RightWaitAndOrder*:

$\vdash f \mathcal{W} (g \wedge h) \longrightarrow f \mathcal{W} (g \mathcal{W} h)$

by (*metis Prop03 WaitAbsorption WaitLeftDistOr inteq-reflection*)

lemma *UntilOrder*:

$\vdash ((\neg f) \mathcal{U} g \vee (\neg g) \mathcal{U} f) = \Diamond(f \vee g)$

proof –

have 1: $\vdash ((\neg f) \mathcal{U} g \vee (\neg g) \mathcal{U} f) \longrightarrow \Diamond(f \vee g)$
using *UntilAbsorpAndDiamond UntilOrDist* **by** *fastforce*
have 2: $\vdash \Diamond(f \vee g) = \#True \mathcal{U} (f \vee g)$
by (*metis DiamondEqvTrueUntil*)
have 3: $\vdash \#True \mathcal{U} (f \vee g) = (\neg (f \vee g)) \mathcal{U} (f \vee g)$
using 2 *UntilRule* **by** *fastforce*
have 4: $\vdash (\neg (f \vee g)) \mathcal{U} (f \vee g) = (\neg f \wedge \neg g) \mathcal{U} (f \vee g)$
by (*metis UntilAbsorp-c int-eq int-simps(14) int-simps(33)*)
have 5: $\vdash (\neg f \wedge \neg g) \mathcal{U} (f \vee g) \longrightarrow (\neg f \wedge \neg g) \mathcal{U} f \vee (\neg f \wedge \neg g) \mathcal{U} g$
by (*simp add: UntilOrDist int-iffD1*)
have 6: $\vdash (\neg f \wedge \neg g) \mathcal{U} f \longrightarrow (\neg g) \mathcal{U} f$
by (*metis UntilAndRule int-iffD2 inteq-reflection lift-and-com*)
have 7: $\vdash (\neg f \wedge \neg g) \mathcal{U} g \longrightarrow (\neg f) \mathcal{U} g$
by (*metis UntilAndRule int-iffD2 inteq-reflection*)
have 8: $\vdash (\neg f \wedge \neg g) \mathcal{U} f \vee (\neg f \wedge \neg g) \mathcal{U} g \longrightarrow (\neg f) \mathcal{U} g \vee (\neg g) \mathcal{U} f$
using 6 7 **by** *fastforce*
have 9: $\vdash \Diamond(f \vee g) \longrightarrow (\neg f) \mathcal{U} g \vee (\neg g) \mathcal{U} f$
using 2 3 4 5 8 **by** *fastforce*
show *?thesis* **using** 1 9 **by** *fastforce*
qed

lemma *WaitOrder*:

$\vdash (\neg f) \mathcal{W} g \vee (\neg g) \mathcal{W} f$


```

proof –
  have 1:  $\vdash (\neg f) \mathcal{W} g = (\Box (\neg f) \vee (\neg f) \mathcal{U} g)$ 
    by (simp add: wait-d-def)
  have 2:  $\vdash (\neg g) \mathcal{W} f = (\Box (\neg g) \vee (\neg g) \mathcal{U} f)$ 
    by (simp add: wait-d-def)
  have 3:  $\vdash ((\Box (\neg f) \vee (\neg f) \mathcal{U} g) \vee (\Box (\neg g) \vee (\neg g) \mathcal{U} f)) =$ 
     $(\Box (\neg f) \vee \Box (\neg g)) \vee ((\neg f) \mathcal{U} g \vee (\neg g) \mathcal{U} f))$ 
    by auto
  have 4:  $\vdash ((\neg f) \mathcal{U} g \vee (\neg g) \mathcal{U} f) = \Diamond(f \vee g)$ 
    using UntilOrder by blast
  have 5:  $\vdash (\Box (\neg f) \vee \Box (\neg g)) = (\neg (\Diamond f) \vee \neg (\Diamond g))$ 
    by (simp add: always-d-def)
  have 6:  $\vdash \Diamond(f \vee g) = (\Diamond f \vee \Diamond g)$ 
    by (simp add: ChopOrEqv sometimes-d-def)
  have 7:  $\vdash (\Box (\neg f) \vee \Box (\neg g)) \vee \Diamond(f \vee g)$ 
    using 5 6 by fastforce
  show ?thesis
  using 1 2 4 7 by fastforce
qed

```

lemma *WaitImpOrder*:

$\vdash f \mathcal{W} g \wedge (\neg g) \mathcal{W} h \longrightarrow f \mathcal{W} h$

proof –

```

  have 1:  $\vdash f \mathcal{W} g = (\Box f \vee f \mathcal{U} g)$ 
    by (simp add: wait-d-def)
  have 2:  $\vdash f \mathcal{W} h = (\Box f \vee f \mathcal{U} h)$ 
    by (simp add: wait-d-def)
  have 3:  $\vdash (\neg g) \mathcal{W} h = (\Box (\neg g) \vee (\neg g) \mathcal{U} h)$ 
    by (simp add: wait-d-def)
  have 4:  $\vdash \Box f \wedge (\Box (\neg g) \vee (\neg g) \mathcal{U} h) \longrightarrow (\Box f \vee f \mathcal{U} h)$ 
    by auto
  have 5:  $\vdash (\Box f \vee f \mathcal{U} g) \wedge \Box (\neg g) \longrightarrow (\Box f \vee f \mathcal{U} h)$ 
    using 1 WaitAndBoxImpBox by fastforce
  have 6:  $\vdash f \mathcal{U} g \wedge (\neg g) \mathcal{U} h \longrightarrow (\Box f \vee f \mathcal{U} h)$ 
    by (simp add: Prop05 UntilNotImp)
  have 7:  $\vdash \Box f \wedge (\neg g) \mathcal{U} h \longrightarrow (\Box f \vee f \mathcal{U} h)$ 
    by auto
  have 8:  $\vdash (\Box f \vee f \mathcal{U} g) \wedge (\neg g) \mathcal{U} h \longrightarrow (\Box f \vee f \mathcal{U} h)$ 
    using 6 7 by fastforce
  have 9:  $\vdash (\Box f \vee f \mathcal{U} g) \wedge (\Box (\neg g) \vee (\neg g) \mathcal{U} h) \longrightarrow (\Box f \vee f \mathcal{U} h)$ 
    using 5 8 by fastforce
  show ?thesis by (simp add: 9 wait-d-def)
qed

```

end

15 Pi operator

theory *Pi*

imports *IFilter UntilSince*

begin

This theory introduces the Pi operator [10, 7]. The Pi operator is defined in terms of the ifilter operator introduced in IFilter.thy. We prove the soundness of the rules and axiom system. The until operator from UntilSince.thy is used as there is a striking similarity of the expressiveness of the until and the Pi operator [7].

15.1 Definitions

definition *sfxfilt* :: 'a interval \Rightarrow ('a:: world) formula \Rightarrow 'a interval interval
where *sfxfilt* *xs* *f* = (ifilter (λ *ys*. *ys* \models *f*) (suffixes *xs*))

definition *pfxfilt* :: 'a interval \Rightarrow ('a:: world) formula \Rightarrow 'a interval interval
where *pfxfilt* *xs* *f* = (ifilter (λ *ys*. *ys* \models *f*) (prefixes *xs*))

definition *pfilt* :: 'a interval \Rightarrow ('a:: world) formula \Rightarrow 'a interval
where *pfilt* *xs* *f* = (imap (λ *s*. inth *s* 0) (sfxfilt *xs* *f*))

definition *rpifilt* :: 'a interval \Rightarrow ('a:: world) formula \Rightarrow 'a interval
where *rpifilt* *xs* *f* = (imap (λ *s*. ilast *s*) (pfxfilt *xs* *f*))

definition *pi-d* :: ('a:: world) formula \Rightarrow 'a formula \Rightarrow 'a formula
where *pi-d* *F* *G* $\equiv \lambda s$. ((\exists *i* \leq *ilen* *s*. (suffix *i* *s*) \models *F*) \wedge ((*pfilt* *s* *F*) \models *G*))

definition *rpi-d* :: ('a:: world) formula \Rightarrow 'a formula \Rightarrow 'a formula
where *rpi-d* *F* *G* $\equiv \lambda s$. ((\exists *i* \leq *ilen* *s*. (prefix *i* *s*) \models *F*) \wedge ((*rpifilt* *s* *F*) \models *G*))

syntax

-*pi-d* :: [*lift*,*lift*] \Rightarrow *lift* ((- Π -) [84,84] 83)
 -*rpi-d* :: [*lift*,*lift*] \Rightarrow *lift* ((- Π^p -) [84,84] 83)

syntax (ASCII)

-*pi-d* :: [*lift*,*lift*] \Rightarrow *lift* ((- *PI* -) [84,84] 83)
 -*rpi-d* :: [*lift*,*lift*] \Rightarrow *lift* ((- *RPI* -) [84,84] 83)

translations

-*pi-d* \equiv *CONST* *pi-d*
 -*rpi-d* \equiv *CONST* *rpi-d*

definition *upi-d* :: ('a:: world) formula \Rightarrow 'a formula \Rightarrow 'a formula
where *upi-d* *F* *G* \equiv *LIFT*(\neg (*F* Π (\neg *G*)))

syntax

-*upi-d* :: [*lift*,*lift*] \Rightarrow *lift* ((- Π^u -) [84,84] 83)

syntax (ASCII)

-*upi-d* :: [*lift*,*lift*] \Rightarrow *lift* ((- *UPI* -) [84,84] 83)

translations

$-upi-d \quad \rightleftharpoons \quad CONST \text{ upi-d}$

15.2 Time reversal

lemma *pifilt-irev-exists*:

$(\exists k \leq \text{ilen } (\text{irev } \sigma). (\text{suffix } k (\text{irev } \sigma)) \models f) =$
 $(\exists k \leq \text{ilen } \sigma. (\text{prefix } k \sigma) \models f^r)$

by (*simp add: reverse-d-def*)

(*metis diff-diff-cancel irev-prefix suffix-ilen-bound suffix-ilen*)

lemma *PiRevsem*:

$(\sigma \models (f \Pi g)^r) = (\sigma \models f^r \Pi^p g^r)$

proof –

have 1: $(\sigma \models (f \Pi g)^r) =$

$((\exists i \leq \text{ilen } \sigma. f (\text{suffix } i (\text{irev } \sigma))) \wedge g (\text{pifilt } (\text{irev } \sigma) f))$

by (*simp add: pi-d-def reverse-d-def*)

have 2: $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i (\text{irev } \sigma))) =$

$(\exists k \leq \text{ilen } \sigma. (\text{prefix } k \sigma) \models f^r)$

using *pifilt-irev-exists* **by** *auto*

have 3: $(\text{pifilt } (\text{irev } \sigma) f) = (\text{imap } (\lambda s. \text{inth } s \ 0) (\text{sfxfilt } (\text{irev } \sigma) f))$

by (*simp add: pifilt-def*)

have 4: $(\text{sfxfilt } (\text{irev } \sigma) f) = (\text{ifilter } (\lambda ys. ys \models f) (\text{suffixes } (\text{irev } \sigma)))$

by (*simp add: sfxfilt-def*)

have 5: $(\text{suffixes } (\text{irev } \sigma)) = \text{irev}(\text{imap } \text{irev } (\text{prefixes } \sigma))$

using *suffixes-irev* **by** *blast*

have 6: $(\text{ifilter } (\lambda ys. ys \models f) (\text{suffixes } (\text{irev } \sigma))) =$

$(\text{ifilter } (\lambda ys. ys \models f) (\text{irev}(\text{imap } \text{irev } (\text{prefixes } \sigma))))$

using 5 **by** *auto*

have 7: $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i (\text{irev } \sigma))) \implies$

$(\text{ifilter } (\lambda ys. ys \models f) (\text{irev}(\text{imap } \text{irev } (\text{prefixes } \sigma)))) =$

$\text{irev } (\text{ifilter } (\lambda ys. ys \models f) (\text{imap } \text{irev } (\text{prefixes } \sigma)))$

by (*metis 5 ifilter-irev ilen-imap ilen-prefixes inth-iset inth-suffixes irev-ilen iset-irev*)

have 8: $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i (\text{irev } \sigma))) \implies$

$(\text{ifilter } (\lambda ys. ys \models f) (\text{imap } \text{irev } (\text{prefixes } \sigma))) =$

$\text{imap } \text{irev } (\text{ifilter } ((\lambda ys. ys \models f) \circ \text{irev}) ((\text{prefixes } \sigma)))$

by (*metis 5 ifilter-imap in-iset-suffixes iset-irev osfx-suffix*)

have 9: $((\lambda ys. ys \models f) \circ \text{irev}) = (\lambda ys. ys \models f^r)$

by (*auto simp add: reverse-d-def*)

have 10: $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i (\text{irev } \sigma))) \implies$

$\text{irev } (\text{ifilter } (\lambda ys. ys \models f) (\text{imap } \text{irev } (\text{prefixes } \sigma))) =$

$\text{irev } (\text{imap } \text{irev } (\text{ifilter } (\lambda ys. ys \models f^r) ((\text{prefixes } \sigma))))$

using 8 9 **by** *auto*

have 11: $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i (\text{irev } \sigma))) \implies$

$(\text{pifilt } (\text{irev } \sigma) f) =$

$(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{irev } (\text{imap } \text{irev } (\text{ifilter } (\lambda ys. ys \models f^r) ((\text{prefixes } \sigma))))))$

by (*simp add: 10 3 4 6 7*)

have 12: $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i (\text{irev } \sigma))) \implies$

$(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{irev } (\text{imap } \text{irev } (\text{ifilter } (\lambda ys. ys \models f^r) ((\text{prefixes } \sigma)))))) =$

$$\text{by } (\text{simp add: irev-imap})$$

$$\text{have 13: } (\text{imap } (\lambda s. \text{inth } s \ 0) \ (\text{imap } \text{irev } (\text{ifilter } (\lambda \text{ys. } \text{ys} \models f^r) \ ((\text{prefixes } \sigma))) \)) =$$

$$(\text{imap } (\lambda s. \text{ilast } s) \ (\text{ifilter } (\lambda \text{ys. } \text{ys} \models f^r) \ ((\text{prefixes } \sigma))) \)$$

$$\text{by } (\text{auto simp add: irev-inth})$$

$$\text{have 14: } (\exists i \leq \text{ilen } \sigma. f \ (\text{suffix } i \ (\text{irev } \sigma))) \implies$$

$$(\text{pfilt } (\text{irev } \sigma) \ f) =$$

$$\text{irev } (\text{imap } (\lambda s. \text{ilast } s) \ (\text{ifilter } (\lambda \text{ys. } \text{ys} \models f^r) \ ((\text{prefixes } \sigma))) \)$$

$$\text{using 11 12 13 by auto}$$

$$\text{have 15: } (\exists i \leq \text{ilen } \sigma. f \ (\text{suffix } i \ (\text{irev } \sigma))) \implies$$

$$\text{irev } (\text{imap } (\lambda s. \text{ilast } s) \ (\text{ifilter } (\lambda \text{ys. } \text{ys} \models f^r) \ ((\text{prefixes } \sigma))) \) =$$

$$\text{irev } (\text{rpifilt } \sigma \ (\text{LIFT}(f^r)))$$

$$\text{by } (\text{simp add: rpifilt-def pxfilt-def})$$

$$\text{have 16: } (\exists i \leq \text{ilen } \sigma. f \ (\text{suffix } i \ (\text{irev } \sigma))) \implies$$

$$((\text{pfilt } (\text{irev } \sigma) \ f) \models g) =$$

$$((\text{rpifilt } \sigma \ (\text{LIFT}(f^r))) \models g^r)$$

$$\text{by } (\text{metis } (\text{mono-tags, lifting}) \ 14 \ 15 \ \text{reverse-d-def})$$

$$\text{have 17: } (\sigma \models f^r \ \Pi^p \ g^r) =$$

$$((\exists k \leq \text{ilen } \sigma. (\text{prefix } k \ \sigma) \models f^r) \wedge (\text{irev}(\text{rpifilt } \sigma \ (\text{LIFT}(f^r))) \models g))$$

$$\text{by } (\text{simp add: rpi-d-def reverse-d-def})$$

$$\text{show ?thesis}$$

$$\text{using 1 14 15 17 2 by force}$$

$$\text{qed}$$

lemma *PiRev*:

$\vdash (f \ \Pi \ g)^r = (f^r \ \Pi^p \ g^r)$

by (*simp add: PiRevsem Valid-def*)

15.3 Semantic Lemmas

lemma *sfxfilter-help*:

$(\exists \text{ys} \in \text{iset } (\text{suffixes } xs) . f \ \text{ys}) = (\exists i \leq \text{ilen } xs. f \ (\text{suffix } i \ xs))$

using *iset-suffixes-sfx* **by** *auto*

lemma *pfiltinit-help*:

$(\exists y \in \text{iset } (xs) . w \ \langle y \rangle) = (\exists i \leq \text{ilen } xs. w \ \langle \text{inth } xs \ i \rangle)$

by (*metis inth-and-iset*)

lemma *sfxfilt-INil*:

$\text{sfxfilt } \langle x \rangle \ f = \langle \langle x \rangle \rangle$

by (*auto simp:sfxfilt-def*)

lemma *pfilt-INil*:

$\text{pfilt } \langle x \rangle \ f = \langle x \rangle$

by (*auto simp: pfilt-def sfxfilt-INil*)

lemma *sfxfilt-ICons*:

shows $(\text{sfxfilt } (x \odot xs) \ f) =$

$(\text{if } (\exists i \leq \text{ilen } xs. f \ (\text{suffix } i \ xs)) \ \text{then}$

$\text{if } f \ (x \odot xs) \ \text{then}$

```

      (x⊙xs)⊙ (sfxfilt (xs) f)
    else (sfxfilt xs f) )
  else ⟨x⊙xs⟩)
proof –
have 1: (sfxfilt (x⊙xs) f) =
  ifilter (λys. f ys) (suffixes (x⊙xs))

  by (simp add: sfxfilt-def)
have 2: suffixes (x⊙xs) = (x⊙xs) ⊙ suffixes xs
  by simp
have 3: ifilter (λys. f ys) (suffixes (x⊙xs)) =
  (if (∃ ys ∈ iset (suffixes xs). f ys) then
    (if f (x⊙xs) then (x⊙xs)⊙ (ifilter (λys. f ys) (suffixes (xs)))
      else (ifilter (λys. f ys) (suffixes (xs)))) )
  else ⟨x⊙xs⟩)

  by auto
have 4: (if (∃ ys ∈ iset (suffixes xs). f ys) then
  (if f (x⊙xs) then (x⊙xs)⊙ (ifilter (λys. f ys) (suffixes (xs)))
    else (ifilter (λys. f ys) (suffixes (xs)))) )
  else ⟨x⊙xs⟩) =
  (if (∃ i ≤ ilen xs. f (suffix i xs)) then
    (if f (x⊙xs) then
      (x⊙xs)⊙ (sfxfilt (xs) f)
    else (sfxfilt xs f) )
  else ⟨x⊙xs⟩)

  by (auto simp add: sfxfilt-def iset-suffixes-sfx)
show ?thesis by (simp add: 1 4)
qed

```

lemma *pfilt-ICons*:

shows (*pfilt* (x⊙xs) f) =
 (if (∃ i ≤ ilen xs. f (suffix i xs)) then
 (if f (x⊙xs) then
 (x)⊙ (*pfilt* (xs) f)
 else (*pfilt* xs f))
 else ⟨x⟩)

proof –

```

have 1: (pfilt (x⊙xs) f) = imap (λs. (inth s 0)) (sfxfilt (x⊙xs) f)
  by (simp add: pfilt-def)
have 2: imap (λs. (inth s 0)) (sfxfilt (x⊙xs) f) =
  (if (∃ i ≤ ilen xs. f (suffix i xs)) then
    (if f (x⊙xs) then
      imap (λs. (inth s 0))((x⊙xs)⊙ (sfxfilt (xs) f))
    else imap (λs. (inth s 0)) (sfxfilt xs f) )
  else imap (λs. (inth s 0)) ⟨x⊙xs⟩)

```

using 1

by (*metis* *sfxfilt-ICons*)

have 3: $\text{imap } (\lambda s. (\text{inth } s \ 0))((x \odot xs) \odot (\text{sfxfilt } (xs) \ f)) =$
 $(x) \odot (\text{pifilt } (xs) \ f)$
by (*simp add: pifilt-def*)
have 4: $\text{imap } (\lambda s. (\text{inth } s \ 0)) \langle x \odot xs \rangle = \langle x \rangle$
by *simp*
have 5: $\text{imap } (\lambda s. (\text{inth } s \ 0)) (\text{sfxfilt } xs \ f) = (\text{pifilt } (xs) \ f)$
by (*simp add: pifilt-def*)
have 6: (*if* $(\exists i \leq \text{ilen } xs. f (\text{suffix } i \ xs))$ *then*
if $f (x \odot xs)$ *then*
 $\text{imap } (\lambda s. (\text{inth } s \ 0))((x \odot xs) \odot (\text{sfxfilt } (xs) \ f))$
else $\text{imap } (\lambda s. (\text{inth } s \ 0)) (\text{sfxfilt } xs \ f)$)
else $\text{imap } (\lambda s. (\text{inth } s \ 0)) \langle x \odot xs \rangle =$
if $(\exists i \leq \text{ilen } xs. f (\text{suffix } i \ xs))$ *then*
if $f (x \odot xs)$ *then*
 $(x) \odot (\text{pifilt } (xs) \ f)$
else $(\text{pifilt } xs \ f)$)
else $\langle x \rangle$)
using 5 **by** *auto*
show ?thesis **by** (*simp add: 1 2 6*)
qed

lemma *sfxfilt-inth-ICons:*

$\text{inth } (\text{sfxfilt } (x \odot xs) \ f) \ j =$
if $(\exists i \leq \text{ilen } xs. f (\text{suffix } i \ xs))$ *then*
if $f (x \odot xs)$ *then*
 $(\text{if } j = 0 \text{ then } (x \odot xs) \text{ else } \text{inth } (\text{sfxfilt } (xs) \ f) \ (j-1))$
else $\text{inth } (\text{sfxfilt } (xs) \ f) \ j$
else $(x \odot xs)$

by *simp*

(*metis inth.simps(1) One-nat-def inth-ICons-a inth-zero sfxfilt-ICons*)

lemma *pifilt-inth-ICons:*

$\text{inth } (\text{pifilt } (x \odot xs) \ f) \ i =$
if $(\exists i \leq \text{ilen } xs. f (\text{suffix } i \ xs))$ *then*
if $f (x \odot xs)$ *then*
 $(\text{if } i = 0 \text{ then } x \text{ else } \text{inth } (\text{pifilt } (xs) \ f) \ (i-1))$
else $\text{inth } (\text{pifilt } (xs) \ f) \ i$
else x

by (*metis inth.simps(1) One-nat-def Suc-pred inth-Suc inth-zero not-gr0 pifilt-ICons*)

lemma *sfxfilt-inth:*

assumes $(\exists i \leq \text{ilen } \sigma. (\text{suffix } i \ \sigma) \models f)$
 $i \leq \text{ilen } (\text{sfxfilt } \sigma \ f)$

shows $(\text{inth } (\text{sfxfilt } \sigma \ f) \ i) \models f$

using *assms in-iset-suffixes osfx-suffix sfxfilter-inth*

by (*simp add: sfxfilt-def ifilter-inth-aa sfxfilter-help*)

lemma *pifilt-exists:*

assumes $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma))$
shows $(\exists i \leq \text{ilen}(\text{sfxfilt } \sigma f). (\text{inth } (\text{sfxfilt } \sigma f) i) \models f)$
using *assms*
using *sfxfilt-inth* **by** *blast*

lemma *sfxfilt-pifilt-ilen*:
assumes $(\exists i \leq \text{ilen } \sigma. (\text{suffix } i \sigma) \models f)$
shows $\text{ilen } (\text{pifilt } \sigma f) = \text{ilen } (\text{imap } (\lambda s. \text{inth } s 0) (\text{sfxfilt } \sigma f))$
using *assms* **by** (*simp add: pifilt-def*)

lemma *sfxfilt-pifilt-inth*:
assumes $(\exists i \leq \text{ilen } \sigma. (\text{suffix } i \sigma) \models f)$
 $j \leq \text{ilen } (\text{pifilt } \sigma f)$
shows $\text{inth } (\text{pifilt } \sigma f) j = \text{inth } (\text{imap } (\lambda s. \text{inth } s 0) (\text{sfxfilt } \sigma f)) j$
using *assms*
by (*simp add: pifilt-def*)

lemma *sfxfilt-pifilt*:
assumes $(\exists i \leq \text{ilen } \sigma. (\text{suffix } i \sigma) \models f)$
shows $(\text{imap } (\lambda s. \text{inth } s 0) (\text{sfxfilt } \sigma f)) = \text{pifilt } \sigma f$
using *assms* **by** (*simp add: pifilt-def*)

lemma *sfxfilt-ilen-bound*:
assumes $(\exists i \leq \text{ilen } xs. f (\text{suffix } i xs))$
shows $\text{ilen } (\text{sfxfilt } xs f) \leq \text{ilen } xs$
using *assms*
by (*simp add: sfxfilt-def*)
 $(\text{metis } \text{ilen-ifilter-le } \text{ilen-suffixes})$

lemma *pifilt-ilen-bound*:
assumes $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma))$
shows $\text{ilen } (\text{pifilt } \sigma f) \leq \text{ilen } \sigma$
using *assms* **by** (*simp add: pifilt-def sfxfilt-ilen-bound*)

lemma *sfxfilt-ilen-inth-bound*:
assumes $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma))$
 $j \leq \text{ilen } (\text{sfxfilt } \sigma f)$
shows $\text{ilen } (\text{inth } (\text{sfxfilt } \sigma f) j) \leq \text{ilen } \sigma$
using *assms*
by (*simp add: sfxfilt-def sfxfilter-help sfxfilter-inth-bound*)

lemma *sfxfilt-pifilt-inth-suffix*:
assumes $(\exists i \leq \text{ilen } \sigma. (\text{suffix } i \sigma) \models f)$
 $j \leq \text{ilen } (\text{pifilt } \sigma f)$
shows $\text{inth } (\text{sfxfilt } \sigma f) j = \text{suffix } (\text{ilen } \sigma - (\text{ilen } (\text{inth } (\text{sfxfilt } \sigma f) j))) \sigma$
proof –
have 1: $\text{inth } (\text{sfxfilt } \sigma f) j = \text{inth } (\text{ifilter } f (\text{suffixes } \sigma)) j$
by (*simp add: sfxfilt-def*)
have 2: $\text{suffix } (\text{ilen } \sigma - \text{ilen } (\text{inth } (\text{ifilter } f (\text{suffixes } \sigma)) j)) \sigma =$
 $\text{suffix } (\text{ilen } \sigma - (\text{ilen } (\text{inth } (\text{sfxfilt } \sigma f) j))) \sigma$

```

  by (simp add: sfxfilt-def)
have 3:  $j \leq \text{ilen} (\text{ifilter } f (\text{suffixes } \sigma))$ 
  using assms by (simp add: pifilt-def sfxfilt-def)
have 4:  $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma))$ 
  using assms by auto
have 5:  $(\exists ys \in \text{iset} (\text{suffixes } \sigma). f \text{ } ys)$ 
  using 4
  using in-iset-suffixes osfx-suffix by blast
have 6:  $\text{inth} (\text{ifilter } f (\text{suffixes } \sigma)) \text{ } j =$ 
   $\text{suffix} (\text{ilen } \sigma - \text{ilen} (\text{inth} (\text{ifilter } f (\text{suffixes } \sigma)) \text{ } j)) \text{ } \sigma$ 

  using 3 5 sfxfilter-inth-suffix[of  $\sigma$   $f$   $j$ ] by blast
show ?thesis using 1 6 by auto
qed

```

```

lemma pifilt-inth:
  assumes  $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma))$ 
     $i \leq \text{ilen} (\text{pifilt } \sigma \text{ } f)$ 
shows  $(\exists k \leq \text{ilen } \sigma. \text{inth} (\text{pifilt } \sigma \text{ } f) \text{ } i = \text{inth } \sigma \text{ } k)$ 
  using assms sfxfilt-pifilt-inth-suffix[of  $\sigma$   $f$   $i$ ]
  by (simp add: pifilt-def)
  (metis diff-le-self ifirst-suffix inth-imap )

```

```

lemma sfxfilt-ilen-imp:
  assumes  $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma) \wedge g (\text{suffix } i \sigma))$ 
shows  $\text{ilen} (\text{sfxfilt } \sigma (\text{LIFT}(f \wedge g))) \leq \text{ilen} (\text{sfxfilt } \sigma \text{ } f)$ 
proof -
  have 1:  $\text{ilen} (\text{sfxfilt } \sigma (\text{LIFT}(f \wedge g))) =$ 
     $\text{ilen} (\text{ifilter} (\lambda ys. f \text{ } ys \wedge g \text{ } ys) (\text{suffixes } \sigma))$ 
  by (simp add: sfxfilt-def)
  have 2:  $\text{ilen} (\text{ifilter } f (\text{suffixes } \sigma)) = \text{ilen} (\text{sfxfilt } \sigma \text{ } f)$ 
  by (simp add: sfxfilt-def)
  have 3:  $\exists x \in \text{iset} (\text{suffixes } \sigma). f \text{ } x \wedge g \text{ } x$ 
  using assms by auto
  have 4:  $\text{ilen} (\text{ifilter} (\lambda x. f \text{ } x \wedge g \text{ } x) (\text{suffixes } \sigma)) \leq \text{ilen} (\text{ifilter } f (\text{suffixes } \sigma))$ 
  using 3 ifilter-ilen-imp[of  $\text{suffixes } \sigma$   $f$   $g$ ] by auto
  show ?thesis using 1 2 4 by auto
qed

```

```

lemma pifilt-ilen-imp:
  assumes  $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma) \wedge g (\text{suffix } i \sigma))$ 
shows  $\text{ilen} (\text{pifilt } \sigma (\text{LIFT}(f \wedge g))) \leq \text{ilen} (\text{pifilt } \sigma \text{ } f)$ 
  using assms
  by (simp add: sfxfilt-ilen-imp pifilt-def)

```

```

lemma interval-iset-sfxfilt [simp]:
  assumes  $(\exists i \leq \text{ilen } xs. f (\text{suffix } i \text{ } xs))$ 
shows  $(\text{iset} (\text{sfxfilt } xs \text{ } f)) = \{ys. ys \in \text{iset} (\text{suffixes } xs) \wedge f \text{ } (ys)\}$ 
  using assms
  proof -

```


have 1: $\text{iset } (\text{sfxfilt } xs \ f) = \text{iset } (\text{ifilter } f \ (\text{suffixes } xs))$
by (simp add: sfxfilt-def)
have 2: $\exists x \in \text{iset } (\text{suffixes } xs). f \ x$
using assms **using** in-iset-suffixes osfx-suffix **by** blast
have 3: $\text{iset } (\text{ifilter } f \ (\text{suffixes } xs)) =$
 $\{ys. ys \in \text{iset } (\text{suffixes } xs) \wedge f \ ys\}$

using 2 iset-ifilter[of suffixes xs f] **by** auto
show ?thesis **by** (simp add: 1 3)
qed

lemma interval-subset-sfxfilt [simp]:
assumes $(\exists i \leq \text{ilen } xs. f \ (\text{suffix } i \ xs))$
shows $(\text{iset } (\text{sfxfilt } xs \ f)) \leq (\text{iset } (\text{sfxfilt } xs \ (\text{LIFT}(f \vee g))))$
proof –
have 1: $\exists x \in \text{iset } (\text{suffixes } xs). f \ x$
using assms **using** in-iset-suffixes osfx-suffix **by** blast
have 2: $\text{iset } (\text{sfxfilt } xs \ f) = \text{iset } (\text{ifilter } f \ (\text{suffixes } xs))$
by (simp add: sfxfilt-def)
have 3: $\text{iset } (\text{sfxfilt } xs \ (\text{LIFT}(f \vee g))) = \text{iset } (\text{ifilter } (\lambda x. f \ x \vee g \ x) \ (\text{suffixes } xs))$
by (simp add: sfxfilt-def)
have 4: $\text{iset } (\text{ifilter } f \ (\text{suffixes } xs)) \leq \text{iset } (\text{ifilter } (\lambda x. f \ x \vee g \ x) \ (\text{suffixes } xs))$
using 1 subset-ifilter[of suffixes xs f g] **by** auto
show ?thesis **using** 2 3 4 **by** blast
qed

lemma interval-iset-pifilt [simp]:
assumes $(\exists i \leq \text{ilen } xs. f \ (\text{suffix } i \ xs))$
shows $(\text{iset } (\text{pifilt } xs \ f)) = \{(\text{inth } ys \ 0) \mid ys. ys \in \text{iset}(\text{suffixes } xs) \wedge f \ (ys)\}$
proof –
have 1: $(\text{iset } (\text{pifilt } xs \ f)) = (\text{iset } (\text{imap } (\lambda s. \text{inth } s \ 0) \ (\text{sfxfilt } xs \ f)))$
by (simp add: pifilt-def)
have 2: $(\text{iset } (\text{imap } (\lambda s. \text{inth } s \ 0) \ (\text{sfxfilt } xs \ f))) =$
 $\{(\text{inth } ys \ 0) \mid ys. ys \in \text{iset } (\text{sfxfilt } xs \ f)\}$
by (induction xs) auto
have 3: $\{(\text{inth } ys \ 0) \mid ys. ys \in \text{iset } (\text{sfxfilt } xs \ f)\} =$
 $\{(\text{inth } ys \ 0) \mid ys. ys \in \text{iset}(\text{suffixes } xs) \wedge f \ (ys)\}$

using assms **by** auto
show ?thesis **using** 1 2 3 **by** auto
qed

lemma interval-inth-sfxfilt-in-iset:
 $x \in \text{iset } (\text{sfxfilt } \sigma \ (\text{LIFT}(f \vee g))) =$
 $(\exists i \leq \text{ilen } (\text{sfxfilt } \sigma \ (\text{LIFT}(f \vee g))). x = (\text{inth } (\text{sfxfilt } \sigma \ (\text{LIFT}(f \vee g))) \ i))$
by (metis inth-and-iset)

lemma *sfxfilt-inth-or*:
assumes $(\exists i \leq \text{ilen } \sigma. (\text{suffix } i \ \sigma) \models f)$
shows $(\exists i \leq \text{ilen } (\text{sfxfilt } \sigma (\text{LIFT}(f \vee g))). (\text{inth } (\text{sfxfilt } \sigma (\text{LIFT}(f \vee g))) \ i) \models f)$
proof –
have 1: $\exists x \in \text{iset } (\text{suffixes } \sigma). f \ x$
using *assms* **by** *auto*
have 2: $\text{sfxfilt } \sigma (\text{LIFT}(f \vee g)) = \text{ifilter } (\lambda x. f \ x \vee g \ x) (\text{suffixes } \sigma)$
by (*simp add: sfxfilt-def*)
have 3: $\exists x \in \text{iset}(\text{ifilter } (\lambda x. f \ x \vee g \ x) (\text{suffixes } \sigma)). f \ x$
using 1 *ifilter-inth-or*[*of suffixes* $\sigma \ f \ g$] **by** *auto*
from 2 3 **show** ?thesis **by** (*metis interval-inth-sfxfilt-in-iset*)
qed

lemma *NotPiFalse*:
 $\sigma \models \neg ((\# \text{False}) \ \Pi \ f)$
by (*simp add: pi-d-def*)

lemma *pifilt-true*:
 $\text{pifilt } \sigma (\text{LIFT}(\# \text{True})) = \sigma$
by (*simp add: pifilt-def sfxfilt-def ifilter-True*)

lemma *pifilt-init-INil*:
 $(\text{pifilt } \langle x \rangle (\text{LIFT}(\text{init } w))) = \langle x \rangle$
by (*auto simp add: pifilt-def sfxfilt-def*)

lemma *pifilt-init-ICons*:
 $(\text{pifilt } (x \odot xs) (\text{LIFT}(\text{init } w))) =$
 $(\text{if } (\exists i \leq \text{ilen } xs. w \langle \text{inth } xs \ i \rangle) \text{ then}$
 $\quad (\text{if } w \langle x \rangle \text{ then } x \odot (\text{pifilt } xs (\text{LIFT}(\text{init } w)))$
 $\quad \text{else } (\text{pifilt } xs (\text{LIFT}(\text{init } w))))$
 $\text{else } \langle x \rangle)$
by (*simp add: pifilt-def sfxfilt-def init-defs*)
(metis ifirst-suffix in-iset-suffixes interval-sfx-1 osfx-suffix pifiltinit-help)

lemma *PiStatesem*:
 $(\sigma \models (\text{init } w) \ \Pi \ f) =$
 $((\exists i \leq \text{ilen } \sigma. w \langle \text{inth } \sigma \ i \rangle) \wedge f (\text{pifilt } \sigma (\text{LIFT}(\text{init } w))))$
by (*simp add: pi-d-def init-defs*)

15.4 Soundness of Axioms

15.4.1 PiK

lemma *PiKsem*:
 $\sigma \models f1 \ \Pi^u \ (f \longrightarrow g) \longrightarrow (f1 \ \Pi^u \ f \longrightarrow f1 \ \Pi^u \ g)$
by (*simp add: upi-d-def init-defs pi-d-def*) *auto*

lemma *PiK*:
 $\vdash f1 \ \Pi^u \ (f \longrightarrow g) \longrightarrow (f1 \ \Pi^u \ f \longrightarrow f1 \ \Pi^u \ g)$
using *PiKsem Valid-def* **by** *blast*

15.4.2 PiDc

lemma *PiDcsem*:

$\sigma \models f \Pi g \longrightarrow f \Pi^u g$

by (*simp add: upi-d-def init-defs pi-d-def*)

lemma *PiDc*:

$\vdash f \Pi g \longrightarrow f \Pi^u g$

using *PiDcsem Valid-def* **by** *blast*

15.4.3 PiN

lemma *PiN*:

assumes $\vdash g$

shows $\vdash f \Pi^u g$

using *assms* **by** (*simp add: Valid-def pi-d-def upi-d-def*)

15.4.4 PiTrueEqvDiamond

lemma *PiTrueEqvDiamond*:

$\vdash f \Pi \#True = \Diamond f$

by (*simp add: Valid-def pi-d-def sometimes-defs*)

15.4.5 PiOr

lemma *PiOr*:

$\vdash f \Pi (g1 \vee g2) = (f \Pi g1 \vee f \Pi g2)$

by (*simp add: Valid-def pi-d-def*) *blast*

15.4.6 UPiFalseEqvBoxNot:

lemma *UPiFalseEqvBoxNot*:

$\vdash f \Pi^u \#False = \Box (\neg f)$

by (*simp add: Valid-def upi-d-def pi-d-def always-defs*)

15.4.7 BoxEqvImpPiEqv

lemma *BoxEqvImpPiEqvsem*:

assumes $(\sigma \models \Box (f1 = f2))$

shows $(\sigma \models (f1 \Pi g = f2 \Pi g))$

proof –

show $(\sigma \models (f1 \Pi g = f2 \Pi g))$

proof –

have 1: $\forall n \leq \text{ilen } \sigma. f1 (\text{suffix } n \sigma) = f2 (\text{suffix } n \sigma)$

using *assms* **by** (*simp add: always-defs*)

have 2: $(\sigma \models (f1 \Pi g)) = ((\exists i \leq \text{ilen } \sigma. f1 (\text{suffix } i \sigma)) \wedge g (\text{pifilt } \sigma f1))$

by (*simp add: pi-d-def*)

have 3: $(\exists i \leq \text{ilen } \sigma. f1 (\text{suffix } i \sigma)) = (\exists i \leq \text{ilen } \sigma. f2 (\text{suffix } i \sigma))$

using 1 **by** *blast*

have 4: $(\text{sfxfilt } \sigma f1) = (\text{sfxfilt } \sigma f2)$

using 1

proof (*induct* σ)

```

case (INil x)
then show ?case by (simp add: sxfilt-INil)
next
case (ICons x1a σ)
then show ?case
  proof –
    have 41: (sxfilt (x1a ⊙ σ) f1) =
      (if (∃ i ≤ ilen σ. (suffix i σ) ⊢ f1) then
        (if ((x1a ⊙ σ) ⊢ f1) then (x1a ⊙ σ) ⊙ (sxfilt σ f1)
          else (sxfilt σ f1))
        else ⟨x1a ⊙ σ⟩)
    using sxfilt-ICons by blast
    have 42: sxfilt σ f1 = sxfilt σ f2
    by (metis ICons.hyps ICons.premys Suc-le-mono suffix-suc ilen.simps(2)
      plus-1-eq-Suc)
    have 43: ((x1a ⊙ σ) ⊢ f1) = ((x1a ⊙ σ) ⊢ f2)
    using ICons.premys by auto
    have 44: (∃ i ≤ ilen σ. (suffix i σ) ⊢ f1) =
      (∃ i ≤ ilen σ. (suffix i σ) ⊢ f2)
    by (metis Suc-le-mono suffix-suc ilen.simps(2) ICons(2) plus-1-eq-Suc)
    have 45: (if (∃ i ≤ ilen σ. (suffix i σ) ⊢ f1) then
      (if ((x1a ⊙ σ) ⊢ f1) then (x1a ⊙ σ) ⊙ (sxfilt σ f1)
        else (sxfilt σ f1))
      else ⟨x1a ⊙ σ⟩) =
      (if (∃ i ≤ ilen σ. (suffix i σ) ⊢ f2) then
        (if ((x1a ⊙ σ) ⊢ f2) then (x1a ⊙ σ) ⊙ (sxfilt σ f2)
          else (sxfilt σ f2))
        else ⟨x1a ⊙ σ⟩)
    using 42 43 44 by auto
    have 46: (if (∃ i ≤ ilen σ. (suffix i σ) ⊢ f2) then
      (if ((x1a ⊙ σ) ⊢ f2) then (x1a ⊙ σ) ⊙ (sxfilt σ f2)
        else (sxfilt σ f2))
      else ⟨x1a ⊙ σ⟩) = (sxfilt (x1a ⊙ σ) f2)
    by (simp add: sxfilt-ICons)
    show ?thesis
    using 41 45 46 by presburger
  qed
qed
have 47: (pifilt σ f1) = (pifilt σ f2)
  by (simp add: 4 pifilt-def)
have 5: ((∃ i ≤ ilen σ. f1 (suffix i σ)) ∧ g (pifilt σ f1)) =
  ((∃ i ≤ ilen σ. f2 (suffix i σ)) ∧ g (pifilt σ f2))
  by (simp add: 3 47)
show ?thesis by (simp add: 5 pi-d-def)
qed
qed

lemma BoxEqvImpPiEqv:
  ⊢ □ (f1 = f2) ⟶ (f1 Π g = f2 Π g)
using BoxEqvImpPiEqvsem by (simp add: Valid-def, auto)

```

15.4.8 PiDiamondImpDiamond

lemma *PiDiamondImpDiamondsem*:

$\sigma \models f \Pi (\Diamond (\text{init } w)) \longrightarrow \Diamond (\text{init } w)$

using *pifilt-inth* **by** (*simp add: Valid-def pi-d-def sometimes-defs init-defs*) *fastforce*

lemma *PiDiamondImp*:

$\vdash f \Pi (\Diamond (\text{init } w)) \longrightarrow \Diamond (\text{init } w)$

using *PiDiamondImpDiamondsem Valid-def* **by** *blast*

15.4.9 PiAssoc

lemma *PiAssocsem1*:

assumes $i \leq \text{ilen } \sigma$

$f (\text{suffix } i \sigma)$

$ia \leq \text{ilen } (\text{pifilt } \sigma f)$

$w \langle \text{inth } (\text{pifilt } \sigma f) ia \rangle$

shows $\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma) \wedge w \langle \text{inth } (\text{suffix } i \sigma) 0 \rangle$

proof –

have 1: $(\text{inth } (\text{pifilt } \sigma f) ia) = (\text{inth } (\text{imap } (\lambda s. \text{inth } s 0) (\text{sfxfilt } \sigma f)) ia)$

using *assms(1) assms(2) assms(3) sfxfilt-pifilt-inth* **by** *blast*

have 2: $(\text{inth } (\text{imap } (\lambda s. \text{inth } s 0) (\text{sfxfilt } \sigma f)) ia) =$

$(\lambda s. \text{inth } s 0) (\text{inth } (\text{sfxfilt } \sigma f) ia)$

using *inth-imap* **by** *auto*

have 3: $f (\text{inth } (\text{sfxfilt } \sigma f) ia)$

using *sfxfilt-inth*

by (*metis assms(1) assms(2) assms(3) ilen-imap pifilt-def*)

have 4: $\text{inth } (\text{sfxfilt } \sigma f) ia = \text{suffix } (\text{ilen } \sigma - \text{ilen } (\text{inth } (\text{sfxfilt } \sigma f) ia)) \sigma$

using *sfxfilt-pifilt-inth-suffix assms(1) assms(2) assms(3)* **by** *blast*

have 5: $w \langle \text{inth } (\text{suffix } (\text{ilen } \sigma - \text{ilen } (\text{inth } (\text{sfxfilt } \sigma f) ia)) \sigma) 0 \rangle$

using 1 2 4 *assms(4)* **by** *auto*

show *?thesis*

by (*metis 3 4 5 diff-le-self*)

qed

lemma *PiAssocsem2*:

assumes $i \leq \text{ilen } \sigma$

$f (\text{suffix } i \sigma)$

$w \langle \text{inth } \sigma i \rangle$

shows $\exists j \leq \text{ilen } (\text{pifilt } \sigma f). w \langle \text{inth } (\text{pifilt } \sigma f) j \rangle$

proof –

have 1: $\exists j \leq \text{ilen } (\text{sfxfilt } \sigma f). f (\text{inth } (\text{sfxfilt } \sigma f) j)$

using *assms pifilt-exists* **by** *blast*

have 2: $(\text{LIFT } (\text{init } w)) (\text{suffix } i \sigma)$

by (*simp add: assms init-defs*)

have 3: $\exists j \leq \text{ilen } (\text{sfxfilt } \sigma (\text{LIFT } (\text{init } w))). (\text{LIFT } (\text{init } w)) (\text{inth } (\text{sfxfilt } \sigma (\text{LIFT } (\text{init } w)))) j$

using *pifilt-exists 2 assms* **by** *blast*

have 4: $(\text{LIFT } (f \wedge \text{init } w)) (\text{suffix } i \sigma)$

by (*simp add: assms init-defs*)

have 5: $\exists j \leq \text{ilen } (\text{sfxfilt } \sigma (\text{LIFT } (f \wedge \text{init } w))).$

$(\text{LIFT } (f \wedge \text{init } w)) (\text{inth } (\text{sfxfilt } \sigma (\text{LIFT } (f \wedge \text{init } w)))) j$

using *pfilt-exists* 4 **assms by** *blast*
have 6: $\exists i \leq \text{ilen } \sigma. \text{suffix } i \sigma \models f \wedge \text{init } w$
using 4 **assms by** *blast*
have 7: $\exists j \leq \text{ilen } (\text{sfxfilt } \sigma (\text{LIFT}((f \wedge \text{init } w) \vee (f \wedge \neg(\text{init } w))) \))).$
 $(\text{LIFT}(f \wedge \text{init } w)) (\text{inth } (\text{sfxfilt } \sigma (\text{LIFT}((f \wedge \text{init } w) \vee (f \wedge \neg(\text{init } w))) \)) j)$
using 6 *sfxfilt-inth-or[of σ LIFT($f \wedge \text{init } w$) LIFT($f \wedge \neg(\text{init } w)$)]*
by *auto*
have 8: $\bigwedge \sigma . (\sigma \models ((f \wedge \text{init } w) \vee (f \wedge \neg(\text{init } w)))) = (\sigma \models f)$
by *auto*
have 9: $(\text{sfxfilt } \sigma (\text{LIFT}((f \wedge \text{init } w) \vee (f \wedge \neg(\text{init } w))) \)) =$
 $(\text{sfxfilt } \sigma f)$
using 8 **by** (*simp add: sfxfilt-def*)
have 10: $\exists j \leq \text{ilen } (\text{sfxfilt } \sigma f).$
 $(\text{LIFT}(f \wedge \text{init } w)) (\text{inth } (\text{sfxfilt } \sigma f) j)$
using 7 9 **by** *auto*
have 11: $\text{ilen } (\text{sfxfilt } \sigma f) = \text{ilen } (\text{pfilt } \sigma f)$
by (*simp add: pfilt-def*)
have 12: $\exists j \leq \text{ilen } (\text{pfilt } \sigma f).$
 $(\text{LIFT}(\text{init } w)) (\text{inth } (\text{sfxfilt } \sigma f) j)$
using 10 11 **by** *auto*
from 12 11 **show** *?thesis*
by (*simp add: init-defs*)
 $(\text{metis inth-imap pfilt-def})$
qed

lemma *PiAssocsema:*

$((\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma)) \wedge$
 $(\exists i \leq \text{ilen } (\text{pfilt } \sigma f). w \langle \text{inth } (\text{suffix } i (\text{pfilt } \sigma f)) 0 \rangle)) =$
 $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma) \wedge w \langle \text{inth } (\text{suffix } i \sigma) 0 \rangle)$
using *PiAssocsem1 PiAssocsem2 by fastforce*

lemma *PiAssocsemb:*

$((\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma)) \wedge$
 $(\exists i \leq \text{ilen } (\text{pfilt } \sigma f). (\text{LIFT}(\text{init } w)) (\text{suffix } i (\text{pfilt } \sigma f)) \)) =$
 $(\exists i \leq \text{ilen } \sigma. (\text{LIFT}(f \wedge \text{init } w)) (\text{suffix } i \sigma))$
using *PiAssocsem1 PiAssocsem2*
by (*simp add: init-defs fastforce*)

lemma *pfilt-INil-help:*

$(\exists x \in \text{iset } (\text{suffixes } xs) . (\text{LIFT}(\text{init } w)) x) = (\exists x \in \text{iset } xs. w \langle x \rangle)$
proof (*auto simp add: init-defs*)
show $\bigwedge x. \text{osfx } x xs \implies w \langle \text{inth } x 0 \rangle \implies \exists x \in \text{iset } xs. w \langle x \rangle$
using *in-iset-suffixes interval-sfx-1 by blast*
show $\bigwedge x. x \in \text{iset } xs \implies w \langle x \rangle \implies \exists x \in \text{iset } (\text{suffixes } xs). w \langle \text{inth } x 0 \rangle$
by (*metis in-iset-suffixes ifirst-suffix inth-and-iset osfx-suffix*)
qed

lemma *pfilt-init:*

assumes $(\exists i \leq \text{ilen } xs. (\text{LIFT}(\text{init } w)) (\text{suffix } i xs))$

```

shows (pifilt xs (LIFT(init w))) = ifilter (λy. w (⟨y⟩)) xs
using asms
proof (induct xs)
case (INil x)
then show ?case
by (simp add: pifilt-init-INil)
next
case (ICons x1a xs)
then show ?case
  proof –
    have 1: pifilt (x1a⊙xs) (LIFT(init w)) =
      imap (λxs . (inth xs 0)) (sfxfilt (x1a⊙xs) (LIFT(init w)))

      using sfxfilt-pifilt by (simp add: pifilt-def)
    have 2: sfxfilt (x1a⊙xs) (LIFT(init w)) =
      (ifilter (λ ys. (LIFT(init w)) ys) (suffixes (x1a ⊙ xs)))

      using sfxfilt-def by blast
    have 3: suffixes (x1a ⊙ xs) = (x1a⊙xs)⊙ (suffixes xs)
      by simp
    have 4: (ifilter (λ ys. (LIFT(init w)) ys) (suffixes (x1a ⊙ xs))) =
      (if (∃ x ∈ iset (suffixes xs) . (LIFT(init w)) x) then
        (if (LIFT(init w)) (x1a⊙xs) then (x1a⊙xs)⊙ (ifilter (LIFT(init w)) (suffixes xs))
          else (ifilter (LIFT(init w)) (suffixes xs)))
        else ⟨x1a ⊙ xs⟩)

      by simp
    have 5: imap (λxs . (inth xs 0)) (ifilter (λ ys. (LIFT(init w)) ys) (suffixes (x1a ⊙ xs))) =
      (if (∃ x ∈ iset (suffixes xs) . (LIFT(init w)) x) then
        (if (LIFT(init w)) (x1a⊙xs)
          then (x1a) ⊙ imap (λxs . (inth xs 0)) (ifilter (LIFT(init w)) (suffixes xs))
          else imap (λxs . (inth xs 0)) (ifilter (LIFT(init w)) (suffixes xs)))
        else ⟨x1a⟩)

      by auto
    have 6: ifilter (λy. w (⟨y⟩)) (x1a ⊙ xs) =
      (if (∃ x ∈ iset xs. w (⟨x⟩)) then
        (if w (⟨x1a⟩) then x1a⊙ (ifilter (λy. w (⟨y⟩)) xs) else (ifilter (λy. w (⟨y⟩)) xs))
        else ⟨x1a⟩)

      by simp
    have 61: (∃ x ∈ iset (suffixes xs) . (LIFT(init w)) x) =
      (∃ x ∈ iset xs. w (⟨x⟩))

      by (auto simp: init-defs interval-sfx-1)
      (metis init-defs prefix-zero-ifirst pifilt-INil-help)
    have 62: (LIFT(init w)) (x1a⊙xs) = w (⟨x1a⟩)
      by (simp add: init-defs)
    have 63: (∃ x ∈ iset xs. w (⟨x⟩)) ⟶
      imap (λxs . (inth xs 0)) (ifilter (LIFT(init w)) (suffixes xs)) =

```

```

      (ifilter (λy. w ⟨y⟩) xs)
  by (auto simp add: inth-and-iset )
      (metis ICons.hyps interval.distinct(1) pifilt-ICons pifilt-def pifilt-init-ICons sfxfilt-def)
  have 7: (if (∃ x ∈ iset (suffixes xs) . (LIFT(init w)) x) then
    (if (LIFT(init w)) (x1a⊙xs)
      then (x1a) ⊙ imap (λxs . (inth xs 0)) (ifilter (LIFT(init w)) (suffixes xs))
      else imap (λxs . (inth xs 0)) (ifilter (LIFT(init w)) (suffixes xs)))
    else ⟨x1a⟩) =
    (if (∃ x ∈ iset xs. w (⟨x⟩)) then
      (if w (⟨x1a⟩) then x1a⊙ (ifilter (λy. w ⟨y⟩) xs) else (ifilter (λy. w ⟨y⟩) xs))
      else ⟨x1a⟩)

  by (simp add: 61 62 63)
  show ?thesis using 1 2 5 7 by auto
qed
qed

```

lemma *pifilt-init-a*:

```

  assumes (∃ i ≤ len xs. w ⟨ inth xs i ⟩)
  shows (pifilt xs (λs. w ⟨ inth s 0 ⟩)) = ifilter (λy. w (⟨y⟩)) xs
  using assms pifilt-init by (auto simp add: init-defs pifilt-def sfxfilt-def)

```

lemma *pifilt-pifilt* :

```

  assumes (∃ i ≤ len xs. f (suffix i xs))
    (∃ i ≤ len (pifilt xs f). w ⟨ inth (suffix i (pifilt xs f)) 0 ⟩)
  shows (pifilt (pifilt xs f) (LIFT(init w))) = pifilt xs (LIFT(f ∧ init w))
  proof -
    have 1: ∃ i ≤ len (pifilt xs f). (LIFT(init w)) (suffix i (pifilt xs f))
      using assms by (simp add: init-defs)
    have 2: (pifilt (pifilt xs f) (LIFT(init w))) =
      ifilter (λy. w (⟨y⟩)) (pifilt xs f)

```

```

      using 1 pifilt-init[of (pifilt xs f) w ] by auto

```

```

  have 3: (pifilt xs f) =
    imap (λs. inth s 0) (sfxfilt xs f)
    by (simp add: assms sfxfilt-pifilt)
  have 4: (sfxfilt xs f) = ifilter (λ ys. f ys) (suffixes xs)
    using sfxfilt-def by blast
  have 5: (pifilt xs f) = imap (λs. inth s 0) (ifilter (λ ys. f ys) (suffixes xs))
    by (simp add: 3 4)
  have 6: ifilter (λy. w (⟨y⟩)) (pifilt xs f) =
    ifilter (λy. w (⟨y⟩)) (imap (λs. inth s 0) (ifilter (λ ys. f ys) (suffixes xs)))

```

```

    using 5 by simp

```

```

  have 7: ifilter (λy. w (⟨y⟩)) (imap (λs. inth s 0) (ifilter (λ ys. f ys) (suffixes xs))) =
    imap (λs. inth s 0) (ifilter ((λy. w (⟨y⟩))∘(λs. inth s 0)) (ifilter (λ ys. f ys) (suffixes xs)))
    using assms by (metis 3 4 ifilter-imag in-iset-suffixes interval-sfx-1 osfx-suffix)
  have 8: ∃ x ∈ iset (ifilter (λ ys. f ys) (suffixes xs)). ((λy. w (⟨y⟩))∘(λs. inth s 0)) x
    using assms 3 4
    by simp-all

```


$(metis \text{ ilen-map } inth-map \text{ inth-iset})$
have 9: $\exists x \in iset \text{ (suffixes } xs). (\lambda ys. f \text{ } ys) \text{ } x$
using *assms in-iset-suffixes osfx-suffix* **by** *blast*
have 10: $\exists x \in iset \text{ (suffixes } xs). ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } x \wedge (\lambda ys. f \text{ } ys) \text{ } x$
using *8 9 by auto*
have 11: $ifilter \text{ } ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } (ifilter \text{ } (\lambda ys. f \text{ } ys) \text{ } (suffixes \text{ } xs)) =$
 $ifilter \text{ } (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs) \text{ } (suffixes \text{ } xs)$

using *ifilter-ifilter[of* $(\lambda ys. f \text{ } ys) \text{ } (suffixes \text{ } xs) \text{ } ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0))$ *]*
 $8 \text{ } 10 \text{ by } blast$
have 12: $\exists i \leq ilen \text{ } xs. (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs) \text{ } (suffix \text{ } i \text{ } xs)$
by *(metis 10 ilen-suffixes inth-and-iset inth-suffixes)*
have 13: $(ifilter \text{ } (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs) \text{ } (suffixes \text{ } xs))$
 $= (sfxfilt \text{ } xs \text{ } (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs))$

by *(simp add: sfxfilt-def)*
have 14: $\exists i \leq ilen \text{ } xs. (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs) \text{ } (suffix \text{ } i \text{ } xs)$
using *12 by blast*
have 15: $imap \text{ } (\lambda s. inth \text{ } s \text{ } 0)$
 $((sfxfilt \text{ } xs \text{ } (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs)) \text{ }) =$
 $pfilt \text{ } xs \text{ } (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs)$

using *14 by (simp add: sfxfilt-pfilt)*
have 16: $\bigwedge xs. (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs) \text{ } xs =$
 $(LIFT(f \wedge init \text{ } w)) \text{ } xs$

by *(auto simp add: init-defs)*
have 17: $pfilt \text{ } xs \text{ } (\lambda zs. ((\lambda y. w \text{ } (\langle y \rangle)) \circ (\lambda s. inth \text{ } s \text{ } 0)) \text{ } zs \wedge (\lambda ys. f \text{ } ys) \text{ } zs) =$
 $pfilt \text{ } xs \text{ } (LIFT(f \wedge init \text{ } w))$

using *16 by presburger*
show *?thesis*
using *11 13 15 17 2 3 4 7 by auto*
qed

lemma *PiAssocsem:*

$\sigma \models f \text{ } \Pi \text{ } ((init \text{ } w) \text{ } \Pi \text{ } g) = (f \wedge (init \text{ } w)) \text{ } \Pi \text{ } g$

proof *(auto simp add: pi-d-def init-defs)*

fix *i*

fix *ia*

assume *a0:* $g \text{ } (pfilt \text{ } (pfilt \text{ } \sigma \text{ } f) \text{ } (LIFT(init \text{ } w)))$

assume *a1:* $i \leq ilen \text{ } \sigma$

assume *a2:* $f \text{ } (suffix \text{ } i \text{ } \sigma)$

assume *a3:* $ia \leq ilen \text{ } (pfilt \text{ } \sigma \text{ } f)$

assume *a4:* $w \text{ } \langle inth \text{ } (pfilt \text{ } \sigma \text{ } f) \text{ } ia \rangle$

show $\exists i \leq ilen \text{ } \sigma. f \text{ } (suffix \text{ } i \text{ } \sigma) \wedge w \text{ } \langle inth \text{ } \sigma \text{ } i \rangle$

using *a0 a1 a2 a3 a4 PiAssocsem1 by fastforce*

next

fix *i*

```

fix ia
assume a0:  $g \text{ (pifilt (pifilt } \sigma \text{ f) (LIFT(init w)) )}$ 
assume a1:  $i \leq \text{ilen } \sigma$ 
assume a2:  $f \text{ (suffix i } \sigma)$ 
assume a3:  $ia \leq \text{ilen (pifilt } \sigma \text{ f)}$ 
assume a4:  $w \langle \text{inth (pifilt } \sigma \text{ f) ia} \rangle$ 
show  $g \text{ (pifilt } \sigma \text{ (LIFT(f } \wedge \text{ init w)) )}$ 
  using a0 a1 a2 a3 a4 by (metis ifirst-suffix pifilt-pifilt)
next
fix i
assume a0:  $g \text{ (pifilt } \sigma \text{ (LIFT(f } \wedge \text{ init w)) )}$ 
assume a1:  $i \leq \text{ilen } \sigma$ 
assume a2:  $f \text{ (suffix i } \sigma)$ 
assume a3:  $w \langle \text{inth } \sigma \text{ i} \rangle$ 
show  $\exists i \leq \text{ilen (pifilt } \sigma \text{ f). } w \langle \text{inth (pifilt } \sigma \text{ f) i} \rangle$ 
  using a0 a1 a2 a3 by (metis PiAssocsem2)
next
fix i
assume a0:  $g \text{ (pifilt } \sigma \text{ (LIFT(f } \wedge \text{ init w)) )}$ 
assume a1:  $i \leq \text{ilen } \sigma$ 
assume a2:  $f \text{ (suffix i } \sigma)$ 
assume a3:  $w \langle \text{inth } \sigma \text{ i} \rangle$ 
show  $g \text{ (pifilt (pifilt } \sigma \text{ f) (LIFT(init w)) )}$ 
  using a0 a1 a2 a3
  by (metis PiAssocsem2 ifirst-suffix pifilt-pifilt)
qed

```

```

lemma PiAssoc:
 $\vdash f \Pi ((\text{init } w) \Pi g) = (f \wedge (\text{init } w)) \Pi g$ 
using PiAssocsem Valid-def by blast

```

15.4.10 PiNotEqvDiamondAndNotPi

```

lemma PiNotEqvDiamondAndNotPisem:
 $\sigma \models f \Pi (\neg g) = (\Diamond f \wedge \neg(f \Pi g))$ 
by (simp add: pi-d-def sometimes-defs) blast

```

```

lemma PiNotEqvDiamondAndNotPi:
 $\vdash f \Pi (\neg g) = (\Diamond f \wedge \neg(f \Pi g))$ 
using PiNotEqvDiamondAndNotPisem Valid-def by blast

```

15.4.11 PiChopDist

```

lemma iset-fuse:
assumes  $\text{ilast } xs = \text{ifirst } ys$ 
shows  $\text{iset (fuse } xs \text{ ys)} = \text{iset } xs \cup \text{iset } ys$ 
using assms
proof (induction xs arbitrary: ys)
case (INil x)
then show ?case

```

```

by (metis fuse-INil fuse-rightneutral iapp-assoc iset-iapp opfx-code(1) opfx-def sup.idem)
next
case (ICons x1a xs)
then show ?case by simp
qed

```

lemma ifilter-chop:

```

assumes ilast xs = ifirst ys  $\wedge$  P (ilast xs)
      ( $\exists x \in \text{iset } (\text{fuse } xs \text{ } ys). P \ x$ )
      ( $\exists x \in \text{iset } xs. P \ x$ )
      ( $\exists x \in \text{iset } ys. P \ x$ )
shows ifilter P (fuse xs ys) = fuse (ifilter P xs) (ifilter P ys)
using assms
proof (induction xs arbitrary: ys)
case (INil x)
then show ?case
by simp
next
case (ICons x1a xs)
then show ?case
  proof (cases ( $\exists x \in \text{iset } xs. P \ x$ ))
  case True
  then show ?thesis
    using ICons.IH ICons.prem1(1) ICons.prem1(4) iset-fuse by fastforce
  next
  case False
  then show ?thesis
    using ICons.prem1(1) inth-iset order-refl by force
  qed
qed

```

lemma ifilter-chop1:

```

assumes n  $\leq$  ilen xs  $\wedge$  P (ilast (prefix n xs))
      ( $\exists x \in \text{iset } xs. P \ x$ )
      ( $\exists x \in \text{iset } (\text{prefix } n \text{ } xs). P \ x$ )
      ( $\exists x \in \text{iset } (\text{suffix } n \text{ } xs). P \ x$ )
shows ifilter P xs = fuse (ifilter P (prefix n xs)) (ifilter P (suffix n xs))
proof -
  have 1: ( $\exists x \in \text{iset } xs. P \ x$ ) = ( $\exists x \in \text{iset } (\text{fuse } (\text{prefix } n \text{ } xs) (\text{suffix } n \text{ } xs)). P \ x$ )
    by (simp add: assms(1) fuse-prefix-suffix)
  have 2: ilast (prefix n xs) = ifirst (suffix n xs)
    using ilast-ifirst by blast
  have 3: xs = fuse (prefix n xs) (suffix n xs)
    by (simp add: assms(1) fuse-prefix-suffix)
  have 4: ifilter P (fuse (prefix n xs) (suffix n xs)) =
    fuse (ifilter P (prefix n xs)) (ifilter P (suffix n xs))
    using assms 2 3 ifilter-chop[of (prefix n xs) (suffix n xs) P]
    by auto
  show ?thesis using 3 4 by auto
qed

```

lemma *ifilter-chop1-prefix*:
assumes $n \leq \text{ilen } xs$
 $P (\text{ilast } (\text{prefix } n \ xs))$
 $(\exists x \in \text{iset } xs. P \ x)$
 $(\exists x \in \text{iset } (\text{prefix } n \ xs). P \ x)$
 $(\exists x \in \text{iset } (\text{suffix } n \ xs). P \ x)$
shows $\text{prefix } (\text{ilen } (\text{ifilter } P \ (\text{prefix } n \ xs))) \ (\text{ifilter } P \ xs) =$
 $(\text{ifilter } P \ (\text{prefix } n \ xs))$
proof –
have 2: $\text{ifilter } P \ xs = \text{fuse } (\text{ifilter } P \ (\text{prefix } n \ xs)) \ (\text{ifilter } P \ (\text{suffix } n \ xs))$
using *assms ifilter-chop1* **by** *blast*
have 3: $\text{ilast } (\text{ifilter } P \ (\text{prefix } n \ xs)) = \text{ifirst } (\text{ifilter } P \ (\text{suffix } n \ xs))$
using *assms* **by** (*metis ilast-ifirst ifilter-ifirst ifilter-ilast*)
have 4: $\text{prefix } (\text{ilen } (\text{ifilter } P \ (\text{prefix } n \ xs)))$
 $(\text{fuse } (\text{ifilter } P \ (\text{prefix } n \ xs)) \ (\text{ifilter } P \ (\text{suffix } n \ xs))) =$
 $(\text{ifilter } P \ (\text{prefix } n \ xs))$

using *prefix-fuse* **using** 3 **by** *blast*
show ?thesis **by** (*simp add: 2 4*)
qed

lemma *ifilter-chop1-suffix*:
assumes $n \leq \text{ilen } xs$
 $P (\text{ilast } (\text{prefix } n \ xs))$
 $(\exists x \in \text{iset } xs. P \ x)$
 $(\exists x \in \text{iset } (\text{prefix } n \ xs). P \ x)$
 $(\exists x \in \text{iset } (\text{suffix } n \ xs). P \ x)$
shows $\text{suffix } (\text{ilen } (\text{ifilter } P \ (\text{prefix } n \ xs))) \ (\text{ifilter } P \ xs) =$
 $(\text{ifilter } P \ (\text{suffix } n \ xs))$
proof –
have 1: $\text{ifilter } P \ xs = \text{fuse } (\text{ifilter } P \ (\text{prefix } n \ xs)) \ (\text{ifilter } P \ (\text{suffix } n \ xs))$
using *assms ifilter-chop1* **by** *blast*
have 3: $\text{ilast } (\text{ifilter } P \ (\text{prefix } n \ xs)) = \text{ifirst } (\text{ifilter } P \ (\text{suffix } n \ xs))$
using *assms* **by** (*metis ilast-ifirst ifilter-ifirst ifilter-ilast*)
have 4: $\text{suffix } (\text{ilen } (\text{ifilter } P \ (\text{prefix } n \ xs)))$
 $(\text{fuse } (\text{ifilter } P \ (\text{prefix } n \ xs)) \ (\text{ifilter } P \ (\text{suffix } n \ xs))) =$
 $(\text{ifilter } P \ (\text{suffix } n \ xs))$

using *suffix-fuse* **using** 3 **by** *blast*
show ?thesis **by** (*simp add: 1 4*)
qed

lemma *PiChopDistsema*:
assumes $(\sigma \models (\text{init } w) \ \Pi \ (g;h))$
shows $(\sigma \models ((\text{init } w) \ \Pi \ g);((\text{init } w) \wedge ((\text{init } w) \ \Pi \ h)))$
proof –
have 1: $(\sigma \models (\text{init } w) \ \Pi \ (g;h))$

```

using assms by auto
have 2:  $((\exists i \leq \text{ilen } \sigma. (\text{LIFT}(\text{init } w)) (\text{suffix } i \ \sigma)) \wedge$ 
 $((\text{pifilt } \sigma (\text{LIFT}(\text{init } w))) \models g;h)$ 
 $)$ 
using 1 by (simp add: pi-d-def)
have 3:  $(\exists i \leq \text{ilen } \sigma. (\text{LIFT}(\text{init } w)) (\text{suffix } i \ \sigma))$ 
using 2 by auto
have 4:  $((\text{pifilt } \sigma (\text{LIFT}(\text{init } w))) \models g;h)$ 
using 2 by auto
have 5:  $\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma \models g;h$ 
using pifilt-init
using 2 by fastforce
have 6:  $\exists n \leq \text{ilen } (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma).$ 
 $g \ (\text{prefix } n \ (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma)) \wedge$ 
 $h \ (\text{suffix } n \ (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma))$ 
using 5 by (simp add: chop-defs)
obtain n where 7:  $n \leq \text{ilen } (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma) \wedge$ 
 $g \ (\text{prefix } n \ (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma)) \wedge$ 
 $h \ (\text{suffix } n \ (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma))$ 
using 6 by auto
have 8:  $\exists i \leq \text{ilen } \sigma. w \langle \text{inth } \sigma \ i \rangle$ 
using 3 by (auto simp add: init-defs)
have 9:  $\exists x \in \text{iset } \sigma. w \langle x \rangle$ 
using 8 inth-iset by blast
have 10:  $(\text{prefix } n \ (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma)) =$ 
 $(\text{ifilter } (\lambda y. w \langle y \rangle) (\text{prefix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma) \ ))$ 
by (simp add: 7 9 ifilter-nfilter-prefix-1)
have 11:  $(\text{suffix } n \ (\text{ifilter } (\lambda y. w \langle y \rangle) \ \sigma)) =$ 
 $(\text{ifilter } (\lambda y. w \langle y \rangle) (\text{suffix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma) \ ))$ 
by (simp add: 7 9 ifilter-nfilter-suffix-1)
have 12:  $g \ (\text{ifilter } (\lambda y. w \langle y \rangle) (\text{prefix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma) \ ))$ 
using 10 7 by auto
have 13:  $h \ (\text{ifilter } (\lambda y. w \langle y \rangle) (\text{suffix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma) \ ))$ 
using 11 7 by auto
have 14:  $((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \leq \text{ilen } \sigma$ 
by (metis 7 9 add-cancel-right-left nfilter-ilen nfilter-upper-bound)
have 15:  $w \langle \text{inth } (\text{suffix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma) \ 0) \rangle$ 
by (metis (no-types, lifting) 14 7 9 ifilter-inth-aa ifirst-suffix
 $\text{nfilter-ifilter nfilter-ilen nfilter-inth-n-zero}$ )
have 16:  $(\exists i \leq \text{ilen } (\text{prefix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma)).$ 
 $w \langle \text{inth } (\text{suffix } i \ (\text{prefix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma)) \ 0) \rangle)$ 
using 14 15 by auto
have 17:  $(\exists i \leq \text{ilen } (\text{suffix } ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma)).$ 
 $w \langle \text{inth } (\text{suffix } (i + ((\text{inth } (\text{nfilter } (\lambda y. w \langle y \rangle) \ \sigma \ 0) \ n) \ \sigma) \ 0) \rangle)$ 
using 15 by auto
have 18:  $(\exists n \leq \text{ilen } \sigma.$ 
 $(\exists i \leq \text{ilen } (\text{prefix } n \ \sigma). w \langle \text{inth } (\text{suffix } i \ (\text{prefix } n \ \sigma)) \ 0 \rangle) \wedge$ 
 $g \ (\text{ifilter } (\lambda y. w \langle y \rangle) (\text{prefix } n \ \sigma)) \wedge$ 
 $w \langle \text{inth } (\text{suffix } n \ \sigma) \ 0 \rangle \wedge$ 
 $(\exists i \leq \text{ilen } (\text{suffix } n \ \sigma). w \langle \text{inth } (\text{suffix } (i + n) \ \sigma) \ 0 \rangle) \wedge$ 

```

$h (ifilter (\lambda y. w \langle y \rangle)) (suffix\ n\ \sigma))$)
using 12 13 14 15 16 17 **by** blast
have 181: $(\exists n \leq ilen\ \sigma. (\exists i \leq ilen\ (prefix\ n\ \sigma). w \langle inth\ (prefix\ n\ \sigma)\ i \rangle))$)
using 18 **by** auto
have 182: $(\exists n \leq ilen\ \sigma. (\exists i \leq ilen\ (suffix\ n\ \sigma). w \langle inth\ \sigma\ (i + n) \rangle))$)
using 18 **by** auto
have 19: $(\exists n \leq ilen\ \sigma.$
 $(\exists i \leq ilen\ (prefix\ n\ \sigma). w \langle inth\ (suffix\ i\ (prefix\ n\ \sigma))\ 0 \rangle) \wedge$
 $g\ (pifilt\ (prefix\ n\ \sigma)\ (LIFT(init\ w))) \wedge$
 $w \langle inth\ (suffix\ n\ \sigma)\ 0 \rangle \wedge$
 $(\exists i \leq ilen\ (suffix\ n\ \sigma). w \langle inth\ (suffix\ (i + n)\ \sigma)\ 0 \rangle) \wedge$
 $h\ (pifilt\ (suffix\ n\ \sigma)\ (LIFT(init\ w))))$
using 18 *pifilt-init*[*of - w*]
proof –
assume a1: $\bigwedge xs. \exists i \leq ilen\ xs. (suffix\ i\ xs) \models (init\ w) \implies$
 $pifilt\ xs\ (LIFT(init\ w)) = ifilter\ (\lambda y. w \langle y \rangle)\ xs$
obtain nn :: nat **and** nna :: nat **and** nnb :: nat **where**
f2: $h\ (ifilter\ (\lambda a. w \langle a \rangle)\ (suffix\ nn\ \sigma)) \wedge$
 $(w \langle inth\ (suffix\ (nna + nn)\ \sigma)\ 0 \rangle \wedge nna \leq ilen\ (suffix\ nn\ \sigma)) \wedge$
 $w \langle inth\ (suffix\ nn\ \sigma)\ 0 \rangle \wedge$
 $g\ (ifilter\ (\lambda a. w \langle a \rangle)\ (prefix\ nn\ \sigma)) \wedge$
 $(w \langle inth\ (suffix\ nnb\ (prefix\ nn\ \sigma))\ 0 \rangle \wedge nnb \leq ilen\ (prefix\ nn\ \sigma)) \wedge$
 $nn \leq ilen\ \sigma$
using $\langle \exists n \leq ilen\ \sigma.$
 $(\exists i \leq ilen\ (prefix\ n\ \sigma). w \langle inth\ (suffix\ i\ (prefix\ n\ \sigma))\ 0 \rangle) \wedge$
 $g\ (ifilter\ (\lambda y. w \langle y \rangle)\ (prefix\ n\ \sigma)) \wedge$
 $w \langle inth\ (suffix\ n\ \sigma)\ 0 \rangle \wedge$
 $(\exists i \leq ilen\ (suffix\ n\ \sigma). w \langle inth\ (suffix\ (i + n)\ \sigma)\ 0 \rangle) \wedge$
 $h\ (ifilter\ (\lambda y. w \langle y \rangle)\ (suffix\ n\ \sigma)) \rangle$ **by** blast
then have $\exists n\ na. w\ (prefix\ 0\ (suffix\ na\ (suffix\ nn\ \sigma))) \wedge$
 $h\ (pifilt\ (suffix\ nn\ \sigma)\ (LIFT(init\ w))) \wedge$
 $na \leq ilen\ (suffix\ nn\ \sigma) \wedge$
 $g\ (pifilt\ (prefix\ nn\ \sigma)\ (LIFT(init\ w))) \wedge$
 $n \leq ilen\ (prefix\ nn\ \sigma) \wedge$
 $w\ (prefix\ 0\ (suffix\ n\ (prefix\ nn\ \sigma)))$
using a1 **by** (*metis* (*no-types*) *init-defs prefix-zero-ifirst suffix-suffix*)
then show ?thesis
using f2 **by** blast
qed
have 20: $((\exists n \leq ilen\ \sigma.$
 $(\exists i \leq ilen\ (prefix\ n\ \sigma). (LIFT(init\ w))\ (suffix\ i\ (prefix\ n\ \sigma))) \wedge$
 $g\ (pifilt\ (prefix\ n\ \sigma)\ (LIFT(init\ w))) \wedge$
 $(LIFT(init\ w))\ (suffix\ n\ \sigma) \wedge (\exists i \leq ilen\ (suffix\ n\ \sigma). (LIFT(init\ w))\ (suffix\ (i + n)\ \sigma))$
 $\wedge h\ (pifilt\ (suffix\ n\ \sigma)\ (LIFT(init\ w))))$
by (*metis* 19 *init-defs prefix-zero-ifirst*)
have 21: $(\sigma \models ((init\ w) \Pi g); ((init\ w) \wedge ((init\ w) \Pi h)))$
using 20 **by** (*simp add: chop-defs pi-d-def*)
show ?thesis
using 21 **by** auto
qed

lemma *PiChopDistsemb*:

assumes $(\sigma \models ((init\ w) \Pi\ g); ((init\ w) \wedge ((init\ w) \Pi\ h)))$

shows $(\sigma \models (init\ w) \Pi\ (g;h))$

proof –

have 1: $(\sigma \models ((init\ w) \Pi\ g); ((init\ w) \wedge ((init\ w) \Pi\ h)))$

using *assms* **by** *auto*

have 2: $\exists\ n \leq\ ilen\ \sigma.$

$((prefix\ n\ \sigma) \models ((init\ w) \Pi\ g)) \wedge$

$((suffix\ n\ \sigma) \models ((init\ w) \wedge ((init\ w) \Pi\ h)))$

using *assms chop-defs* **by** *blast*

obtain *n* **where** 3: $n \leq\ ilen\ \sigma \wedge ((prefix\ n\ \sigma) \models ((init\ w) \Pi\ g)) \wedge$

$((suffix\ n\ \sigma) \models ((init\ w) \wedge ((init\ w) \Pi\ h)))$

using 2 **by** *auto*

have 4: $((\exists\ i \leq\ ilen\ (prefix\ n\ \sigma). (LIFT(init\ w)) (suffix\ i\ (prefix\ n\ \sigma))) \wedge$

$((pifilt\ (prefix\ n\ \sigma) (LIFT(init\ w))) \models g)$

)

by (*meson* 3 *pi-d-def*)

have 5: $(\exists\ i \leq\ ilen\ (prefix\ n\ \sigma). (LIFT(init\ w)) (suffix\ i\ (prefix\ n\ \sigma)))$

using 4 **by** *auto*

have 6: $g\ (pifilt\ (prefix\ n\ \sigma) (LIFT(init\ w)))$

using 4 **by** *auto*

have 7: $g\ (ifilter\ (\lambda y. w\ (\langle y \rangle))\ (prefix\ n\ \sigma))$

using 5 6 *pifilt-init* **by** (*metis*)

have 8: $((\exists\ i \leq\ ilen\ (suffix\ n\ \sigma). (LIFT(init\ w)) (suffix\ i\ (suffix\ n\ \sigma))) \wedge$

$((pifilt\ (suffix\ n\ \sigma) (LIFT(init\ w))) \models h)$

)

by (*metis* 3 *intensional-rews*(3) *pi-d-def*)

have 9: $(\exists\ i \leq\ ilen\ (suffix\ n\ \sigma). (LIFT(init\ w)) (suffix\ i\ (suffix\ n\ \sigma)))$

using 8 **by** *auto*

have 10: $h\ (pifilt\ (suffix\ n\ \sigma) (LIFT(init\ w)))$

using 8 **by** *auto*

have 11: $h\ (ifilter\ (\lambda y. w\ (\langle y \rangle))\ (suffix\ n\ \sigma))$

using 10 9 *pifilt-init* **by** *metis*

have 12: $(\lambda y. w\ (\langle y \rangle))\ (ilast\ (prefix\ n\ \sigma))$

by (*metis* 3 *ilast-ifirst init-defs intensional-rews*(3) *prefix-zero-ifirst*)

have 13: $\exists\ x \in\ iset\ \sigma. (\lambda y. w\ (\langle y \rangle))\ x$

using 12 3 *inth-iset* **using** *ilast-prefix* **by** *fastforce*

have 14: $\exists\ x \in\ iset\ (prefix\ n\ \sigma) . (\lambda y. w\ (\langle y \rangle))\ x$

using 12 *inth-iset* **by** (*metis order-refl*)

have 15: $\exists\ x \in\ iset\ (suffix\ n\ \sigma) . (\lambda y. w\ (\langle y \rangle))\ x$

by (*metis* 12 3 *ifirst-suffix ilast-prefix ilen-gr-zero inth-iset*)

have 16: $(ifilter\ (\lambda y. w\ (\langle y \rangle))\ (prefix\ n\ \sigma)) =$

$prefix\ (ilen\ (ifilter\ (\lambda y. w\ (\langle y \rangle))\ (prefix\ n\ \sigma)))\ (ifilter\ (\lambda y. w\ (\langle y \rangle))\ \sigma)$

using 12 13 14 15 3

$ifilter\ chop1\ prefix[of\ n\ \sigma\ (\lambda y. w\ \langle y \rangle)]$ **by** *auto*

have 17: $(ifilter\ (\lambda y. w\ (\langle y \rangle))\ (suffix\ n\ \sigma)) =$

$\text{suffix } (\text{ilen } (\text{ifilter } (\lambda y. w \langle y \rangle)) (\text{prefix } n \sigma)) (\text{ifilter } (\lambda y. w \langle y \rangle)) \sigma$

using 12 13 14 15 3
 $\text{ifilter-chop1-suffix}[of\ n\ \sigma\ (\lambda y. w \langle y \rangle)]$ **by** *auto*
have 18: $\exists n \leq \text{ilen } (\text{ifilter } (\lambda y. w \langle y \rangle)) \sigma$.
 $g\ (\text{prefix } n\ (\text{ifilter } (\lambda y. w \langle y \rangle)) \sigma) \wedge$
 $h\ (\text{suffix } n\ (\text{ifilter } (\lambda y. w \langle y \rangle)) \sigma)$
by (*metis* 11 16 17 7 *prefix-ilen-bound*)
have 19: $\text{ifilter } (\lambda y. w \langle y \rangle)) \sigma \models g;h$
by (*simp add: 18 chop-defs*)
have 20: $((\text{pifilt } \sigma\ (\text{LIFT}(\text{init } w))) \models g;h)$
by (*metis* (*mono-tags*, *lifting*) 18 3 *ilast-ifirst intensional-rews*(3)
chop-fuse fuse-prefix-suffix pifilt-init)
have 21: $(\exists i \leq \text{ilen } \sigma. (\text{LIFT}(\text{init } w)) (\text{suffix } i \sigma))$
using 3 **by** *auto*
show ?thesis
by (*simp add: 20 21 pi-d-def*)
qed

lemma *PiChopDistsem*:

$\sigma \models (\text{init } w) \Pi (g;h) = ((\text{init } w) \Pi g);((\text{init } w) \wedge ((\text{init } w) \Pi h))$

using *PiChopDistsema PiChopDistsemb unl-lift2* **by** *blast*

lemma *PiChopDist*:

$\vdash (\text{init } w) \Pi (g;h) = ((\text{init } w) \Pi g);((\text{init } w) \wedge ((\text{init } w) \Pi h))$

using *PiChopDistsem Valid-def* **by** *blast*

15.4.12 PiProp

lemma *Pistate*:

$(\sigma \models (\text{init } w) \Pi f) =$
 $((\exists x \in \text{iset } \sigma. w \langle x \rangle) \wedge ((\text{ifilter } (\lambda y. w \langle y \rangle)) \sigma \models f))$

proof –

have 1: $(\sigma \models (\text{init } w) \Pi f) =$
 $((\exists i \leq \text{ilen } \sigma. w \langle \text{inth } \sigma\ i \rangle) \wedge ((\text{pifilt } \sigma\ (\text{LIFT}(\text{init } w))) \models f))$

by (*auto simp add: pi-d-def init-defs*)

have 2: $(\exists i \leq \text{ilen } \sigma. (\text{LIFT}(\text{init } w)) (\text{suffix } i \sigma)) =$
 $(\exists i \leq \text{ilen } \sigma. w \langle \text{inth } \sigma\ i \rangle)$

by (*auto simp add: init-defs*)

have 3: $(\exists i \leq \text{ilen } \sigma. w \langle \text{inth } \sigma\ i \rangle) \longrightarrow$
 $(\text{pifilt } \sigma\ (\text{LIFT}(\text{init } w))) = (\text{ifilter } (\lambda y. w \langle y \rangle)) \sigma$

using *pifilt-init* **using** 2 **by** *blast*

have 4: $(\exists i \leq \text{ilen } \sigma. w \langle \text{inth } \sigma\ i \rangle) = (\exists x \in \text{iset } \sigma. w \langle x \rangle)$

using *inth-and-iset* **by** *force*

show ?thesis

using 1 3 4 **by** *auto*

qed

lemma *PiPropsem1a*:

$(\sigma \models f \Pi \$p) =$
 $((\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma)) \wedge p (\text{inth } \sigma (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0)))$

using *inth-imap*[of $(\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } \sigma)) 0]$
using *nfilter-ifilter*[of *suffixes* σ f 0 0]
by (*simp add: pi-d-def current-val-d-def pifilt-def sfxfilt-def*)
(metis in-iset-suffixes inth-imap imap-first-suffixes osfx-suffix)

lemma *PiPropsem2a*:

$(\sigma \models (\neg f) \mathcal{U} (f \wedge \$p)) =$
 $(\exists k \leq \text{ilen } \sigma. f (\text{suffix } k \sigma) \wedge p (\text{inth } \sigma k) \wedge (\forall j < k. \neg f (\text{suffix } j \sigma)))$
by (*simp add: until-d-def current-val-d-def*)

lemma *PiPropsem3a*:

assumes $(\sigma \models f \Pi \$p)$
shows $(\sigma \models (\neg f) \mathcal{U} (f \wedge \$p))$
proof –
have 1: $((\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma)) \wedge p (\text{inth } \sigma (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0)))$
using *assms PiPropsem1a* **by** *auto*
have 2: $\exists x \in \text{iset}(\text{suffixes } \sigma). f x$
using 1 *in-iset-suffixes osfx-suffix* **by** *blast*
have 3: $\forall x \in \text{iset}(\text{nfilter } f (\text{suffixes } \sigma) 0). f (\text{inth } (\text{suffixes } \sigma) x)$
by (*simp add: 2*)
have 4: $f (\text{suffix } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0) \sigma)$
by (*metis 2 3 add.left-neutral ilen-gr-zero nfilter-upper-bound inth-iset inth-suffixes*)
have 5: $(\forall j < (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0). \neg f (\text{suffix } j \sigma))$
using *nfilter-not-before*[of *suffixes* σ f] 2
proof –
have f1: $\forall n. \neg n < \text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0 \vee \neg f (\text{inth } (\text{suffixes } \sigma) n)$
using $\langle \wedge i. [\exists x \in \text{iset} (\text{suffixes } \sigma). f x; i < \text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0] \implies$
 $\neg f (\text{inth } (\text{suffixes } \sigma) i) \rangle \langle \exists x \in \text{iset} (\text{suffixes } \sigma). f x \rangle$ **by** *blast*
obtain *ii* :: 'a *interval* **where**
f2: $ii \in \text{iset} (\text{suffixes } \sigma) \wedge f ii$
using $\langle \exists x \in \text{iset} (\text{suffixes } \sigma). f x \rangle$ **by** *blast*
obtain *nn* :: 'a *interval interval* \Rightarrow 'a *interval* \Rightarrow nat **where**
 $\forall x0 x1. (\exists v2 \leq \text{ilen } x0. \text{inth } x0 v2 = x1) = (nn x0 x1 \leq \text{ilen } x0 \wedge \text{inth } x0 (nn x0 x1) = x1)$
by *moura*
then have $nn (\text{suffixes } \sigma) ii \leq \text{ilen } (\text{suffixes } \sigma) \wedge \text{inth } (\text{suffixes } \sigma) (nn (\text{suffixes } \sigma) ii) = ii$
using *f2* **by** (*meson inth-and-iset*)
then have $\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0 \leq \text{ilen } (\text{suffixes } \sigma)$
using *f2 f1* **by** (*metis (no-types) dual-order.trans le-less-linear*)
then show *?thesis*
using *f1* **by** (*metis (no-types) dual-order.trans less-or-eq-imp-le inth-suffixes*)
qed
have 6: $(\exists k \leq \text{ilen } \sigma. f (\text{suffix } k \sigma) \wedge p (\text{inth } \sigma k) \wedge (\forall j < k. \neg f (\text{suffix } j \sigma)))$
by (*metis 1 4 5 interval-suf-first neqE*)
show *?thesis* **using** 6 *PiPropsem2a* **by** *metis*
qed

lemma *PiPropsem3b*:

assumes $(\sigma \models (\neg f) \mathcal{U} (f \wedge \$p))$

shows $(\sigma \models f \Pi \$p)$

proof –

have 1: $(\exists k \leq \text{ilen } \sigma. f (\text{suffix } k \sigma) \wedge p (\text{inth } \sigma k) \wedge (\forall j < k. \neg f (\text{suffix } j \sigma)))$

using *assms PiPropsem2a* **by** *auto*

obtain *k* **where** 2: $k \leq \text{ilen } \sigma \wedge f (\text{suffix } k \sigma) \wedge p (\text{inth } \sigma k) \wedge (\forall j < k. \neg f (\text{suffix } j \sigma))$

using 1 **by** *auto*

have 3: $(\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma))$

using 2 **by** *blast*

have 31: $\exists x \in \text{iset}(\text{suffixes } \sigma). f x$

using 2 **by** *auto*

have 32: $f (\text{suffix } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0) \sigma)$

using *nfilter-holds[of suffixes σ f 0] nfilter-not-before[of suffixes σ f]*

by (*metis 31 diff-zero ilen-gr-zero nfilter-upper-bound inth-iset inth-suffixes ordered-cancel-comm-monoid-diff-class.add-diff-inverse*)

have 4: $p (\text{inth } \sigma (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) 0) 0))$

by (*metis 1 31 32 ilen-suffixes linorder-neqE-nat nfilter-not-before inth-suffixes*)

show *?thesis* **using** 4 3 **by** (*simp add: PiPropsem1a*)

qed

lemma *PiPropsema*:

$\sigma \models f \Pi \$p = (\neg f) \mathcal{U} (f \wedge \$p)$

using *PiPropsem3a PiPropsem3b unl-lift2* **by** *blast*

lemma *PiProp*:

$\vdash f \Pi \$p = (\neg f) \mathcal{U} (f \wedge \$p)$

using *PiPropsema Valid-def* **by** *blast*

15.4.13 PiNext

lemma *PiNextsem1*:

$(\sigma \models f \Pi (\bigcirc g)) =$

$((\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma)) \wedge$

$0 < \text{ilen } (\text{ifilter } f (\text{suffixes } \sigma)) \wedge$

$g (\text{suffix } (\text{Suc } 0) (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } \sigma))))))$

by (*simp add: pi-d-def next-defs pifilt-def sxfilt-def*)

lemma *PiNextsem2*:

$(\sigma \models (\neg f) \mathcal{U} (f \wedge \bigcirc(f \Pi g))) =$

$(\exists k \leq \text{ilen } \sigma.$

$f (\text{suffix } k \sigma) \wedge$

$k < \text{ilen } \sigma \wedge$

$(\exists i \leq \text{ilen } \sigma - \text{Suc } k. f (\text{suffix } (\text{Suc } (i + k)) \sigma)) \wedge$

$g (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } (\text{suffix } (\text{Suc } k) \sigma)))) \wedge (\forall j < k. \neg f (\text{suffix } j \sigma)))$

by (*simp add: until-d-def next-defs pi-d-def pifilt-def sxfilt-def*)

lemma *PiNextsem3*:

assumes $(\sigma \models f \Pi (\circ g))$

shows $(\sigma \models (\neg f) \mathcal{U} (f \wedge \circ(f \Pi g)))$

proof –

have 1: $((\exists i \leq \text{ilen } \sigma. f (\text{suffix } i \sigma)) \wedge$

$0 < \text{ilen } (\text{ifilter } f (\text{suffixes } \sigma)) \wedge$

$g (\text{suffix } (\text{Suc } 0) (\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f (\text{suffixes } \sigma))))$

using *assms PiNextsem1* **by** *auto*

have 2: $\exists x \in \text{iset}(\text{suffixes } \sigma). f x$

using 1 *in-iset-suffixes osfx-suffix* **by** *blast*

have 3: $\forall x \in \text{iset}(\text{nfilter } f (\text{suffixes } \sigma) \ 0). f (\text{inth } (\text{suffixes } \sigma) \ x)$

by (*simp add: 2*)

have 4: $f (\text{suffix } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0) \ \sigma)$

by (*metis 2 add.left-neutral ilen-gr-zero nfilter-ifilter nfilter-inth-n-zero*
nfilter-upper-bound inth-suffixes sfxfilter-inth)

have 41: $(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) \in \text{iset}(\text{nfilter } f (\text{suffixes } \sigma) \ 0)$

by (*metis 1 2 One-nat-def Suc-leI nfilter-ilen inth-iset*)

have 42: $(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) \leq \text{ilen } (\text{suffixes } \sigma)$

by (*metis 1 2 One-nat-def Suc-leI add-cancel-right-left nfilter-ilen nfilter-upper-bound*)

have 5: $f (\text{suffix } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) \ \sigma)$

using 3 41 42 *inth-suffixes* **by** *fastforce*

have 6: $(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0) < (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1)$

by (*simp add: 1 2 idx-nfilter-mono nfilter-ilen*)

have 7: $\text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0) \leq \text{ilen } \sigma$

by (*metis 1 3 ilen-ifilter-le Suc-diff-Suc diff-is-0-eq'*
ifilter-nfilter-suffix ilen-gr-zero suffix-ilen ilen-suffixes
not-less-eq-eq inth-iset)

have 8: $0 < \text{ilen } (\text{nfilter } f (\text{suffixes } \sigma) \ 0)$

by (*simp add: 1 2 nfilter-ilen*)

have 9: $(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) \leq \text{ilen } \sigma$

using *nfilter-upper-bound[of suffixes σ f 1 0]*

by (*simp add: 2 8 Suc-leI*)

have 10: $(\text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0)) \leq (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1)$

using 6 *Suc-leI* **by** *blast*

have 11: $(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) =$

$(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) - (\text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0)) +$
 $(\text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0))$

using 10 **by** *auto*

have 12: $(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) - (\text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0)) \leq$
 $\text{ilen } \sigma - \text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0)$

using 9 *diff-le-mono* **by** *blast*

have 13: $\exists i \leq \text{ilen } \sigma - \text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0).$

$(\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 1) = (\text{Suc } (i + (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0)))$

using 11 12 **by** *auto*

have 14: $(\exists i \leq \text{ilen } \sigma - \text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0).$

$f (\text{suffix } (\text{Suc } (i + (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0))) \ \sigma))$

using 13 5 **by** *auto*

have 15: $(\text{suffix } (\text{Suc } 0) (\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f (\text{suffixes } \sigma)))) =$

$(\text{imap } (\lambda s. \text{inth } s \ 0)$

$(\text{ifilter } f (\text{suffixes } (\text{suffix } (\text{Suc } (\text{inth } (\text{nfilter } f (\text{suffixes } \sigma) \ 0) \ 0)) \ \sigma))))$

using 2 8 **by** (*simp add: 1 Suc-leI ifilter-suffixes-imap*)
have 16: $g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix (Suc (inth (nfilter } f \text{ (suffixes } \sigma) \ 0) \ 0)) \ \sigma)))$)
using 1 15 **by** *auto*
have 17: $\forall j < (\text{inth (nfilter } f \text{ (suffixes } \sigma) \ 0) \ 0). \neg f \text{ (suffix } j \ \sigma)$
by (*metis 7 Suc-leD Suc-leI in-iset-suffixes ilen-suffixes le-trans nfilter-not-before inth-suffixes osfx-suffix*)
have 18: $(\exists k \leq \text{ilen } \sigma. f \text{ (suffix } k \ \sigma) \wedge k < \text{ilen } \sigma \wedge (\exists i \leq \text{ilen } \sigma - \text{Suc } k. f \text{ (suffix (Suc (i + k)) } \sigma)) \wedge g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix (Suc } k) \ \sigma)))) \wedge (\forall j < k. \neg f \text{ (suffix } j \ \sigma)))$
using 14 16 17 4 7 *Suc-leD Suc-le-lessD* **by** *blast*
show ?thesis **using** 18 **by** (*simp add: PiNextsem2*)
qed

lemma *PiNextsem4*:

assumes $(\sigma \models (\neg f) \ \mathcal{U} \ (f \wedge \bigcirc(f \ \Pi \ g)))$

shows $(\sigma \models f \ \Pi \ (\bigcirc \ g))$

proof –

have 1: $(\exists k \leq \text{ilen } \sigma. f \text{ (suffix } k \ \sigma) \wedge k < \text{ilen } \sigma \wedge (\exists i \leq \text{ilen } \sigma - \text{Suc } k. f \text{ (suffix (Suc (i + k)) } \sigma)) \wedge g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix (Suc } k) \ \sigma)))) \wedge (\forall j < k. \neg f \text{ (suffix } j \ \sigma)))$
using *assms* **by** (*simp add: PiNextsem2*)
obtain k **where** 2: $k \leq \text{ilen } \sigma \wedge f \text{ (suffix } k \ \sigma) \wedge k < \text{ilen } \sigma \wedge (\exists i \leq \text{ilen } \sigma - \text{Suc } k. f \text{ (suffix (Suc (i + k)) } \sigma)) \wedge g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix (Suc } k) \ \sigma)))) \wedge (\forall j < k. \neg f \text{ (suffix } j \ \sigma))$
using 1 **by** *auto*
have 3: $\exists x \in \text{iset (suffixes } \sigma). f \ x$
using 1 *in-iset-suffixes osfx-suffix* **by** *blast*
have 4: $\forall x \in \text{iset(nfilter } f \text{ (suffixes } \sigma) \ 0). f \text{ (inth (suffixes } \sigma) \ x)$
by (*simp add: 3*)
have 5: $f \text{ (suffix (inth (nfilter } f \text{ (suffixes } \sigma) \ 0) \ 0) \ \sigma)$
by (*metis 3 4 add.left-neutral ilen-gr-zero nfilter-upper-bound inth-iset inth-suffixes*)
have 6: $0 < \text{ilen (ifilter } f \text{ (suffixes } \sigma))$
using *ifilter-ilen-zero-conv-a[of suffixes } \sigma]*
by (*metis 2 3 Nat.le-diff-conv2 Suc-leI Suc-n-not-le-n add-Suc-right ilen-suffixes le-add2 neq0-conv inth-suffixes*)
have 61: $(\text{inth (nfilter } f \text{ (suffixes } \sigma) \ 0) \ 1) \in \text{iset(nfilter } f \text{ (suffixes } \sigma) \ 0)$
by (*metis 3 6 One-nat-def Suc-leI nfilter-ilen inth-iset*)
have 62: $(\text{inth (nfilter } f \text{ (suffixes } \sigma) \ 0) \ 1) \leq \text{ilen (suffixes } \sigma)$
by (*metis 3 6 One-nat-def Suc-leI add-cancel-right-left nfilter-ilen nfilter-upper-bound*)
have 7: $f \text{ (suffix (inth (nfilter } f \text{ (suffixes } \sigma) \ 0) \ 1) \ \sigma)$
using 4 61 62 *inth-suffixes* **by** *fastforce*
have 8: $(\exists i \leq \text{ilen } \sigma. f \text{ (suffix } i \ \sigma))$
using 1 **by** *blast*

have 9: $g \text{ (suffix (Suc 0) (imap (\lambda s. \text{inth } s \ 0) (\text{ifilter } f \text{ (suffixes } \sigma))))}$
by (*metis* 2 3 5 6 *Suc-leI* *ilen-suffixes* *linorder-neqE-nat*
nfilter-not-before *inth-suffixes* *ifilter-suffixes-imap*)
have 10: $((\exists i \leq \text{ilen } \sigma. f \text{ (suffix } i \ \sigma)) \wedge$
 $0 < \text{ilen} (\text{ifilter } f \text{ (suffixes } \sigma)) \wedge$
 $g \text{ (suffix (Suc 0) (imap (\lambda s. \text{inth } s \ 0) (\text{ifilter } f \text{ (suffixes } \sigma))))})$
using 6 8 9 **by** *blast*
show ?thesis **using** 10
by (*simp* *add: PiNextsem1*)
qed

lemma *PiNextsem*:

$(\sigma \models f \ \Pi \ (\bigcirc g) = (\neg f) \ \mathcal{U} \ (f \wedge \bigcirc(f \ \Pi \ g)))$
using *PiNextsem3* *PiNextsem4*
using *unl-lift2* **by** *blast*

lemma *PiNext*:

$\vdash f \ \Pi \ (\bigcirc g) = (\neg f) \ \mathcal{U} \ (f \wedge \bigcirc(f \ \Pi \ g))$
using *PiNextsem* *Valid-def* **by** *blast*

15.4.14 PiUntil

lemma *PiUntilDistsem1*:

$(\sigma \models f \ \Pi \ (g \ \mathcal{U} \ h)) =$
 $((\exists i \leq \text{ilen } \sigma. f \text{ (suffix } i \ \sigma)) \wedge$
 $(\exists k \leq \text{ilen} (\text{ifilter } f \text{ (suffixes } \sigma)).$
 $h \text{ (suffix } k \text{ (imap (\lambda s. \text{inth } s \ 0) (\text{ifilter } f \text{ (suffixes } \sigma))))} \wedge$
 $(\forall j < k. g \text{ (suffix } j \text{ (imap (\lambda s. \text{inth } s \ 0) (\text{ifilter } f \text{ (suffixes } \sigma))))}))$
by (*simp* *add: pi-d-def* *pifilt-def* *sxfilt-def* *until-d-def*)

lemma *PiUntilDistsem2*:

$(\sigma \models (f \ \Pi \ g) \ \mathcal{U} \ (f \ \Pi \ h)) =$
 $(\exists k \leq \text{ilen } \sigma.$
 $(\exists i \leq \text{ilen } \sigma - k. f \text{ (suffix (i + k) } \sigma)) \wedge$
 $h \text{ (imap (\lambda s. \text{inth } s \ 0) (\text{ifilter } f \text{ (suffixes (suffix } k \ \sigma))))} \wedge$
 $(\forall j < k. (\exists i \leq \text{ilen } \sigma - j. f \text{ (suffix (i + j) } \sigma)) \wedge$
 $g \text{ (imap (\lambda s. \text{inth } s \ 0) (\text{ifilter } f \text{ (suffixes (suffix } j \ \sigma))))}))$
by (*simp* *add: until-d-def* *pi-d-def* *pifilt-def* *sxfilt-def*)

lemma *cover*:

assumes $(\exists i < k. \text{ff}(i) < (j::\text{nat}) \wedge j \leq \text{ff}(\text{Suc } i))$
 $(\forall i < k. \text{ff}(i) < \text{ff}(\text{Suc } i))$
shows $\text{ff}(0) < j \wedge j \leq \text{ff}(k)$
using *assms*
proof (*induct* *k* *arbitrary:j*)
case 0

```

then show ?case by simp
next
case (Suc k)
then show ?case
proof -
  have 1:  $(\exists i < k. \text{ff}(i) < (j::\text{nat}) \wedge j \leq \text{ff}(\text{Suc } i)) \vee (\text{ff}(k) < j \wedge j \leq \text{ff}(\text{Suc } k))$ 
    using Suc.prem1(1) less-SucE by blast
  have 2:  $(\forall i < k. \text{ff}(i) < \text{ff}(\text{Suc } i))$ 
    using Suc.prem1(2) by auto
  have 3:  $\text{ff } k < \text{ff}(\text{Suc } k)$ 
    by (simp add: Suc.prem1(2))
  have 4:  $(\text{ff}(0) < j \wedge j \leq \text{ff}(k)) \vee (\text{ff}(k) < j \wedge j \leq \text{ff}(\text{Suc } k))$ 
    using 1 2 Suc.hyps by blast
  have 41:  $\text{ff}(0) < j$ 
    using 2 4 Suc.hyps order-refl by force
  have 42:  $j \leq \text{ff}(\text{Suc } k)$ 
    using 3 4 by linarith
  have 5:  $\text{ff}(0) < j \wedge j \leq \text{ff}(\text{Suc } k)$ 
    by (simp add: 41 42)
  show ?thesis
    by (simp add: 5)
qed
qed

```

```

lemma cover-a:
  assumes  $(\forall j. (j \leq \text{ff } 0) \vee (\exists i < k. \text{ff}(i) < (j::\text{nat}) \wedge j \leq \text{ff}(\text{Suc } i)) \longrightarrow gg j)$ 

```

```

     $(\forall i < k. \text{ff}(i) < \text{ff}(\text{Suc } i))$ 
  shows  $(\forall j < \text{ff } k. gg j)$ 
proof -
  have 1:  $(\forall j < \text{ff } 0. gg j)$ 
    by (simp add: assms(1))
  have 2:  $(\forall j. \text{ff } 0 < j \wedge j \leq \text{ff } k \longrightarrow gg j)$ 
  proof
    fix j
    show  $\text{ff } 0 < j \wedge j \leq \text{ff } k \longrightarrow gg j$ 
    using assms
    proof (induct k arbitrary: j)
    case 0
    then show ?case by simp
    next
    case (Suc k)
    then show ?case
    proof -
      have 21:  $(\forall j. (j \leq \text{ff } 0) \vee (\exists i < k. \text{ff}(i) < (j::\text{nat}) \wedge j \leq \text{ff}(\text{Suc } i)) \vee (\text{ff } k < j \wedge j \leq \text{ff}(\text{Suc } k)) \longrightarrow gg j)$ 

```

```

     $\longrightarrow gg\ j)$ 
    using Suc.prem1(1) less-SucI by blast
  have 22:  $(\forall i < k. \text{ff}(i) < \text{ff}(\text{Suc } i))$ 
    using Suc.prem1(2) by auto
  have 23:  $\text{ff } k < \text{ff}(\text{Suc } k)$ 
    by (simp add: Suc.prem1(2))
  have 24:  $(\forall j. (j \leq \text{ff } 0) \vee (\text{ff } 0 < j \wedge j \leq \text{ff } k) \vee (\text{ff } k < j \wedge j \leq \text{ff}(\text{Suc } k)))$ 
     $\longrightarrow gg\ j)$ 
    using 21 22 Suc.hyps by blast
  have 25:  $\text{ff } 0 < j \wedge j \leq \text{ff}(\text{Suc } k) \longrightarrow gg\ j$ 
    using 24 not-less by blast
  show ?thesis using 25 by blast
qed
qed
qed
show ?thesis
by (metis 2 asms(1) less-or-eq-imp-le linorder-neqE-nat)
qed

```

lemma PiUntilDistsem3:

```

assumes  $(\sigma \models f \ \Pi \ (g \ \mathcal{U} \ h))$ 
shows  $(\sigma \models (f \ \Pi \ g) \ \mathcal{U} \ (f \ \Pi \ h))$ 
proof -
  have 1:  $((\exists i \leq \text{ilen } \sigma. f(\text{suffix } i \ \sigma)) \wedge (\exists k \leq \text{ilen}(\text{ifilter } f(\text{suffixes } \sigma)). h(\text{suffix } k(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes } \sigma)))) \wedge (\forall j < k. g(\text{suffix } j(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes } \sigma))))))$ 
    using asms PiUntilDistsem1 by blast
  have 2:  $\exists x \in \text{iset}(\text{suffixes } \sigma). f\ x$ 
    using 1 in-iset-suffixes osfx-suffix by blast
  have 3:  $\forall x \in \text{iset}(\text{nfilter } f(\text{suffixes } \sigma) \ 0). f(\text{inth}(\text{suffixes } \sigma) \ x)$ 
    by (simp add: 2)
  have 4:  $(\exists k \leq \text{ilen}(\text{ifilter } f(\text{suffixes } \sigma)). h(\text{suffix } k(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes } \sigma)))) \wedge (\forall j < k. g(\text{suffix } j(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes } \sigma))))$ 
    using 1 by auto
  obtain k where 5:  $k \leq \text{ilen}(\text{ifilter } f(\text{suffixes } \sigma)) \wedge h(\text{suffix } k(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes } \sigma)))) \wedge (\forall j < k. g(\text{suffix } j(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes } \sigma))))$ 
    using 4 by auto
  have 6:  $f(\text{suffix}(\text{inth}(\text{nfilter } f(\text{suffixes } \sigma) \ 0) \ k) \ \sigma)$ 
    using 2 3 5 nfilter-upper-bound[of suffixes  $\sigma$   $f$   $k$  0]
    by (metis (no-types, lifting) 2 3 5 add.left-neutral nfilter-ilen inth-iset inth-suffixes)
  have 7:  $k=0 \longrightarrow ((\exists i \leq \text{ilen } \sigma - k. f(\text{suffix}(i+k) \ \sigma)) \wedge h(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes}(\text{suffix } k \ \sigma)))) \wedge (\forall j < k. (\exists i \leq \text{ilen } \sigma - j. f(\text{suffix}(i+j) \ \sigma)) \wedge g(\text{imap } (\lambda s. \text{inth } s \ 0) (\text{ifilter } f(\text{suffixes}(\text{suffix } j \ \sigma)))))$ 

```

```

using 1 5 by auto
have 71:  $k > 0 \rightarrow (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (k - 1)) \in \text{iset}(\text{nfilter } f \text{ (suffixes } \sigma) 0)$ 
  by (metis 2 5 diff-le-self le-trans nfilter-ilen inth-iset)
have 8:  $k > 0 \rightarrow f \text{ (suffix (inth (nfilter } f \text{ (suffixes } \sigma) 0) (k - 1)) } \sigma)$ 
  by (metis 2 3 71 add.left-neutral inth-and-iset nfilter-upper-bound inth-suffixes)
have 9:  $k > 0 \rightarrow (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (k - 1)) < (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) k)$ 
  by (metis 2 5 One-nat-def Suc-diff-Suc Suc-le-lessD diff-zero idx-nfilter-mono
      nfilter-ilen)
have 10:  $k > 0 \rightarrow (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (k - 1)) \leq \text{ilen } \sigma$ 
  using nfilter-upper-bound[of suffixes  $\sigma$   $f$   $k - 1$  0]
  by (simp add: 2 5 Suc-leD nfilter-ilen)
have 11:  $k > 0 \rightarrow (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) k) \leq \text{ilen } \sigma$ 
  using nfilter-upper-bound[of suffixes  $\sigma$   $f$   $k$  0]
  by (simp add: 2 5 nfilter-ilen)
have 12:  $k > 0 \rightarrow$ 
  h (imap ( $\lambda s. \text{inth } s$  0)
      (ifilter  $f$  (suffixes (suffix (inth (nfilter  $f$  (suffixes  $\sigma$ ) 0)  $k$ )  $\sigma$ ))))
  using ifilter-nfilter-suffix[of  $f$  (suffixes  $\sigma$ )  $k$ ]
  by (metis 11 5 6 ilen-suffixes imap-suffix inth-suffixes
      suffix-suffixes)
have 121:  $k > 0 \rightarrow$ 
  h (imap ( $\lambda s. \text{inth } s$  0)
      (ifilter  $f$  (suffixes (suffix (Suc (inth (nfilter  $f$  (suffixes  $\sigma$ ) 0) ( $k - 1$ )))  $\sigma$ ))))
  by (metis 2 5 Suc-diff-1 ifilter-suffixes-imap)
have 13:  $k > 0 \rightarrow (\exists i \leq \text{ilen } \sigma - (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) k).$ 
   $f \text{ (suffix (i + (inth (nfilter } f \text{ (suffixes } \sigma) 0) k)) } \sigma)$ 
  using 6 by auto
have 131:  $k > 0 \rightarrow (\exists i \leq \text{ilen } \sigma - (\text{Suc (inth (nfilter } f \text{ (suffixes } \sigma) 0) (k - 1))).$ 
   $f \text{ (suffix (i + (Suc (inth (nfilter } f \text{ (suffixes } \sigma) 0) (k - 1))) } \sigma)$ 
  using 11 6 9 diff-le-mono by fastforce
have 14:  $k > 0 \rightarrow (\forall j < (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) k).$ 
   $(\exists i \leq \text{ilen } \sigma - j. f \text{ (suffix (i + j) } \sigma))$ 
  using 11 6 diff-le-mono by fastforce
have 141:  $k > 0 \rightarrow (\forall j < (\text{Suc (inth (nfilter } f \text{ (suffixes } \sigma) 0) (k - 1))).$ 
   $(\exists i \leq \text{ilen } \sigma - j. f \text{ (suffix (i + j) } \sigma))$ 
  using 14 9 by auto
have 15:  $(\forall j < k. g \text{ (suffix } j \text{ (imap } (\lambda s. \text{inth } s$  0) (ifilter  $f$  (suffixes  $\sigma$ ))))
  using 5 by blast
have 151:  $(\forall j < k. g \text{ (imap } (\lambda s. \text{inth } s$  0) (suffix  $j$  (ifilter  $f$  (suffixes  $\sigma$ ))))
  by (simp add: 5 less-le-trans less-or-eq-imp-le imap-suffix)
have 152:  $(\forall j < k. (\text{suffix } j \text{ (ifilter } f \text{ (suffixes } \sigma))) =$ 
   $(\text{ifilter } f \text{ (suffix (inth (nfilter } f \text{ (suffixes } \sigma) 0) j) (suffixes } \sigma)))$ 
  by (meson 2 5 ifilter-nfilter-suffix-1 le-trans less-imp-le-nat)
have 16:  $(\forall j < k. g \text{ (imap } (\lambda s. \text{inth } s$  0)
   $(\text{ifilter } f \text{ (suffix (inth (nfilter } f \text{ (suffixes } \sigma) 0) j) (suffixes } \sigma))))$ 
  using 151 152 ifilter-nfilter-suffix by simp
have 1610:  $(\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) k) \leq \text{ilen}(\text{suffixes } \sigma)$ 
  by (metis 2 5 diff-zero ilen-gr-zero nfilter-ilen
      nfilter-upper-bound ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 1611:  $(\forall j < k. (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) j) < (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) k))$ 

```



```

by (simp add: 2 5 idx-nfilter-gr nfilter-ilen)
have 1612:  $(\forall j < k. \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) j) \leq \text{ilen}(\text{suffixes } \sigma))$ 
  using 1610 1611 by auto
have 161:  $(\forall j < k. ( (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) j) (\text{suffixes } \sigma))) =$ 
   $( (\text{suffixes} (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) j) \sigma)))$ 
  using suffix-suffixes using 1612 by blast
have 17:  $(\forall j < k. g \text{ (imap } (\lambda s. \text{inth } s 0)$ 
   $(\text{ifilter } f (\text{suffixes} (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) j) \sigma))))$ 
  using 16 161 by auto
have 18:  $k > 0 \longrightarrow$ 
   $g \text{ (imap } (\lambda s. \text{inth } s 0)$ 
   $(\text{ifilter } f (\text{suffixes} (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (k-1)) \sigma)))$ 
  using 17 by simp
have 19:  $k > 0 \longrightarrow$ 
   $g \text{ (imap } (\lambda s. \text{inth } s 0)$ 
   $(\text{ifilter } f (\text{suffixes} (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0) \sigma)))$ 
  using 17 by blast
have 20:  $k > 0 \longrightarrow (\text{ifilter } f (\text{suffixes} (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0) \sigma))) =$ 
   $(\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0) (\text{suffixes } \sigma)))$ 
  using 161 by auto
have 21:  $k > 0 \longrightarrow (\forall j \leq (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0).$ 
   $( (\text{suffixes} (\text{suffix } j \sigma))) =$ 
   $( (\text{suffix } j (\text{suffixes } \sigma)))$ 
  using 1612 le-trans suffix-suffixes by fastforce
have 22:  $k > 0 \longrightarrow (\forall j \leq (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0).$ 
   $(\text{ifilter } f (\text{suffixes} (\text{suffix } j \sigma))) =$ 
   $(\text{ifilter } f (\text{suffix } j (\text{suffixes } \sigma)))$ 
  using 21 by auto
have 23:  $k > 0 \longrightarrow$ 
   $(\forall j \leq (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0).$ 
   $(\text{imap } (\lambda s. \text{inth } s 0)$ 
   $(\text{ifilter } f (\text{suffixes} (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0) \sigma)))) =$ 
   $(\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes} (\text{suffix } j \sigma))))$ 
  )
  by (simp add: 2 21 ifilter-suffixes-imap-help-0)
have 24:  $k > 0 \longrightarrow$ 
   $(\forall j \leq (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0).$ 
   $g \text{ (imap } (\lambda s. \text{inth } s 0)$ 
   $(\text{ifilter } f (\text{suffixes} (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0) \sigma)))) =$ 
   $g \text{ (imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes} (\text{suffix } j \sigma))))$ 
  )
  using 23 by auto
have 241:  $k > 0 \longrightarrow (\forall j \leq (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) 0).$ 
   $g \text{ (imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes} (\text{suffix } j \sigma))))$ 
  using 19 24 by blast
have 25:  $k > 0 \longrightarrow (\forall i < k - 1.$ 
   $(\forall l. l \leq (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i) ) \wedge$ 
   $(\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i) < l \longrightarrow$ 
   $(\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma))) =$ 
   $(\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))$ 
  ) ) )

```

proof

assume $k > 0$

show $(\forall i < k - 1.$

$(\forall l. l \leq \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) \wedge$
 $(\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i) < l \longrightarrow$
 $(\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)) =$
 $(\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))))$

proof

fix i

show $i < k - 1 \longrightarrow$

$(\forall l. l \leq \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i) \wedge$
 $\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i < l \longrightarrow$
 $\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)) =$
 $\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))$

proof –

have 251: $k=1 \longrightarrow i < k - 1 \longrightarrow$

$(\forall l. l \leq \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i) \wedge$
 $\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i < l \longrightarrow$
 $\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)) =$
 $\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))$

by *auto*

have 252: $k > 1 \longrightarrow i < k - 1 \longrightarrow$

$(\forall l. l \leq \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i) \wedge$
 $\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i < l \longrightarrow$
 $\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)) =$
 $\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))$

proof

assume $k > 1$

show $i < k - 1 \longrightarrow$

$(\forall l. l \leq \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i) \wedge$
 $\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i < l \longrightarrow$
 $\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)) =$
 $\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))$
 $)$

proof

assume $i < k - 1$

show $(\forall l. l \leq \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i) \wedge$

$\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i < l \longrightarrow$
 $\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)) =$
 $\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))$

proof

fix l

show $l \leq \text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i) \wedge$

$\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) i < l \longrightarrow$
 $\text{ifilter } f (\text{suffix} (\text{inth} (\text{nfilter } f (\text{suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)) =$
 $\text{ifilter } f (\text{suffix } l (\text{suffixes } \sigma)))$

using *ifilter-suffixes-imap-help-j*[*of l f suffixes σ i*]

using 2 5 $\langle i < k - 1 \rangle$ **by** *linarith*

qed

qed

```

    qed
  show ?thesis
  by (simp add: 252)
  qed
  qed
  qed
have 261:  $k > 0 \longrightarrow (\forall i < k - 1.
  (\forall l. l \leq (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) \wedge
    (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) i) < l \longrightarrow
    l \leq \text{ilen } (\text{suffixes } \sigma)))$ 

  by (metis 1612 Suc-diff-1 Suc-mono le-eq-less-or-eq less-le-trans)
have 262:  $k > 0 \longrightarrow (\forall i < k - 1.
  (\forall l. l \leq (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) \wedge
    (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) i) < l \longrightarrow
    (\text{suffix } l \text{ (suffixes } \sigma)) = (\text{suffixes } (\text{suffix } l \sigma)))$ 

  using 261 suffix-suffixes by blast
have 26:  $k > 0 \longrightarrow (\forall i < k - 1.
  (\forall l. l \leq (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) \wedge
    (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) i) < l \longrightarrow
    (\text{imap } (\lambda s. \text{inth } s 0)
      (\text{ifilter } f \text{ (suffix } (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma)))) =
      (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f \text{ (suffixes } (\text{suffix } l \sigma))))))$ 

  using 25 262 by auto
have 27:  $k > 0 \longrightarrow (\forall i < k - 1.
  (\forall l. l \leq (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) \wedge
    (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) i) < l \longrightarrow
    g (\text{imap } (\lambda s. \text{inth } s 0)
      (\text{ifilter } f
        (\text{suffix } (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) (\text{suffixes } \sigma))))))$ 

  by (simp add: 16)
have 28:  $k > 0 \longrightarrow (\forall i < k - 1.
  (\forall l. l \leq (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) \wedge
    (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) i) < l \longrightarrow
    g (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f \text{ (suffixes } (\text{suffix } l \sigma))))))$ 

  using 25 262 27 by auto
have 281:  $k > 0 \longrightarrow
  (\forall j.
    (j \leq (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) 0) \vee
      (\exists i. i < k - 1 \wedge
        j \leq (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i)) \wedge
        (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) i) < j) \longrightarrow
        g (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f \text{ (suffixes } (\text{suffix } j \sigma))))))$ 

  using 241 28 by blast
have 282:  $k > 0 \longrightarrow (\forall i < k - 1.
  \text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) i < \text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (\text{Suc } i))$ 

  by (metis 2 5 Suc-diff-1 Suc-leI idx-nfilter-gr le-SucI le-trans lessI nfilter-ilen)
have 29:  $k > 0 \longrightarrow (\forall j < (\text{Suc } (\text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) (k - 1))).
  g (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f \text{ (suffixes } (\text{suffix } j \sigma))))$ 

  using 281 282 cover-a[of  $\lambda x. \text{inth } (\text{nfilter } f \text{ (suffixes } \sigma) 0) x$ ]  $k - 1$ 

```

$(\lambda j. g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } j \ \sigma))))))]$

using 18 less-antisym **by** blast
have 30: $k > 0 \longrightarrow (\exists k \leq \text{ilen } \sigma.$
 $(\exists i \leq \text{ilen } \sigma - k. f \text{ (suffix } (i + k) \ \sigma)) \wedge$
 $h \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } k \ \sigma)))) \wedge$
 $(\forall j < k. (\exists i \leq \text{ilen } \sigma - j. f \text{ (suffix } (i + j) \ \sigma)) \wedge$
 $g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } j \ \sigma))))))$
using 29 121 131 141
by (meson 11 9 Suc-leI less-le-trans)
have 31: $(\exists k \leq \text{ilen } \sigma.$
 $(\exists i \leq \text{ilen } \sigma - k. f \text{ (suffix } (i + k) \ \sigma)) \wedge$
 $h \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } k \ \sigma)))) \wedge$
 $(\forall j < k. (\exists i \leq \text{ilen } \sigma - j. f \text{ (suffix } (i + j) \ \sigma)) \wedge$
 $g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } j \ \sigma))))))$
using 30 7 **by** blast
show ?thesis **using** 31
by (simp add: PiUntilDistsem2)
qed

lemma PiUntilDistsem4:

assumes $(\sigma \models (f \ \Pi \ g) \ \mathcal{U} \ (f \ \Pi \ h))$

shows $(\sigma \models f \ \Pi \ (g \ \mathcal{U} \ h))$

proof –

have 1: $(\exists k \leq \text{ilen } \sigma.$
 $(\exists i \leq \text{ilen } \sigma - k. f \text{ (suffix } (i + k) \ \sigma)) \wedge$
 $h \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } k \ \sigma)))) \wedge$
 $(\forall j < k. (\exists i \leq \text{ilen } \sigma - j. f \text{ (suffix } (i + j) \ \sigma)) \wedge$
 $g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } j \ \sigma))))))$
using assms **by** (simp add: PiUntilDistsem2)
obtain k **where** 2: $k \leq \text{ilen } \sigma \wedge (\exists i \leq \text{ilen } \sigma - k. f \text{ (suffix } (i + k) \ \sigma)) \wedge$
 $h \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } k \ \sigma)))) \wedge$
 $(\forall j < k. (\exists i \leq \text{ilen } \sigma - j. f \text{ (suffix } (i + j) \ \sigma)) \wedge$
 $g \text{ (imap } (\lambda s. \text{inth } s \ 0) \text{ (ifilter } f \text{ (suffixes (suffix } j \ \sigma))))))$
using 1 **by** auto
have 3: $(\exists i \leq \text{ilen } \sigma - k. f \text{ (suffix } (i + k) \ \sigma))$
using 2 **by** auto
have 4: $k \leq \text{ilen } \sigma$
using 2 **by** auto
obtain i **where** 5: $i \leq \text{ilen } \sigma - k \wedge f \text{ (suffix } (i + k) \ \sigma)$
using 3 **by** auto
have 6: $i + k \leq \text{ilen } \sigma$
using 4 5 Nat.le-diff-conv2 **by** blast
have 61: $\exists x \in \text{iset } (\text{suffixes } (\text{suffix } k \ \sigma)). f \ x$
by (metis 5 in-iset-suffixes suffix-suffix osfx-suffix)
have 7: $(\exists i \leq \text{ilen } \sigma. f \text{ (suffix } i \ \sigma))$
using 5 6 **by** blast
have 71: $\exists x \in \text{iset } (\text{suffixes } \sigma). f \ x$
using 5 6 in-iset-suffixes osfx-suffix **by** blast

have 72: $\exists x \in \text{iset}(\text{nfilter } f (\text{suffixes } \sigma) 0). f (\text{inth } (\text{suffixes } \sigma) x)$
by (metis 5 6 cancel-comm-monoid-add-class.diff-cancel ilen-suffixes le-refl
nfilter-holds-not-a inth-iset inth-suffixes
ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
have 73: $f (\text{suffix } (\text{inth } (\text{nfilter } f (\text{suffixes } (\text{suffix } k \sigma)) 0) 0) (\text{suffix } k \sigma))$
using nfilter-holds[of suffixes (suffix k σ) f 0]
by (metis 61 add.left-neutral diff-zero inth-iset inth-suffixes nfilter-upper-bound zero-le)
have 74: $f (\text{suffix } ((\text{inth } (\text{nfilter } f (\text{suffixes } (\text{suffix } k \sigma)) 0) 0) + k) \sigma)$
using 73 **by** auto
have 75: $f (\text{suffix } k (\text{suffix } (\text{inth } (\text{nfilter } f (\text{suffixes } (\text{suffix } k \sigma)) 0) 0) \sigma))$
by (metis 73 add commute suffix-suffix)
have 8: $h (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } (\text{suffix } k \sigma))))$
using 2 **by** auto
have 9: $(\text{ifilter } f (\text{suffixes } (\text{suffix } k \sigma))) = (\text{ifilter } f (\text{suffix } k (\text{suffixes } \sigma)))$
by (simp add: 4 suffix-suffixes)
have 10: $h (\text{imap } (\lambda s. \text{inth } s 0)$
 $(\text{suffix } (\text{ilen}(\text{ifilter } f (\text{suffixes } \sigma)) - \text{ilen } (\text{ifilter } f (\text{suffixes } (\text{suffix } k \sigma))))$
 $(\text{ifilter } f (\text{suffixes } \sigma))))$
using 2 61 sfxfilter-suffix-suffix **by** fastforce
have 11: $h (\text{suffix } (\text{ilen}(\text{ifilter } f (\text{suffixes } \sigma)) - \text{ilen } (\text{ifilter } f (\text{suffixes } (\text{suffix } k \sigma))))$
 $(\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } \sigma))))$
by (metis 10 diff-le-self imap-suffix)
have 12: $(\forall j < k. (\exists i \leq \text{ilen } \sigma - j. f (\text{suffix } (i + j) \sigma)) \wedge$
 $g (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } (\text{suffix } j \sigma))))$
using 2 **by** blast
have 13: $(\forall j < k. (\exists x \in \text{iset}(\text{suffixes } (\text{suffix } j \sigma)). f x) \wedge$
 $g (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } (\text{suffix } j \sigma))))$
by (metis 2 suffix-ilen suffix-suffix ilen-suffixes inth-iset inth-suffixes)
have 14: $(\forall j < k. (\exists x \in \text{iset}(\text{suffixes } (\text{suffix } j \sigma)). f x) \wedge$
 $g (\text{imap } (\lambda s. \text{inth } s 0)$
 $(\text{suffix } (\text{ilen}(\text{ifilter } f (\text{suffixes } \sigma)) - \text{ilen } (\text{ifilter } f (\text{suffixes } (\text{suffix } j \sigma))))$
 $(\text{ifilter } f (\text{suffixes } \sigma))))$
using 13 sfxfilter-suffix-suffix
by (metis (no-types, lifting) 2 dual-order.strict-iff-order less-le-trans)
have 15: $(\forall j < k. (\exists x \in \text{iset}(\text{suffixes } (\text{suffix } j \sigma)). f x) \wedge$
 $g (\text{suffix } (\text{ilen}(\text{ifilter } f (\text{suffixes } \sigma)) - \text{ilen } (\text{ifilter } f (\text{suffixes } (\text{suffix } j \sigma))))$
 $(\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } \sigma))))$
by (metis 14 diff-le-self imap-suffix)
have 150: $k=0 \longrightarrow (\forall jj < (\text{ilen}(\text{ifilter } f (\text{suffixes } \sigma)) - \text{ilen } (\text{ifilter } f (\text{suffixes } (\text{suffix } k \sigma))))$
 $g (\text{suffix } jj (\text{imap } (\lambda s. \text{inth } s 0) (\text{ifilter } f (\text{suffixes } \sigma))))$
by auto
have 151: $(\forall jj < (\text{ilen}(\text{ifilter } f (\text{suffixes } \sigma)) - \text{ilen } (\text{ifilter } f (\text{suffixes } (\text{suffix } k \sigma))))$
 $(\text{suffix } jj (\text{ifilter } f (\text{suffixes } \sigma))) =$
 $\text{ifilter } f (\text{suffixes } (\text{suffix } ((\text{ilen } \sigma) - \text{ilen } (\text{inth } (\text{ifilter } f (\text{suffixes } \sigma)) jj)) \sigma))$
 $)$
by (simp add: 71 sfxfilter-suffix-suffix-a)
have 153: $(\forall jj < (\text{ilen}(\text{ifilter } f (\text{suffixes } \sigma)) - \text{ilen } (\text{ifilter } f (\text{suffixes } (\text{suffix } k \sigma))))$
 $g (\text{imap } (\lambda s. \text{inth } s 0)$
 $(\text{ifilter } f$
 $(\text{suffixes } (\text{suffix } ((\text{ilen } \sigma) - \text{ilen } (\text{inth } (\text{ifilter } f (\text{suffixes } \sigma)) jj)) \sigma))))$

)
using 13 sfx-suffix-upperbound **by** blast
have 1530: $(\forall jj < (ilen(ifilter\ f\ (suffixes\ \sigma)) - ilen\ (ifilter\ f\ (suffix\ k\ \sigma))))$.
 $g\ (imap\ (\lambda s. inth\ s\ 0)\ (suffix\ jj\ (ifilter\ f\ (suffixes\ \sigma))))$
by (simp add: 151 153)
have 1531: $(\forall jj < (ilen(ifilter\ f\ (suffixes\ \sigma)) - ilen\ (ifilter\ f\ (suffix\ k\ \sigma))))$.
 $g\ (suffix\ jj\ (imap\ (\lambda s. inth\ s\ 0)\ (ifilter\ f\ (suffixes\ \sigma))))$
by (metis 1530 diff-le-self suffix-ilen suffix-ilen-code less-le-trans
less-numeral-extra(3) imap-suffix zero-less-diff)
have 154: $((\exists i \leq ilen\ \sigma. f\ (suffix\ i\ \sigma)) \wedge$
 $(\exists k \leq ilen\ (ifilter\ f\ (suffixes\ \sigma)).$
 $h\ (suffix\ k\ (imap\ (\lambda s. inth\ s\ 0)\ (ifilter\ f\ (suffixes\ \sigma)))) \wedge$
 $(\forall j < k. g\ (suffix\ j\ (imap\ (\lambda s. inth\ s\ 0)\ (ifilter\ f\ (suffixes\ \sigma))))))$
using 11 1531 7 diff-le-self **by** blast
show ?thesis
by (simp add: 154 PiUntilDistsem1)
qed

lemma PiUntilDistsem:

$\sigma \models f \Pi (g \mathcal{U} h) = (f \Pi g) \mathcal{U} (f \Pi h)$
using PiUntilDistsem3 PiUntilDistsem4 **using** unl-lift2 **by** blast

lemma PiUntilDist:

$\vdash f \Pi (g \mathcal{U} h) = (f \Pi g) \mathcal{U} (f \Pi h)$
using PiUntilDistsem Valid-def **by** blast

15.4.15 PiChopstar

lemma wnextboxnotstatesem:

assumes $k \leq ilen\ \sigma$
shows $(\forall j \leq ilen\ \sigma. k < j \longrightarrow \neg (\lambda y. w\ \langle y \rangle) (inth\ \sigma\ j)) =$
 $(LIFT(wnext\ (\Box (init\ (\neg w)))) (suffix\ k\ \sigma))$

using assms

proof (auto simp add: always-defs init-defs wnext-defs)

show $\bigwedge j. j \leq ilen\ \sigma \implies k < j \implies w\ \langle inth\ \sigma\ j \rangle \implies$
 $\forall n \leq ilen\ \sigma - Suc\ k. \neg w\ \langle inth\ \sigma\ (Suc\ (n + k)) \rangle \implies False$

proof (cases k)

show $\bigwedge j. j \leq ilen\ \sigma \implies k < j \implies w\ \langle inth\ \sigma\ j \rangle \implies$
 $\forall n \leq ilen\ \sigma - Suc\ k. \neg w\ \langle inth\ \sigma\ (Suc\ (n + k)) \rangle \implies k = 0 \implies False$

by simp

(metis Suc-le-mono Suc-pred less-le-trans)

show $\bigwedge j\ nat. j \leq ilen\ \sigma \implies k < j \implies w\ \langle inth\ \sigma\ j \rangle \implies$
 $\forall n \leq ilen\ \sigma - Suc\ k. \neg w\ \langle inth\ \sigma\ (Suc\ (n + k)) \rangle \implies k = Suc\ nat \implies False$

proof –

fix j :: nat **and** nat :: nat

assume a1: $k < j$

assume a2: $\forall n \leq ilen\ \sigma - Suc\ k. \neg w\ \langle inth\ \sigma\ (Suc\ (n + k)) \rangle$

assume a3: $j \leq ilen\ \sigma$

assume a4: $w\ \langle inth\ \sigma\ j \rangle$

have $Suc\ (k + (j - Suc\ k)) = j$
using $a1$ **by** $simp$
then show $False$
using $a4\ a3\ a2$ **by** ($metis\ diff-add-inverse2\ diff-le-mono$
 $linordered-semidom-class.add-diff-inverse\ not-add-less2$)
qed
qed
qed

lemma *NotStateUntilStateAndsem:*

$(\sigma \models (init\ (\neg w))\ \mathcal{U}\ ((init\ w) \wedge f)) =$
 $(\exists k \leq ilen\ \sigma. w\ \langle inth\ \sigma\ k \rangle \wedge f\ (suffix\ k\ \sigma) \wedge (\forall j < k. \neg w\ \langle inth\ \sigma\ j \rangle))$

by ($auto\ simp\ add: until-d-def\ init-defs$)

lemma *StateUntilEqvWPrevChopsem:*

$\sigma \models (init\ w)\ \mathcal{U}\ f = (wprev\ (\Box\ (init\ w)));f$
by ($auto\ simp\ add: until-d-def\ wprev-defs\ always-defs\ init-defs\ chop-defs$)

lemma *StateUntilEqvWPrevChop:*

$\vdash (init\ w)\ \mathcal{U}\ f = (wprev\ (\Box\ (init\ w)));f$
using *StateUntilEqvWPrevChopsem Valid-def* **by** $blast$

lemma *UntilChopDist:*

$\vdash (init\ w)\ \mathcal{U}\ (g;h) = ((init\ w)\ \mathcal{U}\ g);h$
by ($metis\ ChopAssoc\ StateUntilEqvWPrevChop\ inteq-reflection$)

lemma *PiEmptysem:*

$\sigma \models (init\ w)\ \Pi\ empty = (init\ (\neg w))\ \mathcal{U}\ ((init\ w) \wedge wnext\ (\Box\ (init\ (\neg w))))$
proof –

have 1: $(\sigma \models (init\ w)\ \Pi\ empty) =$
 $((\exists x \in iset\ \sigma. w\ \langle x \rangle) \wedge ilen\ (ifilter\ (\lambda y. w\ \langle y \rangle)\ \sigma) = 0)$

by ($simp\ add: init-defs\ empty-defs\ Pistate$)

have 2: $((\exists x \in iset\ \sigma. w\ \langle x \rangle) \wedge ilen\ (ifilter\ (\lambda y. w\ \langle y \rangle)\ \sigma) = 0) =$
 $(\exists k \leq ilen\ \sigma. (\lambda y. w\ \langle y \rangle)\ (inth\ \sigma\ k) \wedge$
 $(\forall j. j < k \longrightarrow \neg (\lambda y. w\ \langle y \rangle)\ (inth\ \sigma\ j)) \wedge$
 $(\forall j \leq ilen\ \sigma. k < j \longrightarrow \neg (\lambda y. w\ \langle y \rangle)\ (inth\ \sigma\ j)))$

by ($simp\ add: ifilter-ilen-zero-conv-2$)

have 3: $(\exists k \leq ilen\ \sigma. (\lambda y. w\ \langle y \rangle)\ (inth\ \sigma\ k) \wedge$
 $(\forall j. j < k \longrightarrow \neg (\lambda y. w\ \langle y \rangle)\ (inth\ \sigma\ j)) \wedge$
 $(\forall j \leq ilen\ \sigma. k < j \longrightarrow \neg (\lambda y. w\ \langle y \rangle)\ (inth\ \sigma\ j))) =$
 $(\exists k \leq ilen\ \sigma.$
 $w\ \langle inth\ \sigma\ k \rangle \wedge$
 $(LIFT(wnext\ (\Box\ (init\ (\neg w)))))\ (suffix\ k\ \sigma) \wedge$
 $(\forall j < k. \neg w\ \langle inth\ \sigma\ j \rangle))$

```

    using wnextboornotstatesem
  by metis
have 4: ( $\exists k \leq \text{ilen } \sigma.$ 
   $w \langle \text{inth } \sigma \ k \rangle \wedge$ 
   $(\text{LIFT}(\text{wnext } (\Box (\text{init } (\neg w)))) (\text{suffix } k \ \sigma) \wedge$ 
   $(\forall j < k. \neg w \langle \text{inth } \sigma \ j \rangle)) =$ 
   $(\sigma \models (\text{init } (\neg w)) \mathcal{U} ((\text{init } w) \wedge \text{wnext } (\Box (\text{init } (\neg w))))))$ 

  by (simp add: NotStateUntilStateAndsem)
from 1 2 3 4 show ?thesis by auto
qed

```

```

lemma PiEmpty:
 $\vdash (\text{init } w) \Pi \text{ empty} = (\text{init } (\neg w)) \mathcal{U} ((\text{init } w) \wedge \text{wnext } (\Box (\text{init } (\neg w))))$ 
using PiEmptysem Valid-def by blast

```

```

lemma StatePiBoxStatesem:
 $\sigma \models (\text{init } w) \Pi f = (\text{init } w) \Pi (f \wedge \Box (\text{init } w))$ 
proof -
  have 1: ( $\sigma \models (\text{init } w) \Pi f =$ 
     $((\exists x \in \text{iset } \sigma. w \langle x \rangle) \wedge ((\text{ifilter } (\lambda y. w \langle y \rangle) \sigma) \models f))$ )

    by (metis Pistate)
  have 2: ( $\sigma \models (\text{init } w) \Pi (f \wedge \Box (\text{init } w)) =$ 
     $((\exists x \in \text{iset } \sigma. w \langle x \rangle) \wedge ((\text{ifilter } (\lambda y. w \langle y \rangle) \sigma) \models f \wedge \Box (\text{init } w)))$ )

    by (metis Pistate)
  have 3: ( $((\text{ifilter } (\lambda y. w \langle y \rangle) \sigma) \models f \wedge \Box (\text{init } w))$ 
     $= (f (\text{ifilter } (\lambda y. w \langle y \rangle) \sigma) \wedge$ 
     $(\forall n \leq \text{ilen } (\text{ifilter } (\lambda y. w \langle y \rangle) \sigma). w \langle \text{inth } (\text{ifilter } (\lambda y. w \langle y \rangle) \sigma) \ n \rangle))$ )
    by (simp add: always-defs init-defs)
  have 4: ( $((\exists x \in \text{iset } \sigma. w \langle x \rangle) \wedge (f (\text{ifilter } (\lambda y. w \langle y \rangle) \sigma) \wedge$ 
     $(\forall n \leq \text{ilen } (\text{ifilter } (\lambda y. w \langle y \rangle) \sigma). w \langle \text{inth } (\text{ifilter } (\lambda y. w \langle y \rangle) \sigma) \ n \rangle))) =$ 
     $((\exists x \in \text{iset } \sigma. w \langle x \rangle) \wedge f (\text{ifilter } (\lambda y. w \langle y \rangle) \sigma))$ )

    by (meson ifilter-inth-aa)
  show ?thesis using 1 2 3 4 by auto
qed

```

```

lemma StatePiBoxState:
 $\vdash (\text{init } w) \Pi f = (\text{init } w) \Pi (f \wedge \Box (\text{init } w))$ 
using StatePiBoxStatesem Valid-def by blast

```

```

lemma StatePiUntil1:
 $\vdash ((\text{init } (\neg w)) \mathcal{U} ((\text{init } w) \wedge (\text{init } w) \Pi f)) =$ 
 $(w\text{prev } (\Box (\text{init } (\neg w)))) ; ((\text{init } w) \wedge (\text{init } w) \Pi f)$ 
using StateUntilEqvWPrevChop by blast

```


lemma *StatePiUntilsem2*:

$(\sigma \models (wprev (\Box (init (\neg w))))); (init w) \wedge (init w) \Pi f) =$
 $(\sigma \models ((wprev (\Box (init (\neg w))))); ((init w) \wedge empty)) ; ((init w) \wedge (init w) \Pi f))$
by (*auto simp add: chop-defs init-defs empty-defs*)
fastforce

lemma *StatePiUntil2*:

$\vdash (wprev (\Box (init (\neg w)))); (init w) \wedge (init w) \Pi f =$
 $((wprev (\Box (init (\neg w)))); ((init w) \wedge empty)) ; ((init w) \wedge (init w) \Pi f)$
by (*simp add: StatePiUntilsem2 Valid-def*)

lemma *StatePiUntil3*:

$\vdash ((wprev (\Box (init (\neg w))))); ((init w) \wedge empty) =$
 $(((init (\neg w)) \mathcal{U} ((init w) \wedge wnext (\Box (init (\neg w))))) ; ((init w) \wedge empty))$

proof –

have 1: $\vdash (wprev (\Box (init (\neg w)))); ((init w) \wedge empty) =$
 $(init (\neg w)) \mathcal{U} ((init w) \wedge empty)$
by (*meson Prop11 StateUntilEqWPrevChop*)
have 2: $\vdash ((init w) \wedge empty) = ((init w) \wedge wnext (\Box (init (\neg w)))); ((init w) \wedge empty)$
by (*auto simp add: Valid-def init-defs empty-defs wnext-defs always-defs chop-defs*)
 $(metis Suc-pred eq-imp-le ilen-gr-zero le-neq-trans)$
show ?thesis **by** (*metis 1 2 UntilChopDist integ-reflection*)
qed

lemma *StatePiUntilsem4*:

$(\sigma \models ((init (\neg w)) \mathcal{U} ((init w) \wedge wnext (\Box (init (\neg w)))))) ; ((init w) \wedge empty) =$
 $(\sigma \models ((init w) \Pi empty); ((init w) \wedge empty))$
by (*metis PiEmpty integ-reflection*)

lemma *StatePiUntil4*:

$\vdash ((init (\neg w)) \mathcal{U} ((init w) \wedge wnext (\Box (init (\neg w))))) ; ((init w) \wedge empty) =$
 $(((init w) \Pi empty); ((init w) \wedge empty))$
by (*simp add: StatePiUntilsem4 Valid-def*)

lemma *StatePiUntilsem*:

$\sigma \models (init w) \Pi f = (init (\neg w)) \mathcal{U} ((init w) \wedge (init w) \Pi f)$

proof –

have 2: $(\sigma \models (init (\neg w)) \mathcal{U} ((init w) \wedge (init w) \Pi f)) =$
 $(\sigma \models (wprev (\Box (init (\neg w)))); (init w) \wedge (init w) \Pi f)$

using *StateUntilEqWPrevChopsem*[of *LIFT*($\neg w$) *LIFT*(($init w$) \wedge ($init w$) Πf) σ]
by *simp*

have 7: $(\sigma \models (wprev (\Box (init (\neg w)))); (init w) \wedge (init w) \Pi f) =$
 $(\sigma \models (((init w) \Pi empty); ((init w) \wedge empty)); (init w) \wedge (init w) \Pi f)$

by (*metis PiEmpty StatePiUntil2 StatePiUntil3 integ-reflection*)

have 8: $(\sigma \models (((init w) \Pi empty); ((init w) \wedge empty)); (init w) \wedge (init w) \Pi f) =$
 $(\sigma \models (((init w) \Pi empty); (init w) \wedge (init w) \Pi f))$

by (auto simp add: chop-defs init-defs empty-defs)
 fastforce
 have 9: $(\sigma \models ((init\ w) \amalg empty)); (init\ w) \wedge (init\ w) \amalg f =$
 $(\sigma \models (init\ w) \amalg (empty;f))$

 using PiChopDistsema PiChopDistsemb by blast
 have 10: $(\sigma \models (init\ w) \amalg (empty;f)) = (\sigma \models (init\ w) \amalg f)$
 by (metis chop-defs empty-defs prefix-ilen-good suffix-zero pi-d-def zero-order(1))
 show ?thesis
 by (simp add: 10 2 7 8 9)
 qed

lemma StatePiUntil:

$\vdash (init\ w) \amalg f = (init\ (\neg w)) \mathcal{U} ((init\ w) \wedge (init\ w) \amalg f)$
 using StatePiUntilsem by blast

lemma StateAndPiEmpty:

$\vdash ((init\ w) \wedge (init\ w) \amalg empty) = (w \wedge empty) ; (wnext\ (\Box (init\ (\neg w))))$
proof –
 have 1: $\vdash ((init\ w) \wedge (init\ w) \amalg empty) =$
 $((init\ w) \wedge (init\ (\neg w)) \mathcal{U} ((init\ w) \wedge (wnext\ (\Box (init\ (\neg w))))))$
 using PiEmpty by fastforce
 have 2: $\vdash ((init\ w) \wedge (init\ (\neg w)) \mathcal{U} ((init\ w) \wedge (wnext\ (\Box (init\ (\neg w)))))) =$
 $((init\ w) \wedge (wnext\ (\Box (init\ (\neg w)))))$
 by (auto simp add: until-d-def Valid-def init-defs)
 force
 have 3: $\vdash ((init\ w) \wedge (wnext\ (\Box (init\ (\neg w)))) = (w \wedge empty) ; (wnext\ (\Box (init\ (\neg w))))$
 by (metis InitAndEmptyEqvAndEmpty StateAndEmptyChop inteq-reflection)
 show ?thesis
 using 1 2 3 by fastforce
 qed

lemma PiPowerExpandsem:

$(\sigma \models (\exists k. (init\ w) \amalg (power\ f\ k))) =$
 $(\sigma \models (init\ w) \amalg empty \vee (\exists k. (init\ w) \amalg (f;(power\ f\ (k)))))$
proof –
 have 1: $(\sigma \models (\exists k. (init\ w) \amalg (power\ f\ k))) =$
 $(\exists k. (\sigma \models (init\ w) \amalg (power\ f\ k)))$
 by simp
 have 2: $(\exists k. (\sigma \models (init\ w) \amalg (power\ f\ k))) =$
 $((\sigma \models (init\ w) \amalg (power\ f\ 0)) \vee (\exists k. 1 \leq k \wedge (\sigma \models (init\ w) \amalg (power\ f\ (k)))))$
 by (metis One-nat-def diff-Suc-1 le-SucE le-add1 plus-1-eq-Suc)
 have 3: $(\sigma \models (init\ w) \amalg (power\ f\ 0)) = (\sigma \models (init\ w) \amalg empty)$
 by simp
 have 4: $(\exists k. 1 \leq k \wedge (\sigma \models (init\ w) \amalg (power\ f\ (k)))) =$
 $(\exists k. (\sigma \models (init\ w) \amalg (power\ f\ (Suc\ k))))$
 by (metis le-add1 ordered-cancel-comm-monoid-diff-class.add-diff-inverse plus-1-eq-Suc)
 have 5: $(\exists k. (\sigma \models (init\ w) \amalg (power\ f\ (Suc\ k)))) =$

$(\sigma \models (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k))))$

by *simp*

have 6: $((\sigma \models (init\ w) \sqcap empty) \vee (\sigma \models (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k)))) =$
 $(\sigma \models (init\ w) \sqcap empty \vee (\exists k. (init\ w) \sqcap (f;power\ f\ (k))))$

by *auto*

show *?thesis*

using 1 2 3 4 5 6 by *blast*

qed

lemma *PiPowerExpandsem1*:

$\forall \sigma. \sigma \models (\exists k. (init\ w) \sqcap (power\ f\ k)) =$
 $((init\ w) \sqcap empty \vee (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k))))$

proof

fix σ

show $\sigma \models (\exists k. (init\ w) \sqcap (power\ f\ k)) =$
 $((init\ w) \sqcap empty \vee (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k))))$

proof –

have 1: $(\sigma \models (\exists k. (init\ w) \sqcap (power\ f\ k)) =$
 $((init\ w) \sqcap empty \vee (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k))))$
 $= (\sigma \models (\exists k. (init\ w) \sqcap (power\ f\ k))) =$
 $(\sigma \models (init\ w) \sqcap empty \vee (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k))))$

by *auto*

have 2: $(\sigma \models (\exists k. (init\ w) \sqcap (power\ f\ k)) =$
 $(\sigma \models (init\ w) \sqcap empty \vee (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k))))$

using *PiPowerExpandsem[of w f σ]* by *simp*

show *?thesis*

using 1 2 by *blast*

qed

qed

lemma *PiPowerExpand*:

$\vdash (\exists k. (init\ w) \sqcap (power\ f\ k)) =$
 $((init\ w) \sqcap empty \vee (\exists k. (init\ w) \sqcap (power\ f\ (Suc\ k))))$

using *PiPowerExpandsem1[of w f]* by (auto simp add: *Valid-def PiPowerExpandsem1*)

lemma *exists-expand-sem*:

$(\sigma \models (\exists k. (power\ ((init\ w) \sqcap f \wedge fin\ w)\ k))) =$
 $((\sigma \models (power\ ((init\ w) \sqcap f \wedge fin\ w)\ 0)) \vee$
 $(\sigma \models (\exists k. (power\ ((init\ w) \sqcap f \wedge fin\ w)\ (Suc\ k))))$

by (metis (no-types, lifting) *not0-implies-Suc unl-Rex*)

lemma *exists-expand*:

$\vdash (\exists k. (power\ ((init\ w) \sqcap f \wedge fin\ w)\ k)) =$
 $((power\ ((init\ w) \sqcap f \wedge fin\ w)\ 0) \vee (\exists k. (power\ ((init\ w) \sqcap f \wedge fin\ w)\ (Suc\ k))))$

using *exists-expand-sem Valid-def* by *fastforce*

15.4.16 TruePiEqv

lemma *TruePiEqvsem*:

$\sigma \models \#True \Pi f = f$

by (*simp add: pi-d-def pifilt-true*) *auto*

lemma *TruePiEqv*:

$\vdash (\#True) \Pi f = f$

using *TruePiEqvsem* **by** (*auto simp add: Valid-def*)

15.4.17 BoxImpEqvPi

lemma *BoxImpEqvPi*:

$\vdash \Box f \longrightarrow g = f \Pi g$

by (*simp add: Valid-def always-defs pi-d-def pifilt-def sfxfilt-def*)
(*metis ifilter-True ilen-gr-zero inth-and-iset ilen-suffixes*
imap-first-suffixes inth-suffixes)

15.4.18 PiEqvDiamondUPi

lemma *PiEqvDiamondUPi*:

$\vdash f \Pi g = (\Diamond f \wedge f \Pi^u g)$

by (*simp add: Valid-def upi-d-def sometimes-defs pi-d-def,blast*)

15.4.19 PiEqvUntilPi

lemma *PiEqvUntilPi*:

$\vdash (init\ w) \Pi g = (init\ (\neg w)) \mathcal{U} ((init\ w) \Pi g)$

by (*metis StatePiUntil UntilUntil int-eq*)

15.4.20 UPiEqvBoxOrPi

lemma *UPiEqvBoxOrPi*:

$\vdash f \Pi^u g = (\Box (\neg f) \vee f \Pi g)$

by (*simp add: Valid-def upi-d-def always-defs pi-d-def,blast*)

15.5 Theorems

lemma *UPiImpRule*:

assumes $\vdash g1 \longrightarrow g2$

shows $\vdash f \Pi^u g1 \longrightarrow f \Pi^u g2$

using *assms*

by (*meson MP PiK PiN*)

lemma *UPiEqvRule*:

assumes $\vdash g1 = g2$

shows $\vdash f \Pi^u g1 = f \Pi^u g2$

proof –

have *1*: $\vdash g1 \longrightarrow g2$

using *assms* **by** (*simp add: int-iffD1*)

have *2*: $\vdash f \Pi^u g1 \longrightarrow f \Pi^u g2$

using *1 UPiImpRule* **by** *blast*

```

have 3:  $\vdash g2 \longrightarrow g1$ 
  using assms by (simp add: int-iffD2)
have 4:  $\vdash f \Pi^u g2 \longrightarrow f \Pi^u g1$ 
  using 3 UPiImpRule by blast
from 3 4 show ?thesis
by (simp add: 2 int-iffI)
qed

lemma PIEqvNotUPiNot:
 $\vdash f \Pi g = (\neg (f \Pi^u (\neg g)))$ 
by (simp add: upi-d-def)

lemma NotPIEqvNotUPi:
 $\vdash f \Pi (\neg g) = (\neg (f \Pi^u g))$ 
by (simp add: upi-d-def)

lemma UPiEqvNotPiNot:
 $\vdash f \Pi^u g = (\neg (f \Pi (\neg g)))$ 
by (simp add: upi-d-def)

lemma NotUPiEqvNotPi:
 $\vdash f \Pi^u (\neg g) = (\neg (f \Pi g))$ 
by (simp add: upi-d-def)

lemma PIImpRule:
assumes  $\vdash g1 \longrightarrow g2$ 
shows  $\vdash f \Pi g1 \longrightarrow f \Pi g2$ 
proof –
  have 1:  $\vdash \neg g2 \longrightarrow \neg g1$ 
    by (simp add: assms)
  have 2:  $\vdash f \Pi^u (\neg g2) \longrightarrow f \Pi^u (\neg g1)$ 
    using 1 UPiImpRule by blast
  have 3:  $\vdash \neg(f \Pi^u (\neg g1)) \longrightarrow \neg(f \Pi^u (\neg g2))$ 
    using 2 by fastforce
  from 3 show ?thesis using PIEqvNotUPiNot by fastforce
qed

lemma PIEqvRule:
assumes  $\vdash g1 = g2$ 
shows  $\vdash f \Pi g1 = f \Pi g2$ 
proof –
  have 1:  $\vdash g1 \longrightarrow g2$ 
    using assms by (simp add: int-iffD1)
  have 2:  $\vdash f \Pi g1 \longrightarrow f \Pi g2$ 
    using 1 PIImpRule by blast
  have 3:  $\vdash g2 \longrightarrow g1$ 
    using assms by (simp add: int-iffD2)
  have 4:  $\vdash f \Pi g2 \longrightarrow f \Pi g1$ 
    using 3 PIImpRule by blast
  from 2 4 show ?thesis by (simp add: int-iffI)

```

qed

lemma *UPiAndPiImpPiAnd*:

$\vdash f1 \Pi^u f \wedge f1 \Pi (\neg g) \longrightarrow f1 \Pi (f \wedge \neg g)$

proof –

have 1: $\vdash (\neg(f \longrightarrow g)) = (f \wedge \neg g)$

by *fastforce*

have 2: $\vdash (\neg(f1 \Pi^u (f \longrightarrow g))) = f1 \Pi (\neg(f \longrightarrow g))$

by (*simp add: NotPiEqvNotUPi int-iffD1 int-iffD2 int-iffI*)

have 3: $\vdash \neg(f1 \Pi^u f \longrightarrow f1 \Pi^u g) \longrightarrow \neg(f1 \Pi^u (f \longrightarrow g))$

by (*simp add: PiK*)

have 4: $\vdash (\neg(f1 \Pi^u f \longrightarrow f1 \Pi^u g)) = (f1 \Pi^u f \wedge f1 \Pi (\neg g))$

using *NotPiEqvNotUPi[of f1 g]* **by** *fastforce*

have 5: $\vdash f1 \Pi (\neg(f \longrightarrow g)) = f1 \Pi (f \wedge \neg g)$

using 1 **by** (*simp add: PiEqvRule*)

from 1 2 3 4 5 **show** *?thesis* **by** *fastforce*

qed

lemma *UPiAndPiImpPiAndA*:

$\vdash f1 \Pi^u f \wedge f1 \Pi g \longrightarrow f1 \Pi (f \wedge g)$

using *UPiAndPiImpPiAnd[of f1 f LIFT($\neg g$)]* **by** *fastforce*

lemma *PiAndPiImpPiAnd*:

$\vdash f1 \Pi f \wedge f1 \Pi g \longrightarrow f1 \Pi (f \wedge g)$

proof –

have 1: $\vdash f1 \Pi^u f \wedge f1 \Pi g \longrightarrow f1 \Pi (f \wedge g)$

using *UPiAndPiImpPiAndA* **by** *fastforce*

have 2: $\vdash f1 \Pi f \longrightarrow f1 \Pi^u f$

using *PiDc* **by** *blast*

from 1 2 **show** *?thesis* **by** *fastforce*

qed

lemma *PiAnd*:

$\vdash f \Pi (g1 \wedge g2) = (f \Pi g1 \wedge f \Pi g2)$

proof –

have 1: $\vdash f \Pi (g1 \wedge g2) \longrightarrow f \Pi g1$

by (*meson PiImpRule Prop12 int-iffD1 lift-and-com*)

have 2: $\vdash f \Pi (g1 \wedge g2) \longrightarrow f \Pi g2$

by (*meson PiImpRule Prop12 int-iffD1 lift-and-com*)

have 3: $\vdash f \Pi (g1 \wedge g2) \longrightarrow f \Pi g1 \wedge f \Pi g2$

using 1 2 **by** *fastforce*

have 4: $\vdash f \Pi g1 \wedge f \Pi g2 \longrightarrow f \Pi (g1 \wedge g2)$

by (*simp add: PiAndPiImpPiAnd*)

from 3 4 **show** *?thesis* **by** *fastforce*

qed

lemma *UPiAnd*:

$\vdash f \Pi^u (g1 \wedge g2) = (f \Pi^u g1 \wedge f \Pi^u g2)$

proof –

have 1: $\vdash f \Pi (\neg g1 \vee \neg g2) = (f \Pi (\neg g1) \vee f \Pi (\neg g2))$

by (*simp add: PiOr*)

have 2: $\vdash (\neg(f \Pi (\neg g1 \vee \neg g2))) = (\neg(f \Pi (\neg g1) \vee f \Pi (\neg g2)))$

using 1 **by** *fastforce*

have 3: $\vdash (\neg(f \Pi (\neg g1 \vee \neg g2))) = f \Pi^u (\neg(\neg g1 \vee \neg g2))$

by (*meson NotUPiEqvNotPi Prop11*)

have 4: $\vdash (\neg(\neg g1 \vee \neg g2)) = (g1 \wedge g2)$

by *fastforce*

have 5: $\vdash f \Pi^u (\neg(\neg g1 \vee \neg g2)) = f \Pi^u (g1 \wedge g2)$

using 4 **by** (*simp add: UPiEqvRule*)

have 6: $\vdash (\neg(f \Pi (\neg g1) \vee f \Pi (\neg g2))) = (\neg(f \Pi (\neg g1)) \wedge \neg(f \Pi (\neg g2)))$

by *fastforce*

have 7: $\vdash \neg(f \Pi (\neg g1)) = f \Pi^u g1$

by (*simp add: NotPiEqvNotUPi*)

have 8: $\vdash \neg(f \Pi (\neg g2)) = f \Pi^u g2$

by (*simp add: NotPiEqvNotUPi*)

have 9: $\vdash (\neg(f \Pi (\neg g1)) \wedge \neg(f \Pi (\neg g2))) = (f \Pi^u g1 \wedge f \Pi^u g2)$

using 6 7 8 **by** *fastforce*

from 2 3 5 6 9 **show** *?thesis* **by** *fastforce*

qed

lemma *UpiAndImp*:

$\vdash f \Pi^u (g1 \longrightarrow g2) \wedge f \Pi g1 \longrightarrow f \Pi g2$

proof –

have 2: $\vdash f \Pi^u ((\neg g2) \longrightarrow (\neg g1)) \longrightarrow (f \Pi^u (\neg g2) \longrightarrow f \Pi^u (\neg g1))$

using *PiK* **by** *blast*

have 3: $\vdash (f \Pi^u (\neg g2) \longrightarrow f \Pi^u (\neg g1)) = ((\neg(f \Pi^u (\neg g1))) \longrightarrow (\neg(f \Pi^u (\neg g2))))$

by *auto*

have 4: $\vdash (\neg(f \Pi^u (\neg g2))) = f \Pi g2$

by (*simp add: upi-d-def*)

have 5: $\vdash (\neg(f \Pi^u (\neg g1))) = f \Pi g1$

by (*simp add: upi-d-def*)

have 6: $\vdash f \Pi^u ((\neg g2) \longrightarrow (\neg g1)) = f \Pi^u (g1 \longrightarrow g2)$

by *simp*

have 7: $\vdash (f \Pi^u (g1 \longrightarrow g2) \wedge f \Pi g1 \longrightarrow f \Pi g2) =$

$(f \Pi^u (g1 \longrightarrow g2) \longrightarrow (f \Pi g1 \longrightarrow f \Pi g2))$

by *auto*

show *?thesis*

using 2 4 5 **by** *fastforce*

qed

lemma *BoxImpUPiBox*:

$\vdash \Box (init\ w) \longrightarrow f \Pi^u (\Box (init\ w))$

proof –

```

have 1:  $\vdash f \Pi (\Diamond (init (\neg w))) \longrightarrow \Diamond (init (\neg w))$ 
  by (simp add: PiDiamondImp)
have 2:  $\vdash \neg \Diamond (init (\neg w)) \longrightarrow \neg (f \Pi (\Diamond (init (\neg w))))$ 
  using 1 by auto
have 3:  $\vdash (\neg \Diamond (init (\neg w))) = \Box (init w)$ 
  by (metis 2 Initprop(2) Prop10 always-d-def inteq-reflection)
have 4:  $\vdash (\neg (f \Pi (\Diamond (init (\neg w)))) = f \Pi^u (\Box (init w))$ 
  by (simp add: upi-d-def)
  (metis 3 int-simps(4) inteq-reflection)
show ?thesis
using 2 3 4 by fastforce
qed

```

lemma *WPrevPi*:

```

 $\vdash (init w) \Pi f = (wprev (\Box (init (\neg w)))); ((init w) \wedge (init w) \Pi f)$ 
using StatePiUntil StatePiUntil1 by fastforce

```

lemma *PiChopstarhelp2a*:

```

 $\vdash (w \wedge empty); (power (f; (w \wedge empty)) \wedge more) k =$ 
   $(power (((w \wedge empty); f) \wedge more) k); (w \wedge empty)$ 
proof (induction k)
case 0
then show ?case
by (metis ChopEmpty EmptyChop inteq-reflection pow-0)
next
case (Suc k)
then show ?case
  proof –
    have 1:  $\vdash (w \wedge empty); power (f; (w \wedge empty) \wedge more) (Suc k) =$ 
       $(w \wedge empty); ((f; (w \wedge empty) \wedge more); power (f; (w \wedge empty) \wedge more) k)$ 
      by simp
    have 11:  $\vdash (f \wedge more); (w \wedge empty) = (fin w \wedge (f \wedge more))$ 
      by (metis FinEqvTrueChopAndEmpty TrueChopAndEmptyEqvChopAndEmpty inteq-reflection)
    have 12:  $\vdash (f; (w \wedge empty)) = (fin w \wedge f)$ 
      by (meson AndFinEqvChopAndEmpty Prop04 lift-and-com)
    have 13:  $\vdash (f; (w \wedge empty) \wedge more) = ((fin w \wedge f) \wedge more)$ 
      using 12 by auto
    have 14:  $\vdash ((fin w \wedge f) \wedge more) = (fin w \wedge (f \wedge more))$ 
      by fastforce
    have 2:  $\vdash (f; (w \wedge empty) \wedge more) = (f \wedge more); (w \wedge empty)$ 
      using 11 13 by fastforce
    have 20:  $\vdash ((f; (w \wedge empty) \wedge more); power (f; (w \wedge empty) \wedge more) k) =$ 
       $((f \wedge more); (w \wedge empty)); power (f; (w \wedge empty) \wedge more) k$ 
      by (simp add: 2 LeftChopEqvChop)
    have 21:  $\vdash ((f \wedge more); (w \wedge empty)); power (f; (w \wedge empty) \wedge more) k =$ 
       $(f \wedge more); ((w \wedge empty); power (f; (w \wedge empty) \wedge more) k)$ 
      using ChopAssocB by blast
    have 22:  $\vdash (f \wedge more); ((w \wedge empty); power (f; (w \wedge empty) \wedge more) k) =$ 
       $((f \wedge more); (power ((w \wedge empty); f \wedge more) k; (w \wedge empty)))$ 

```


by (simp add: RightChopEqvChop Suc.IH)
 have 3: $\vdash (w \wedge \text{empty}); (f; (w \wedge \text{empty}) \wedge \text{more}); \text{power } (f; (w \wedge \text{empty}) \wedge \text{more}) k =$
 $(w \wedge \text{empty}); (f \wedge \text{more}); (\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty}))$
 by (metis 1 2 21 22 inteq-reflection)
 have 4: $\vdash (w \wedge \text{empty}); (f \wedge \text{more}); (\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty})) =$
 $((w \wedge \text{empty}); f \wedge \text{more}); ((\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty})))$
 using ChopAssoc by blast
 have 5: $\vdash (w \wedge \text{empty}); f \wedge \text{more} = ((w \wedge \text{empty}); f \wedge \text{more})$
 by (auto simp add: Valid-def chop-defs empty-defs more-defs)
 have 6: $\vdash ((w \wedge \text{empty}); f \wedge \text{more}); ((\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty}))) =$
 $((w \wedge \text{empty}); f \wedge \text{more}); ((\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty})))$
 using 5 LeftChopEqvChop by blast
 have 7: $\vdash ((w \wedge \text{empty}); f \wedge \text{more}); ((\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty}))) =$
 $((\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) (\text{Suc } k); (w \wedge \text{empty})))$
 by (simp add: ChopAssoc)
 show ?thesis
 by (metis 1 3 4 5 7 int-eq)
 qed
 qed

lemma PiChopstarhelp2:

$\vdash (w \wedge \text{empty}); (f; (w \wedge \text{empty}))^* = ((w \wedge \text{empty}); f)^*; (w \wedge \text{empty})$
proof –
 have 1: $\vdash (w \wedge \text{empty}); (f; (w \wedge \text{empty}))^* = (w \wedge \text{empty}); (\exists k. \text{power } (f; (w \wedge \text{empty})) \wedge \text{more}) k$
 by (simp add: chopstar-d-def powerstar-d-def)
 have 2: $\vdash (w \wedge \text{empty}); (\exists k. \text{power } (f; (w \wedge \text{empty})) \wedge \text{more}) k =$
 $(\exists k. (w \wedge \text{empty}); (\text{power } (f; (w \wedge \text{empty})) \wedge \text{more}) k)$
 using ChopExist by fastforce
 have 3: $\vdash (\exists k. (w \wedge \text{empty}); (\text{power } (f; (w \wedge \text{empty})) \wedge \text{more}) k) =$
 $(\exists k. (\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty})))$

 by (simp add: ExEqvRule PiChopstarhelp2a)
 have 4: $\vdash (\exists k. (\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k; (w \wedge \text{empty}))) =$
 $(\exists k. (\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k); (w \wedge \text{empty}))$
 using ExistChop by fastforce
 have 5: $\vdash (\exists k. (\text{power } ((w \wedge \text{empty}); f \wedge \text{more}) k); (w \wedge \text{empty})) =$
 $((w \wedge \text{empty}); f)^*; (w \wedge \text{empty})$
 by (simp add: chopstar-d-def powerstar-d-def)
 show ?thesis
 using 1 2 3 4 5 by fastforce
 qed

lemma PiPowerSuca:

$\vdash (\text{init } w) \amalg (\text{power } f (\text{Suc } k)) = ((\text{init } w) \amalg f); ((\text{init } w) \wedge (\text{init } w) \amalg (\text{power } f k))$
 by (simp add: PiChopDist)

lemma PiPowerSuchb:

$\vdash (\text{init } w) \amalg (\text{power } f (\text{Suc } k)) = ((\text{init } w) \amalg f \wedge \text{fin } w); ((\text{init } w) \wedge (\text{init } w) \amalg (\text{power } f k))$
 by (metis (no-types, lifting) AndFinChopEqvStateAndChop ChopAssoc ChopImpDiamond FinChopEqvDiamond)

*FinEqvTrueChopAndEmpty InitAndEmptyEqvAndEmpty PiPowerSuca Prop10 StateAndEmptyChop
inteq-reflection)*

lemma *PiPowerSucc:*

$\vdash (init\ w) \Pi (power\ f\ (Suc\ k)) =$
 $(power\ ((init\ w) \Pi f \wedge fin\ w)\ (Suc\ k));((init\ w) \wedge (init\ w) \Pi empty)$

proof (*induction k*)

case *0*

then show *?case*

by (*metis ChopEmpty PiPowerSuccb inteq-reflection pow-0 pow-Suc*)

next

case (*Suc k*)

then show *?case*

by (*metis (no-types, opaque-lifting) AndFinEqvChopAndEmpty ChopAssocB InitAndEmptyEqvAndEmpty*

PiChopDist StateAndEmptyChop TrueW int-simps(1) inteq-reflection lift-and-com pow-Suc)

qed

lemma *PiPowerSuccd:*

$\vdash (init\ w) \Pi (power\ f\ (Suc\ k)) = (power\ ((init\ w) \Pi f \wedge fin\ w)\ (Suc\ k));((init\ w) \Pi empty)$

proof (*induction k*)

case *0*

then show *?case*

by (*metis (no-types, lifting) AndFinChopEqvStateAndChop AndFinEqvChopAndEmpty*
InitAndEmptyEqvAndEmpty PiPowerSuca int-eq pow-0 pow-Suc)

next

case (*Suc k*)

then show *?case*

by (*metis (no-types, lifting) ChopAssoc PiPowerSucc inteq-reflection pow-Suc*)

qed

lemma *PiChopstar:*

$\vdash (init\ w) \Pi (f^*) = (init\ (\neg w)) \mathcal{U} ((init\ w) \wedge (((init\ w) \Pi f) \wedge fin\ w)^*; wnext(\Box (init\ (\neg w))))$

proof –

have *1:* $\vdash (init\ w) \Pi (f^*) = (init\ w) \Pi (\exists k. power\ f\ k)$

by (*metis ChopstarEqvPowerstar PiEqvRule powerstar-d-def*)

have *2:* $\vdash (init\ w) \Pi (\exists k. power\ f\ k) = (\exists k. (init\ w) \Pi (power\ f\ k))$

by (*simp add: Valid-def pi-d-def*)

have *3:* $\vdash (\exists k. (init\ w) \Pi (power\ f\ k)) =$
 $((init\ w) \Pi empty \vee (\exists k. (init\ w) \Pi (power\ f\ (Suc\ k))))$

using *PiPowerExpand* **by** *auto*

have *4:* $\vdash (\exists k. (init\ w) \Pi (power\ f\ (Suc\ k))) =$
 $(\exists k. (power\ ((init\ w) \Pi f \wedge fin\ w)\ (Suc\ k));((init\ w) \Pi empty))$

by (*meson ExEqvRule PiPowerSuccd*)

have *5:* $\vdash (init\ w) \Pi empty = empty ; ((init\ w) \Pi empty)$

by (*simp add: EmptyChop int-iffD1 int-iffD2 int-iffI*)

have *6:* $\vdash (\exists k. (power\ ((init\ w) \Pi f \wedge fin\ w)\ (Suc\ k));((init\ w) \Pi empty)) =$
 $(\exists k. (power\ ((init\ w) \Pi f \wedge fin\ w)\ (Suc\ k));((init\ w) \Pi empty))$

by (*simp add: Semantics.ExistChop*)

have 7: $\vdash ((init\ w) \Pi\ empty \vee (\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k)); ((init\ w) \Pi\ empty))) =$
 $(\ empty; ((init\ w) \Pi\ empty) \vee$
 $(\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k)); ((init\ w) \Pi\ empty)))$
using 5 6 **by** fastforce
have 8: $\vdash (\ empty; ((init\ w) \Pi\ empty) \vee$
 $(\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k)); ((init\ w) \Pi\ empty)) =$
 $(\ empty \vee (\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k)); ((init\ w) \Pi\ empty)))$
by (meson OrChopEqv Prop11)
have 9: $\vdash\ empty = (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ 0)$
by simp
have 10: $\vdash (\ empty \vee (\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k)))) =$
 $(\ (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ 0) \vee (\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k))))$
by simp
have 11: $\vdash ((power\ ((init\ w) \Pi\ f \wedge fin\ w)\ 0) \vee (\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k)))) =$
 $(\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ k))$
using exists-expand[of w f] **by** fastforce
have 12: $\vdash ((init\ w) \Pi\ empty \vee$
 $(\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (Suc\ k)); ((init\ w) \Pi\ empty))) =$
 $(\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (k)); ((init\ w) \Pi\ empty))$
by (metis 11 7 8 9 inteq-reflection)
have 13: $\vdash (\exists k. (power\ ((init\ w) \Pi\ f \wedge fin\ w)\ (k)); ((init\ w) \Pi\ empty) =$
 $((init\ w) \Pi\ f \wedge fin\ w)^* ; ((init\ w) \Pi\ empty))$
by (metis ChopstarEqvPowerstar LeftChopEqvChop inteq-reflection powerstar-d-def)
have 14: $\vdash ((init\ w) \Pi\ f \wedge fin\ w)^* ; ((init\ w) \Pi\ empty) =$
 $(\ ((init\ w) \Pi\ empty) \vee$
 $((init\ w) \Pi\ f \wedge fin\ w); ((init\ w) \Pi\ f \wedge fin\ w)^* ; ((init\ w) \Pi\ empty))$
using CSChopEqvOrChopPlusChop **by** blast
have 15: $\vdash ((init\ w) \Pi\ empty) = (init\ (\neg w)) \mathcal{U} ((init\ w) \wedge wnext\ (\Box\ (init\ (\neg w))))$
by (simp add: PiEmpty)
have 16: $\vdash (((init\ w) \Pi\ f \wedge fin\ w); ((init\ w) \Pi\ f \wedge fin\ w)^* ; ((init\ w) \Pi\ empty) =$
 $((init\ w) \Pi\ f \wedge fin\ w); ((init\ w) \Pi\ f \wedge fin\ w)^* ; wnext\ (\Box\ (init\ (\neg w))))$
by (metis (no-types, lifting) AndFinEqvChopAndEmpty ChopAssoc InitAndEmptyEqvAndEmpty
PiChopstarhelp2 StateAndEmptyChop StateAndPiEmpty inteq-reflection)
have 17: $\vdash ((init\ w) \Pi\ f \wedge fin\ w) = ((init\ w) \Pi\ f); ((init\ w) \wedge empty)$
by (metis AndFinEqvChopAndEmpty InitAndEmptyEqvAndEmpty inteq-reflection)
have 18: $\vdash ((init\ w) \Pi\ f) = wprev(\Box\ (init\ (\neg w))); ((init\ w) \wedge (init\ w) \Pi\ f)$
by (simp add: WPrevPi)
have 19: $\vdash wprev(\Box\ (init\ (\neg w))); ((init\ w) \wedge (init\ w) \Pi\ f) =$
 $wprev(\Box\ (init\ (\neg w))); ((init\ w) \wedge empty) ; ((init\ w) \Pi\ f)$
by (meson RightChopImpChop StateAndEmptyChop int-iffD1 int-iffD2 int-iffI)
have 20: $\vdash wprev(\Box\ (init\ (\neg w))); (((init\ w) \wedge empty) ; ((init\ w) \Pi\ f)) =$
 $(init\ (\neg w)) \mathcal{U} ((init\ w) \wedge ((init\ w) \Pi\ f))$
using 18 19 StatePiUntil **by** fastforce
have 21: $\vdash (((init\ w) \Pi\ f \wedge fin\ w); ((init\ w) \Pi\ f \wedge fin\ w)^* ; wnext\ (\Box\ (init\ (\neg w)))) =$
 $(init\ (\neg w)) \mathcal{U} ($
 $(init\ w) \wedge$
 $((init\ w) \Pi\ f) \wedge fin\ w; ((init\ w) \Pi\ f \wedge fin\ w)^* ; wnext\ (\Box\ (init\ (\neg w))))$
 $)$
by (metis (no-types, lifting) 17 18 AndFinChopEqvStateAndChop ChopAssocB
StateUntilEqvWPrevChop int-eq)

```

have 22: ⊢ ((init (¬ w))  $\mathcal{U}$  ( (init w) ∧ wnext (□ (init (¬ w)))) ∨
  (init (¬ w))  $\mathcal{U}$  (
    (init w) ∧
    (((init w)  $\Pi$  f) ∧ fin w);((init w)  $\Pi$  f ∧ fin w)*;wnext (□ (init (¬ w)))
  ) ) =
  (init (¬ w))  $\mathcal{U}$  (
    ((init w) ∧ wnext (□ (init (¬ w)))) ∨
    ( (init w) ∧ (((init w)  $\Pi$  f) ∧ fin w);((init w)  $\Pi$  f ∧ fin w)*;wnext (□ (init (¬ w))))
  )
using UntilOrDist by fastforce
have 23: ⊢ (
  ((init w) ∧ wnext (□ (init (¬ w)))) ∨
  ( (init w) ∧ (((init w)  $\Pi$  f) ∧ fin w);((init w)  $\Pi$  f ∧ fin w)*;wnext (□ (init (¬ w)))
) =
  ( (init w) ∧ (((init w)  $\Pi$  f) ∧ fin w)*;wnext(□ (init (¬ w))))
by (metis (no-types, lifting) CSChopEqvOrChopPlusChop ChopOrEqv StateAndEmptyChop
  integ-reflection)
have 24: ⊢ (init w)  $\Pi$  (f*) = ( (init w)  $\Pi$  empty ∨ (∃ k. (init w)  $\Pi$  (power f (Suc k))))
using 1 2 3 by fastforce
have 25: ⊢ ( (init w)  $\Pi$  empty ∨ (∃ k. (init w)  $\Pi$  (power f (Suc k)))) =
  ( (init w)  $\Pi$  empty ∨ (∃ k. (power ((init w)  $\Pi$  f ∧ fin w) (Suc k));((init w)  $\Pi$  empty)))
using 4 by fastforce
have 26: ⊢ ( (init w)  $\Pi$  empty ∨
  (∃ k. (power ((init w)  $\Pi$  f ∧ fin w) (Suc k));((init w)  $\Pi$  empty))) =
  ((init w)  $\Pi$  f ∧ fin w)* ; ((init w)  $\Pi$  empty)
using 12 13 by fastforce
have 27: ⊢ ((init w)  $\Pi$  f ∧ fin w)* ; ((init w)  $\Pi$  empty) =
  (init (¬ w))  $\mathcal{U}$  ((init w) ∧ (((init w)  $\Pi$  f) ∧ fin w)*;wnext(□ (init (¬ w))))
by (metis 14 15 16 21 22 23 integ-reflection)
from 24 25 26 27 show ?thesis by fastforce
qed

```

end

16 Interval Temporal Algebra

```

theory ITA
imports Fuse Semantics TimeReversal
begin

```

16.1 Definition of Set of intervals and Operations on them

```

type-synonym 'a intervals = 'a interval set

```

definition $lan :: ('a :: world) formula \Rightarrow 'a intervals$
where $lan f = \{ \sigma . (\sigma \models f) \}$

definition $fusion :: 'a intervals \Rightarrow 'a intervals \Rightarrow 'a intervals$ (**infixl** \cdot 70)
where $X \cdot Y = \{fuse \ \sigma 1 \ \sigma 2 \mid \sigma 1 \ \sigma 2. \ \sigma 1 \in X \wedge \sigma 2 \in Y \wedge ilast \ \sigma 1 = ifirst \ \sigma 2\}$

definition $reverse :: 'a intervals \Rightarrow 'a intervals$ ($(SRev \ -)$ [85] 85)
where $(SRev \ X) = \{irev \ \sigma \mid \sigma. \ \sigma \in X \}$

definition $empty :: 'a intervals$ ($SEmpty$)
where
 $SEmpty \equiv range \ INil$

definition $smore :: 'a intervals$ ($SMore$)
where
 $SMore \equiv - \ SEmpty$

definition $sskip :: 'a intervals$ ($SSkip$)
where
 $SSkip \equiv -(SEmpty \cup (SMore \cdot SMore))$

definition $sfalse :: 'a intervals$ ($SFalse$)
where
 $SFalse \equiv \{\}$

definition $strue :: 'a intervals$ ($STrue$)
where
 $STrue \equiv -\{\}$

definition $sinit :: 'a intervals \Rightarrow 'a intervals$ ($(SInit \ -)$ [85] 85)
where
 $SInit \ X \equiv (X \cap SEmpty) \cdot STrue$

definition $sfin :: 'a intervals \Rightarrow 'a intervals$ ($(SFin \ -)$ [85] 85)
where
 $SFin \ X \equiv STrue \cdot (X \cap SEmpty)$

definition $ssometime :: 'a intervals \Rightarrow 'a intervals$ ($(SSometime \ -)$ [85] 85)
where
 $SSometime \ X \equiv STrue \cdot X$

definition $salways :: 'a intervals \Rightarrow 'a intervals$ ($(SAlways \ -)$ [85] 85)
where
 $SAlways \ X \equiv -(SSometime \ (-X))$

definition $sdi :: 'a intervals \Rightarrow 'a intervals$ ($(SDi \ -)$ [85] 85)
where
 $SDi \ X \equiv X \cdot STrue$

definition $sbi :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((Sbi \ -) [85] \ 85)$

where

$$Sbi \ X \equiv -(SDi \ (-X))$$

definition $sda :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((SDa \ -) [85] \ 85)$

where

$$SDa \ X \equiv STrue \cdot X \cdot STrue$$

definition $sba :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((SBa \ -) [85] \ 85)$

where

$$SBa \ X \equiv -(SDa \ (-X))$$

definition $snext :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((SNext \ -) [85] \ 85)$

where

$$SNext \ X \equiv SSkip \cdot X$$

definition $swnext :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((SWnext \ -) [85] \ 85)$

where

$$SWnext \ X \equiv (- (SSkip \cdot (-X)))$$

definition $sprev :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((SPrev \ -) [85] \ 85)$

where

$$SPrev \ X \equiv X \cdot SSkip$$

definition $swprev :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((SWprev \ -) [85] \ 85)$

where

$$SWprev \ X \equiv (-((-X) \cdot SSkip))$$

primrec $spower :: 'a \text{ intervals} \Rightarrow \text{nat} \Rightarrow 'a \text{ intervals} ((SPower \ - \ -) [88,88] \ 87)$

where

$$\begin{aligned} \text{pwr-0} & : SPower \ X \ 0 = SEmpty \\ | \text{pwr-Suc} & : SPower \ X \ (Suc \ n) = ((X \cap SMore) \cdot (SPower \ X \ n)) \end{aligned}$$

definition $sstar :: 'a \text{ intervals} \Rightarrow 'a \text{ intervals} ((SStar \ -) [85] \ 85)$

where

$$SStar \ X \equiv (\bigcup n. SPower \ X \ n)$$

16.2 Simplification Lemmas

lemma *snot-elim* :

$$x \in -X \longleftrightarrow x \notin X$$

by *simp*

lemma *sor-elim* :

$$x \in (X \cup Y) \longleftrightarrow (x \in X \vee x \in Y)$$

by *simp*

lemma *sand-elim* :

$$x \in (X \cap Y) \longleftrightarrow (x \in X \wedge x \in Y)$$

by *simp*

lemma *sfalse-elim* :

$\sigma \notin SFalse$

by (*simp add: sfalse-def*)

lemma *strue-elim* :

$\sigma \in STrue$

by (*simp add: strue-def*)

lemma *empty-elim* :

$\sigma \in SEmpty \longleftrightarrow \text{ilen } \sigma = 0$

by (*simp add: image-iff INil-ilen empty-def*)

lemma *smore-elim* :

$\sigma \in SMore \longleftrightarrow \text{ilen } \sigma > 0$

by (*simp add: empty-elim smore-def*)

lemma *fusion-iff*:

$\sigma \in X \cdot Y \longleftrightarrow (\exists \sigma 1 \sigma 2. \sigma = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in X \wedge \sigma 2 \in Y \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2)$

by (*unfold fusion-def*) *auto*

lemma *fusion-iff-1*:

$\sigma \in X \cdot Y \longleftrightarrow (\exists i \leq \text{ilen } \sigma. (\text{prefix } i \sigma) \in X \wedge (\text{suffix } i \sigma) \in Y)$

using *fusion-iff[of σ X Y]*

by (*metis ilast-ifirst fuse-ilen fuse-prefix-suffix prefix-fuse suffix-fuse le-add1*)

lemma *smore-fusion-smore* :

$\sigma \in (SMore \cdot SMore) \longleftrightarrow \text{ilen } \sigma > 1$

using *fusion-iff-1*

by (*metis prefix-ilen-good suffix-ilen-good less-one*

not-less not-less-iff-gr-or-eq smore-elim zero-less-diff)

lemma *sskip-elim* :

$\sigma \in SSkip \longleftrightarrow \text{ilen } \sigma = 1$

using *sskip-def smore-fusion-smore*

by (*metis One-nat-def Suc-lessI Un-iff less-numeral-extra(4) empty-elim smore-def smore-elim snot-elim zero-neq-one*)

lemma *spower-elim-zero* :

$\sigma \in SPower X 0 \longleftrightarrow \sigma \in SEmpty$

by *simp*

lemma *spower-elim-suc* :

$\sigma \in SPower X (Suc n) \longleftrightarrow \sigma \in (X \cap SMore) \cdot (SPower X n)$

by *simp*

lemma *spower-elim-suc-1* :

$\sigma \in (X \cap SMore) \cdot (SPower X n) \longleftrightarrow$

$(\exists \sigma 1 \sigma 2. \sigma = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in X \wedge \text{ilen } \sigma 1 > 0 \wedge \sigma 2 \in (SPower X n) \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2)$

by (*meson IntD1 IntD2 IntI smore-elim fusion-iff*)

lemma *sstar-elim* :

$\sigma \in SStar\ X \longleftrightarrow (\exists\ n.\ \sigma \in SPower\ X\ n)$

by (*simp add: sstar-def*)

lemma *sstar-elim-1* :

$(\exists\ n.\ \sigma \in SPower\ X\ n) \longleftrightarrow$
 $(\sigma \in SPower\ X\ 0 \vee (\exists\ n.\ \sigma \in SPower\ X\ (Suc\ n)))$

by (*metis not0-implies-Suc*)

lemma *spower-suc* :

$(\exists\ n.\ \sigma \in SPower\ X\ (Suc\ n)) \longleftrightarrow$
 $(\exists\ n.\ \sigma \in (X \cap SMore) \cdot (SPower\ X\ n))$

by *simp*

lemma *spower-suc-1* :

$(\exists\ n.\ \sigma \in (X \cap SMore) \cdot (SPower\ X\ n)) \longleftrightarrow$
 $\sigma \in (X \cap SMore) \cdot (SStar\ X)$

by (*metis fusion-iff sstar-elim*)

lemma *sstar-equiv* :

$\sigma \in SStar\ X \longleftrightarrow$
 $(\sigma \in SEmpty \vee \sigma \in (X \cap SMore) \cdot (SStar\ X))$

by (*metis spower.simps(1) spower-elim-suc spower-suc-1 sstar-elim sstar-elim-1*)

lemma *spower-skip-elim* :

$(\sigma \in SPower\ SSkip\ n) \longleftrightarrow\ ilen\ \sigma = n$

proof (*induct n arbitrary: σ*)

case 0

then show ?case **by** (*auto simp add: empty-elim*)

next

case (*Suc n*)

then show ?case

proof *auto*

show $(\bigwedge\sigma. (\sigma \in SPower\ SSkip\ n) = (ilen\ \sigma = n)) \implies$
 $\sigma \in SSkip \cap SMore \cdot SPower\ SSkip\ n \implies ilen\ \sigma = Suc\ n$

by (*metis Suc.hyps fuse-ilen plus-1-eq-Suc spower-elim-suc-1 skip-elim*)

show $(\bigwedge\sigma. (\sigma \in SPower\ SSkip\ n) = (ilen\ \sigma = n)) \implies$
 $ilen\ \sigma = Suc\ n \implies \sigma \in SSkip \cap SMore \cdot SPower\ SSkip\ n$

by (*metis Suc.hyps Compl-Un Int-commute add-diff-cancel-left' fusion-iff-1 inf-sup-aci(4)*
prefix-ilen-good suffix-ilen-good le-add1 plus-1-eq-Suc smore-def
sskip-def skip-elim)

qed

qed

lemma *srev-elim*:

$\sigma \in (SRev\ X) \longleftrightarrow\ irev\ \sigma \in X$

using *irev-swap* by (*auto simp add: reverse-def*)

16.3 Algebraic Laws

16.3.1 Commutative Additive Monoid

lemma *UnionCommute*:

$$(X::'a \text{ intervals}) \cup Y = Y \cup X$$

by (*simp add: Un-commute*)

lemma *UnionSFalse*:

$$X \cup SFalse = X$$

by (*simp add: sfalse-def*)

lemma *UnionAssoc*:

$$(X::'a \text{ intervals}) \cup (Y \cup Z) = (X \cup Y) \cup Z$$

by (*simp add: sup-assoc*)

16.3.2 Boolean algebra

lemma *Huntington*:

$$(X::'a \text{ intervals}) = -(-X \cup -Y) \cup -(-X \cup Y)$$

by *auto*

lemma *Morgan*:

$$(X::'a \text{ intervals}) \cap Y = -(-X \cup -Y)$$

by *auto*

— identities

lemma *STrueTop*:

$$STrue = X \cup -X$$

by (*simp add: strue-def*)

lemma *SFalseBottom*:

$$SFalse = X \cap -X$$

by (*simp add: sfalse-def*)

16.3.3 multiplicative monoid

lemma *FusionSEmptyL* :

$$SEmpty \cdot X = X$$

using *fusion-iff-1 set-eqI*[of *SEmpty.X X*]

by (*metis ilen-gr-zero prefix-ilen-good suffix-zero empty-elim*)

lemma *FusionSEmptyR* :

$$X \cdot SEmpty = X$$

using *fusion-iff-1 set-eqI*[of *X.SEmpty X*]

proof *auto*

show $\bigwedge x. (\bigwedge \sigma \ X \ Y. (\sigma \in X \cdot Y) = (\exists i \leq \text{ilen } \sigma. \text{prefix } i \ \sigma \in X \wedge \text{suffix } i \ \sigma \in Y)) \implies$
 $((\bigwedge x. (x \in X \cdot SEmpty) = (x \in X)) \implies X \cdot SEmpty = X) \implies x \in X \cdot SEmpty \implies x \in X$

by (*metis diff-diff-cancel diff-zero fusion-iff-1 prefix-ilen*
suffix-ilen-good empty-elim)
show $\bigwedge x. (\bigwedge \sigma. X \cdot Y. (\sigma \in X \cdot Y) = (\exists i \leq \text{ilen } \sigma. \text{prefix } i \sigma \in X \wedge \text{suffix } i \sigma \in Y)) \implies$
 $((\bigwedge x. (x \in X \cdot \text{SEmpty}) = (x \in X)) \implies X \cdot \text{SEmpty} = X) \implies x \in X \implies x \in X \cdot \text{SEmpty})$
using *empty-elim fusion-iff-1* **by** *fastforce*
qed

lemma *FusionAssocA*:

assumes $x \in X \cdot (Y \cdot Z)$

shows $x \in (X \cdot Y) \cdot Z$

proof –

have 1: $(\exists \sigma 1 \sigma 2. x = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in X \wedge \sigma 2 \in Y \cdot Z \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2)$

using *assms fusion-iff*[*of x X Y · Z*] **by** *auto*

obtain $\sigma 1 \sigma 2$ **where** 2: $x = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in X \wedge \sigma 2 \in Y \cdot Z \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2$

using 1 **by** *auto*

have 3: $(\exists \sigma 3 \sigma 4. \sigma 2 = \text{fuse } \sigma 3 \sigma 4 \wedge \sigma 3 \in Y \wedge \sigma 4 \in Z \wedge \text{ilast } \sigma 3 = \text{ifirst } \sigma 4)$

using 2 *fusion-iff*[*of σ 2 Y Z*] **by** *auto*

obtain $\sigma 3 \sigma 4$ **where** 4: $\sigma 2 = \text{fuse } \sigma 3 \sigma 4 \wedge \sigma 3 \in Y \wedge \sigma 4 \in Z \wedge \text{ilast } \sigma 3 = \text{ifirst } \sigma 4$

using 3 **by** *auto*

have 5: $x = \text{fuse } \sigma 1 (\text{fuse } \sigma 3 \sigma 4)$

using 2 4 **by** *auto*

have 6: $x = \text{fuse } (\text{fuse } \sigma 1 \sigma 3) \sigma 4$

using 5 2 4 *FusionAssoc ifirst-fuse* **by** *fastforce*

show *?thesis*

by (*metis 2 4 6 fusion-iff ifirst-fuse ilast-fuse*)

qed

lemma *FusionAssocB*:

assumes $x \in (X \cdot Y) \cdot Z$

shows $x \in X \cdot (Y \cdot Z)$

proof –

have 1: $(\exists \sigma 1 \sigma 2. x = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in X \cdot Y \wedge \sigma 2 \in Z \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2)$

using *assms fusion-iff*[*of x X · Y Z*] **by** *auto*

obtain $\sigma 1 \sigma 2$ **where** 2: $x = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in X \cdot Y \wedge \sigma 2 \in Z \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2$

using 1 **by** *auto*

have 3: $(\exists \sigma 3 \sigma 4. \sigma 1 = \text{fuse } \sigma 3 \sigma 4 \wedge \sigma 3 \in X \wedge \sigma 4 \in Y \wedge \text{ilast } \sigma 3 = \text{ifirst } \sigma 4)$

using 2 *fusion-iff*[*of σ 1 X Y*] **by** *auto*

obtain $\sigma 3 \sigma 4$ **where** 4: $\sigma 1 = \text{fuse } \sigma 3 \sigma 4 \wedge \sigma 3 \in X \wedge \sigma 4 \in Y \wedge \text{ilast } \sigma 3 = \text{ifirst } \sigma 4$

using 3 **by** *auto*

have 5: $x = \text{fuse } (\text{fuse } \sigma 3 \sigma 4) \sigma 2$

using 2 4 **by** *auto*

have 6: $x = \text{fuse } \sigma 3 (\text{fuse } \sigma 4 \sigma 2)$

using 2 4 *FusionAssoc ilast-fuse* **by** *force*

show *?thesis*

by (*metis 2 4 6 fusion-iff ifirst-fuse ilast-fuse*)

qed

lemma *FusionAssoc* :

$X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$

using *set-eqI*[*of X · (Y · Z) (X · Y) · Z*]

FusionAssocA FusionAssocB **by** *blast*

— left and right distributivity

lemma *FusionUnionDistL*:

$$(X \cup Y) \cdot Z = (X \cdot Z) \cup (Y \cdot Z)$$

using *fusion-iff*[*of* - $X \cup Y \ Z$] *fusion-iff*[*of* - $X \ Z$] *fusion-iff*[*of* - $Y \ Z$]
set-eqI[*of* $(X \cup Y) \cdot Z \ (X \cdot Z) \cup (Y \cdot Z)$]
by *auto*

lemma *FusionUnionDistR*:

$$X \cdot (Y \cup Z) = (X \cdot Y) \cup (X \cdot Z)$$

using *fusion-iff* *set-eqI*[*of* $X \cdot (Y \cup Z) \ (X \cdot Y) \cup (X \cdot Z)$]
fusion-iff[*of* - $X \ Y \cup Z$] *fusion-iff*[*of* - $X \ Y$] *fusion-iff*[*of* - $X \ Z$]
by *auto*

— left and right annihilation

lemma *SFalseFusion*:

$$SFalse \cdot X = SFalse$$

by (*simp* *add*: *fusion-def* *sfalse-def*)

lemma *FusionSFalse*:

$$X \cdot SFalse = SFalse$$

by (*simp* *add*: *fusion-def* *sfalse-def*)

— idempotency

lemma *UnionIdem*:

$$(X :: 'a \text{ intervals}) \cup X = X$$

by *simp*

16.3.4 Subsumption order

lemma *Subsumption*:

$$((X :: 'a \text{ intervals}) \subseteq Y) = (X \cup Y = Y)$$

by *auto*

16.3.5 Helper lemmas

lemma *FusionRuleR*:

assumes $X \subseteq Y$

shows $Z \cdot X \subseteq Z \cdot Y$

using *assms* *FusionUnionDistR* **by** (*metis* *Subsumption*)

lemma *FusionRuleL*:

assumes $X \subseteq Y$

shows $X \cdot Z \subseteq Y \cdot Z$

using *assms* **by** (*metis* *FusionUnionDistL* *subset-Un-eq*)

lemma *spower-commutes*:

$(X \cap SMore) \cdot (SPower X n) = (SPower X n) \cdot (X \cap SMore)$
proof (*induct n*)
case 0
then show ?*case* **by** (*simp add: FusionSEmptyL FusionSEmptyR*)
next
case (*Suc n*)
then show ?*case* **by** (*simp add: FusionAssoc*)
qed

lemma *fusion-inductl*:
assumes $Y \cup X \cdot Z \subseteq Z$
shows $(SPower X n) \cdot Y \subseteq Z$
using *assms*
proof (*induct n*)
case 0
then show ?*case* **by** (*simp add: FusionSEmptyL*)
next
case (*Suc n*)
then show ?*case*
proof –
have *f1*: $X \cdot (SPower X n \cdot Y) \cup X \cdot Z = X \cdot Z$
by (*metis FusionUnionDistR Suc.hyps assms subset-Un-eq*)
have $X \cdot SPower X n \cdot Y \cup X \cap SMore \cdot SPower X n \cdot Y = X \cdot (SPower X n \cdot Y)$
by (*metis (no-types) FusionAssoc FusionUnionDistL sup-inf-absorb*)
then have $SPower X (Suc n) \cdot Y \cup Z = X \cdot Z \cup (Y \cup Z)$
using *f1 assms* **by** *auto*
then show ?*thesis*
using *assms* **by** *auto*
qed
qed

lemma *fusion-inductr*:
assumes $Y \cup Z \cdot X \subseteq Z$
shows $Y \cdot (SPower X n) \subseteq Z$
using *assms*
proof (*induct n*)
case 0
then show ?*case* **by** (*simp add: FusionSEmptyR*)
next
case (*Suc n*)
then show ?*case*
proof –
have *f1*: $Y \cdot SPower X n \cup Z = Z$
using *Suc.hyps assms* **by** *blast*
have $Y \cdot (X \cap SMore) \cdot SPower X n = Y \cdot (SPower X n \cdot (X \cap SMore))$
by (*metis (no-types) FusionAssoc spower-commutes*)
then have $Y \cdot (X \cap SMore) \cdot SPower X n \subseteq Z$
using *f1* **by** (*metis (no-types) FusionAssoc FusionUnionDistL FusionUnionDistR Un-subset-iff*
assms sup-inf-absorb)
then show ?*thesis*

by (simp add: FusionAssoc)

qed

qed

lemma sstar-contlA:

assumes $x \in Y \cdot (SStar\ X)$

shows $x \in (\bigcup n. Y \cdot (SPower\ X\ n))$

proof –

have 1: $(\exists \sigma 1\ \sigma 2. x = fuse\ \sigma 1\ \sigma 2 \wedge \sigma 1 \in Y \wedge \sigma 2 \in (SStar\ X) \wedge ilast\ \sigma 1 = ifirst\ \sigma 2)$

using assms by (simp add: fusion-iff)

obtain $\sigma 1\ \sigma 2$ where 2: $x = fuse\ \sigma 1\ \sigma 2 \wedge \sigma 1 \in Y \wedge \sigma 2 \in (SStar\ X) \wedge ilast\ \sigma 1 = ifirst\ \sigma 2$

using 1 by auto

have 3: $(\exists n. \sigma 2 \in SPower\ X\ n)$

using 2 sstar-elim by blast

obtain n where 4: $\sigma 2 \in SPower\ X\ n$

using 3 by auto

have 5: $(\exists n. x \in Y \cdot SPower\ X\ n)$

using 2 4 fusion-iff by blast

from 5 show ?thesis by blast

qed

lemma sstar-contlB:

assumes $x \in (\bigcup n. Y \cdot (SPower\ X\ n))$

shows $x \in Y \cdot (SStar\ X)$

proof –

have 1: $\exists n. x \in Y \cdot (SPower\ X\ n)$

using assms by blast

obtain n where 2: $x \in Y \cdot (SPower\ X\ n)$

using 1 by auto

have 3: $(\exists \sigma 1\ \sigma 2. x = fuse\ \sigma 1\ \sigma 2 \wedge \sigma 1 \in Y \wedge \sigma 2 \in (SPower\ X\ n) \wedge ilast\ \sigma 1 = ifirst\ \sigma 2)$

using 2 by (simp add: fusion-iff)

obtain $\sigma 1\ \sigma 2$ where 4: $x = fuse\ \sigma 1\ \sigma 2 \wedge \sigma 1 \in Y \wedge \sigma 2 \in (SPower\ X\ n) \wedge ilast\ \sigma 1 = ifirst\ \sigma 2$

using 3 by auto

have 5: $\sigma 2 \in (SStar\ X)$

using 4 sstar-elim by auto

from 5 4 show ?thesis using fusion-iff by blast

qed

lemma sstar-contl:

$Y \cdot (SStar\ X) = (\bigcup n. Y \cdot (SPower\ X\ n))$

using set-eqI[of $Y \cdot (SStar\ X)$ $(\bigcup n. Y \cdot (SPower\ X\ n))$]

by (metis sstar-contlA sstar-contlB)

lemma sstar-contrA:

assumes $x \in (SStar\ X) \cdot Y$

shows $x \in (\bigcup n. (SPower\ X\ n) \cdot Y)$

proof –

have 1: $(\exists \sigma 1\ \sigma 2. x = fuse\ \sigma 1\ \sigma 2 \wedge \sigma 1 \in (SStar\ X) \wedge \sigma 2 \in Y \wedge ilast\ \sigma 1 = ifirst\ \sigma 2)$

using assms by (simp add: fusion-iff)

obtain $\sigma 1 \sigma 2$ **where** $2: x = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in (SStar X) \wedge \sigma 2 \in Y \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2$
using 1 **by** *auto*
have 3: $\exists n. \sigma 1 \in (SPower X n)$
using 2 *sstar-elim* **by** *blast*
obtain n **where** $4: \sigma 1 \in (SPower X n)$
using 3 **by** *auto*
have 5: $(\exists n. x \in (SPower X n) \cdot Y)$
by (*metis* 2 4 *fusion-iff-1* *fuse-ilen* *prefix-fuse* *suffix-fuse* *le-add1*)
from 5 **show** *?thesis* **by** *blast*
qed

lemma *sstar-contrB*:

assumes $x \in (\bigcup n. (SPower X n) \cdot Y)$

shows $x \in (SStar X) \cdot Y$

proof –

have 1: $\exists n. x \in (SPower X n) \cdot Y$

using *assms* **by** *blast*

obtain n **where** $2: x \in (SPower X n) \cdot Y$

using 1 **by** *auto*

have 3: $(\exists \sigma 1 \sigma 2. x = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in (SPower X n) \wedge \sigma 2 \in Y \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2)$

using 2 **by** (*simp* *add: fusion-iff*)

obtain $\sigma 1 \sigma 2$ **where** $4: x = \text{fuse } \sigma 1 \sigma 2 \wedge \sigma 1 \in (SPower X n) \wedge \sigma 2 \in Y \wedge \text{ilast } \sigma 1 = \text{ifirst } \sigma 2$

using 3 **by** *auto*

have 5: $\sigma 1 \in (SStar X)$

using 4 *sstar-elim* **by** *auto*

from 5 4 **show** *?thesis* **using** *fusion-iff* **by** *blast*

qed

lemma *sstar-contr*:

$(SStar X) \cdot Y = (\bigcup n. (SPower X n) \cdot Y)$

using *set-eqI* [*of* $(SStar X) \cdot Y$ $(\bigcup n. (SPower X n) \cdot Y)$]

by (*metis* *sstar-contrA* *sstar-contrB*)

16.3.6 Kleene Algebra

lemma *UnfoldL*:

$SEmpty \cup X \cdot (SStar X) = (SStar X)$

proof –

have 1: $(SStar X) = SEmpty \cup (X \cap SMore) \cdot (SStar X)$

by (*meson* *Un-iff* *set-eqI* *sstar-eqv*)

have 2: $(X \cap SMore) \cdot (SStar X) \subseteq X \cdot (SStar X)$

by (*simp* *add: FusionRuleL*)

have 3: $(SStar X) \subseteq SEmpty \cup X \cdot (SStar X)$

using 1 2 **by** *blast*

have 4: $SEmpty \subseteq (SStar X)$

using 1 **by** *auto*

have 5: $X \subseteq SEmpty \cup (X \cap SMore)$

by (*simp* *add: Un-Int-distrib* *smore-def*)

have 6: $X \cdot (SStar X) \subseteq (SStar X) \cup (X \cap SMore) \cdot (SStar X)$

using 5 **by** (*metis FusionRuleL FusionUnionDistL FusionSEmptyL*)
have 7: $(SStar\ X) \subseteq SEmpty \cup (X \cap SMore) \cdot (SStar\ X)$
using 1 **by** *auto*
have 8: $X \cdot (SStar\ X) \subseteq SEmpty \cup (X \cap SMore) \cdot (SStar\ X)$
using 6 7 **by** *blast*
hence 9: $X \cdot (SStar\ X) \subseteq (SStar\ X)$
using 1 **by** *auto*
have 10: $SEmpty \cup X \cdot (SStar\ X) \subseteq (SStar\ X)$
using 9 4 **by** *simp*
from 3 10 **show** ?thesis **by** *auto*
qed

— Left induction

lemma *SStarInductL*:

assumes $Y \cup X \cdot Z \subseteq Z$

shows $(SStar\ X) \cdot Y \subseteq Z$

by (*metis UN-least assms fusion-inductl sstar-contr*)

— Right induction

lemma *SStarInductR*:

assumes $Y \cup Z \cdot X \subseteq Z$

shows $Y \cdot (SStar\ X) \subseteq Z$

using *sstar-contl assms fusion-inductr* **by** *blast*

16.3.7 ITL specific Laws

lemma *PwrFusionInterL*:

$(((((SPower\ SSkip\ n) \cap X) \cdot V) \cap (((SPower\ SSkip\ n) \cap Y) \cdot W)) =$
 $(((((SPower\ SSkip\ n) \cap X \cap Y) \cdot (V \cap W))$

using *set-eqI*[of $(((((SPower\ SSkip\ n) \cap X) \cdot V) \cap (((SPower\ SSkip\ n) \cap Y) \cdot W))$
 $(((((SPower\ SSkip\ n) \cap X \cap Y) \cdot (V \cap W))$]

by (*simp add: fusion-iff-1 spower-skip-elim*)
(metis min.absorb1)

lemma *PwrFusionInterR*:

$((V \cdot (((SPower\ SSkip\ n) \cap X)) \cap ((W \cdot (((SPower\ SSkip\ n) \cap Y)))) =$
 $((V \cap W) \cdot (((SPower\ SSkip\ n) \cap X \cap Y)))$

using *set-eqI*[of $((V \cdot (((SPower\ SSkip\ n) \cap X)) \cap ((W \cdot (((SPower\ SSkip\ n) \cap Y))))$
 $((V \cap W) \cdot (((SPower\ SSkip\ n) \cap X \cap Y)))$]

by (*simp add: fusion-iff-1 spower-skip-elim*)
(metis diff-diff-cancel)

lemma *SSkipFusionImpSMore*:

$SSkip \cdot STrue \subseteq SMore$

using *subsetI*[of $SSkip \cdot STrue\ SMore$]

by (*auto simp add: fusion-iff-1 skip-elim smore-elim strue-elim*)

lemma *SStarSkip*:

$(SStar\ SSkip) = STrue$
using *set-eqI*[of (*SStar SSkip*) *STrue*]
by (*simp add: strue-def spower-skip-elim sstar-elim*)

16.4 Derived Laws

16.4.1 Helper Lemmas

lemma *B01*:
assumes $(X:: 'a\ intervals) \subseteq Y$
shows $-Y \subseteq -X$
using *assms* **by** *auto*

lemma *B04*:
 $((X:: 'a\ intervals) = Y) \longleftrightarrow (X \subseteq Y) \wedge (Y \subseteq X)$
by *auto*

lemma *B09*:
assumes $-X \cup Y = STrue$
shows $(X:: 'a\ intervals) \subseteq Y$
using *assms* **using** *strue-def* **by** *auto*

lemma *B20*:
 $(X:: 'a\ intervals) \subseteq Y \cup Z \longleftrightarrow X \cap -Y \subseteq Z$
by *auto*

lemma *B28*:
 $((X:: 'a\ intervals) \cap Y) \cup (X \cap Z) = X \cap (Y \cup Z)$
by *auto*

lemma *CH01*:
 $STrue \cdot STrue = STrue$
by (*metis FusionSEmptyR FusionUnionDistR Int-commute SStarSkip STrueTop UnfoldL inf-sup-absorb*)

lemma *CH07*:
 $((Sskip \cap X) \cdot V) \cap ((Sskip \cap Y) \cdot W) = ((Sskip \cap X \cap Y) \cdot (V \cap W))$
using *PwrFusionInterL*[of 1 *X V Y W*]
by (*simp add: FusionSEmptyR inf-commute smore-def skip-def*)

lemma *CH08*:
 $((V \cdot (Sskip \cap X)) \cap (W \cdot (Sskip \cap Y))) = ((V \cap W) \cdot (Sskip \cap X \cap Y))$
using *PwrFusionInterR*[of *V 1 X W Y*]
by (*simp add: FusionSEmptyR inf-commute smore-def skip-def*)

lemma *CH09*:
 $((X \cap SEmpty) \cdot V) \cap ((Y \cap SEmpty) \cdot W) = ((X \cap Y) \cap SEmpty) \cdot (V \cap W)$
using *PwrFusionInterL*[of 0 *X V Y W*]
by (*metis (no-types, lifting) inf-assoc inf-commute pwr-0*)

lemma *CH10*:
 $((V \cdot (X \cap SEmpty)) \cap (W \cdot (Y \cap SEmpty))) = ((V \cap W) \cdot ((X \cap Y) \cap SEmpty))$

using *PwrFusionInterR*[of *V 0 X W Y*]
by (*metis* (*no-types*, *lifting*) *inf-assoc inf-commute pwr-0*)

lemma *ST13*:
 $((X \cap SEmpty) \cdot Z) \cap ((Y \cap SEmpty) \cdot Z) = ((X \cap Y) \cap SEmpty) \cdot Z$
by (*simp add: CH09*)

lemma *ST15*:
 $(SStar (X \cap SEmpty)) = SEmpty$
by (*metis FusionSEmptyL inf.right-idem inf-le2 UnfoldL*
SStarInductR sup.orderE sup-inf-absorb)

lemma *ST21*:
 $((-X) \cap SEmpty) \cup (X \cap SEmpty) = SEmpty$
by *blast*

lemma *ST24*:
 $(SInit X) \cap (SInit Y) = (SInit (X \cap Y))$
by (*simp add: ST13 sinit-def*)

lemma *ST25*:
 $(SInit STrue) = STrue$
by (*simp add: sinit-def strue-def FusionSEmptyL*)

lemma *ST26*:
 $(SInit (-X)) \cup (SInit X) = STrue$
by (*metis Compl-disjoint2 ST21 ST25 Un-Int-distrib compl-bot-eq FusionUnionDistL*
sinit-def strue-def sup-bot.right-neutral sup-top-right)

lemma *ST28*:
 $(SDi (SInit X)) = (SInit X)$
by (*metis compl-bot-eq FusionAssoc FusionUnionDistR FusionSEmptyR sdi-def*
sinit-def strue-def sup-top-right UnionCommute)

lemma *ST33*:
 $(STrue \cap SEmpty) \cdot SEmpty = SEmpty$
by (*simp add: strue-def FusionSEmptyL*)

lemma *ST36*:
 $(SInit (-X)) \subseteq -(SInit X)$
by (*metis Compl-disjoint ST24 compl-bot-eq disjoint-eq-subset-Compl double-complement*
inf.coboundedI2 inf.orderE sfalse-def SFalseFusion sinit-def strue-def)

lemma *ST37*:
 $-(SInit X) \subseteq (SInit (-X))$
using *B09 ST26* **by** *auto*

lemma *ST38*:
 $-(SInit X) = (SInit (-X))$
using *ST37 ST36* **by** *auto*

lemma *ST47*:

$$X \cup Y \cdot X = (SEmpty \cup Y) \cdot X$$

by (*simp add: FusionUnionDistL FusionSEmptyL*)

lemma *SStar01*:

$$\text{assumes } X \cdot (SStar\ Y) \cup SEmpty \subseteq (SStar\ Y)$$

$$\text{shows } (SStar\ X) \subseteq (SStar\ Y)$$

using *assms*

by (*metis Un-commute FusionSEmptyR SStarInductL*)

lemma *SStar03*:

$$(SStar\ X) \cdot (SStar\ X) \subseteq (SStar\ X)$$

by (*metis SStarInductL Un-absorb UnfoldL sup.absorb-iff1 sup.right-idem*)

lemma *SStar05*:

$$\text{assumes } ((SStar\ X) \cdot (SStar\ X)) \cup SEmpty \subseteq (SStar\ X)$$

$$\text{shows } (SStar\ (SStar\ X)) \subseteq (SStar\ X)$$

using *assms*

by (*simp add: SStar01*)

lemma *SStar12*:

$$(SEmpty \cup (X \cdot (SStar\ X))) \subseteq (SStar\ X)$$

using *UnfoldL* **by** *blast*

lemma *SStar06*:

$$((SStar\ X) \cdot (SStar\ X)) \cup SEmpty \subseteq (SStar\ X)$$

using *SStar03 SStar12* **by** *force*

lemma *SStar07*:

$$(SStar\ X) \subseteq (SStar\ (SStar\ X))$$

by (*metis FusionUnionDistR FusionSEmptyR Subsumption Un-commute UnfoldL ST47 sup.right-idem*)

lemma *SStar08*:

$$(SStar\ X) = (SStar\ (SStar\ X))$$

by (*meson B04 SStar05 SStar06 SStar07*)

lemma *SStar15*:

$$SEmpty \subseteq (SStar\ SSkip)$$

by (*simp add: SStarSkip strue-def*)

lemma *SStar16*:

$$SSkip \subseteq (SStar\ SSkip)$$

by (*simp add: SStarSkip strue-def*)

lemma *SStar22*:

$$(SEmpty \cap X) \cdot (SStar\ (SEmpty \cap X)) = (SEmpty \cap X)$$

by (*metis ST15 FusionSEmptyR inf-commute*)

lemma *SStar23*:

$(SStar (SEmpty \cap X)) = SEmpty$
using *SStar22 UnfoldL* **by** *auto*

lemma *SStar25*:
 $(SStar STrue) = STrue$
by (*metis SStar08 SStarSkip*)

lemma *SStar28*:
 $(SStar X) \cdot X \subseteq X \cdot (SStar X)$
by (*metis B04 FusionUnionDistR FusionSEmptyR UnfoldL SStarInductL*)

lemma *SStar29*:
 $X \cdot (SStar X) \subseteq (SStar X) \cdot X$
by (*metis B04 SStar28 SStarInductR UnfoldL FusionRuleL ST47 sup.mono*)

lemma *SStar17*:
 $(SStar SSkip) \cdot SSkip \subseteq SSkip \cdot (SStar SSkip)$
by (*simp add: SStar28*)

lemma *SStar18*:
 $SSkip \cdot (SStar SSkip) \subseteq (SStar SSkip) \cdot SSkip$
by (*simp add: SStar29*)

lemma *SStar19*:
 $(SStar SSkip) \cdot SSkip = SSkip \cdot (SStar SSkip)$
using *SStar17 SStar18* **by** *auto*

lemma *SStar30*:
 $X \cdot (SStar X) = (SStar X) \cdot X$
using *SStar28 SStar29* **by** *auto*

lemma *SStar34*:
assumes $SEmpty \cup (X \cup Y) \cdot ((SStar X) \cdot (SStar (Y \cdot (SStar X)))) \subseteq (SStar X) \cdot (SStar (Y \cdot (SStar X)))$
shows $(SStar (X \cup Y)) \subseteq (SStar X) \cdot (SStar (Y \cdot (SStar X)))$
by (*metis assms FusionSEmptyR SStarInductL*)

lemma *SStar35*:
 $SEmpty \cup (X \cup Y) \cdot ((SStar X) \cdot (SStar (Y \cdot (SStar X)))) \subseteq (SStar X) \cdot (SStar (Y \cdot (SStar X)))$
by (*simp add: FusionAssoc FusionUnionDistL ST47 UnfoldL UnionAssoc UnionCommute*)

lemma *SStar36*:
 $(SStar (X \cup Y)) \subseteq (SStar X) \cdot (SStar (Y \cdot (SStar X)))$
using *SStar34 SStar35* **by** *blast*

lemma *SStar46*:
 $(SStar X) \cdot (SStar (Y \cdot (SStar X))) \subseteq (SStar (X \cup Y))$
proof –
have $(SEmpty \cup SStar (X \cup Y) \cdot Y) \cdot SStar X \subseteq SStar (X \cup Y)$
by (*metis (no-types) FusionUnionDistR SStar12 SStar30 SStarInductR sup.bounded-iff*)
then show *?thesis* **by** (*simp add: SStarInductR ST47 FusionAssoc*)

qed

lemma *SStar47*:

$(SStar\ Z) = (SStar\ (Z \cap SMore))$

proof –

have 1: $(SStar\ Z) = (SStar\ ((SEmpty \cap Z) \cup (SMore \cap Z)))$

by (*metis Int-Un-distrib2 compl-bot-eq inf-top.left-neutral smore-def strue-def STrueTop*)

have 2: $(SStar\ ((SEmpty \cap Z) \cup (SMore \cap Z))) =$

$(SStar\ (SEmpty \cap Z)) \cdot (SStar\ ((SMore \cap Z) \cdot (SStar\ (SEmpty \cap Z))))$

by (*simp add: SStar36 SStar46 subset-antisym*)

have 3: $(SStar\ (SEmpty \cap Z)) \cdot (SStar\ ((SMore \cap Z) \cdot (SStar\ (SEmpty \cap Z)))) =$
 $(SStar\ (Z \cap SMore))$

by (*simp add: FusionSEmptyL FusionSEmptyR SStar23 inf-commute*)

from 1 2 3 **show** *?thesis* **by** *auto*

qed

lemma *SStar48*:

$(SStar\ SMore) = STrue$

by (*metis Compl-Un Compl-disjoint2 SStar25 SStar47 ST21 ST33 FusionSEmptyR*
inf.right-idem smore-def strue-def)

lemma *SStar50*:

assumes $SSkip \cdot ((-X) \cup ((SStar\ SSkip) \cdot (X \cap (SSkip \cdot (-X))))) \cup (-X)$

$\subseteq (-X) \cup (SStar\ SSkip) \cdot (X \cap (SSkip \cdot (-X)))$

shows $((SStar\ SSkip) \cdot (-X)) \subseteq ((-X) \cup ((SStar\ SSkip) \cdot (X \cap (SSkip \cdot (-X)))))$

using *SStarInductL assms* **by** *blast*

lemma *SStar51*:

$SSkip \cdot ((-X) \cup ((SStar\ SSkip) \cdot (X \cap (SSkip \cdot (-X))))) \cup (-X)$

$\subseteq (-X) \cup (SStar\ SSkip) \cdot (X \cap (SSkip \cdot (-X)))$

using *FusionUnionDistR[of SSkip] UnfoldL[of SSkip]*

proof –

have *f1*: $-X \cup (SEmpty \cup SSkip \cdot SStar\ SSkip) \cdot (X \cap (SSkip \cdot -X)) \subseteq$
 $-X \cup SStar\ SSkip \cdot (X \cap (SSkip \cdot -X))$

by (*simp add: $\langle SEmpty \cup SSkip \cdot SStar\ SSkip = SStar\ SSkip \rangle$*)

have *f2*: $SSkip \cdot -X \subseteq -X \cup SStar\ SSkip \cdot (X \cap (SSkip \cdot -X))$

by (*metis B20 Morgan ST47 UnionIdem $\langle SEmpty \cup SSkip \cdot SStar\ SSkip = SStar\ SSkip \rangle$*
inf-commute inf-idem sup.cobounded1)

have $SSkip \cdot (SStar\ SSkip \cdot (X \cap (SSkip \cdot -X))) \subseteq -X \cup SStar\ SSkip \cdot (X \cap (SSkip \cdot -X))$

using *f1* **by** (*metis (no-types) FusionAssoc ST47 Un-subset-iff*)

then show *?thesis*

using *f2* **by** (*simp add: $\langle \bigwedge Z\ Y. SSkip \cdot (Y \cup Z) = SSkip \cdot Y \cup SSkip \cdot Z \rangle$*)

qed

lemma *SStar52*:

$(SStar\ X) \subseteq SEmpty \cup (X \cap SMore) \cdot (SStar\ X)$

by (*metis B04 SStar47 UnfoldL*)

lemma *SStar53*:

$SEmpty \cup (X \cap SMore) \cdot (SStar X) \subseteq (SStar X)$
by (*metis SStar12 SStar47*)

lemma *BD45*:

$(SBI ((-X) \cup X1)) \cap (X \cdot Y) \subseteq X1 \cdot Y$

proof –

have 1: $(SBI ((-X) \cup X1)) = -(-((-X) \cup X1) \cdot (Y \cup (-Y)))$

by (*metis sbi-def sdi-def STrueTop*)

have 2: $-(-((-X) \cup X1) \cdot (Y \cup (-Y))) \cap (X \cdot Y) \subseteq$
 $-(-((-X) \cup X1) \cdot (Y)) \cap (X \cdot Y)$

using *FusionUnionDistR* **by** *fastforce*

have 3: $-(-((-X) \cup X1) \cdot (Y)) \cap (X \cdot Y) \subseteq (((-X) \cup X1) \cap X) \cdot Y$

by (*metis FusionRuleL FusionUnionDistL dual-order.refl inf-sup-aci(1) shunt1 sup-neg-inf*)

have 4: $(((-X) \cup X1) \cap X) \cdot Y \subseteq X1 \cdot Y$

by (*metis B20 double-compl FusionRuleL inf.right-idem inf-le1*)

from 1 2 3 4 **show** *?thesis* **by** *blast*

qed

lemma *BD46*:

$(SAlways ((-Y) \cup Y1)) \cap (X1 \cdot Y) \subseteq (X1 \cdot Y1)$

proof –

have 1: $(SAlways ((-Y) \cup Y1)) = -((X1 \cup (-X1)) \cdot (-((-Y) \cup Y1)))$

by (*metis salways-def s sometime-def STrueTop*)

have 2: $-((X1 \cup (-X1)) \cdot (-((-Y) \cup Y1))) \cap (X1 \cdot Y) \subseteq$
 $-(X1 \cdot (-((-Y) \cup Y1))) \cap (X1 \cdot Y)$

using *FusionUnionDistL* **by** *fastforce*

have 3: $-(X1 \cdot (-((-Y) \cup Y1))) \cap (X1 \cdot Y) \subseteq X1 \cdot (((-Y) \cup Y1) \cap Y)$

by (*metis (no-types, lifting) B20 B04 compl-inf FusionUnionDistR Huntington*
Morgan Subsumption sup-ge1 UnionCommute)

have 4: $X1 \cdot (((-Y) \cup Y1) \cap Y) \subseteq (X1 \cdot Y1)$

by (*metis B20 double-compl FusionRuleR inf.right-idem inf-le1*)

from 1 2 3 4 **show** *?thesis* **by** *blast*

qed

16.4.2 ITL Axioms derived

lemma *SBoxGen*:

assumes $X = STrue$

shows $(SAlways X) = STrue$

using *assms*

by (*metis double-compl FusionSFalse salways-def sfalse-def s sometime-def strue-def*)

lemma *SBiGen*:

assumes $X = STrue$

shows $(SBI X) = STrue$

using *assms*

by (*metis double-compl sbi-def sdi-def sfalse-def SFalseFusion strue-def*)

lemma *SMP*:

assumes $X \subseteq Y$

assumes $X = STrue$
shows $Y = STrue$
using $assms(1)$ $assms(2)$
using $strue-def$ **by** $blast$

lemma $SChopAssoc$:
 $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$
by ($simp$ add : $FusionAssoc$)

lemma $SOrChopImp$:
 $(X \cup Y) \cdot Z \subseteq (X \cdot Z) \cup (Y \cdot Z)$
by ($simp$ add : $FusionUnionDistL$)

lemma $SChopOrImp$:
 $X \cdot (Y \cup Z) \subseteq (X \cdot Y) \cup (X \cdot Z)$
by ($simp$ add : $FusionUnionDistR$)

lemma $SEmptyChop$:
 $SEmpty \cdot X = X$
by ($simp$ add : $FusionSEmptyL$)

lemma $SChopEmpty$:
 $X \cdot SEmpty = X$
by ($simp$ add : $FusionSEmptyR$)

lemma $SStateImpBi$:
 $(SInit X) \subseteq (SBi (SInit X))$
by ($simp$ add : $ST28 ST38 sbi-def$)

lemma $SNextImpNotNextNot$:
 $(SNext X) \subseteq \neg(SNext (\neg X))$
proof –
have 1: $((SNext X) \subseteq \neg(SNext (\neg X))) = (((SNext X) \cap (SNext (\neg X))) \subseteq SFalse)$
by ($simp$ add : $disjoint-eq-subset-Compl sfalse-def$)
have 2: $((SNext X) \cap (SNext (\neg X))) = SSkip \cdot (X \cap (\neg X))$
by ($metis CH07 SStar16 inf.orderE snext-def$)
have 3: $(SSkip \cdot (X \cap (\neg X))) = SSkip \cdot SFalse$
by ($simp$ add : $sfalse-def$)
have 4: $SSkip \cdot SFalse = SFalse$ **by** ($simp$ add : $FusionSFalse$)
from 1 2 3 4 **show** $?thesis$ **by** $auto$
qed

lemma $SBiBoxChopImpChop$:
 $(SBi ((\neg X) \cup X1)) \cap (SAlways ((\neg Y) \cup Y1)) \cap (X \cdot Y) \subseteq (X1 \cdot Y1)$
using $BD45 BD46$ **by** $blast$

lemma $SBoxInduct$:
 $(SAlways (\neg X \cup (SWnext X))) \cap X \subseteq (SAlways X)$
proof –
have 1: $((SAlways (\neg X \cup (SWnext X))) \cap X) \subseteq (SAlways X) =$

```

      ((SSometime (-X))  $\subseteq$  ((-X)  $\cup$  (SSometime (X  $\cap$  (SNext (-X))) )))
    by (simp add: always-def snext-def swnext-def)
      blast
  have 2: ((SSometime (-X))  $\subseteq$  ((-X)  $\cup$  (SSometime (X  $\cap$  (SNext (-X))) ))) =
    ( ((SStar SSkip) $\cdot$ (-X))  $\subseteq$  ((-X)  $\cup$  ((SStar SSkip) $\cdot$ (X  $\cap$  (SSkip $\cdot$ (-X))))) )
    by (simp add: SStarSkip snext-def ssometime-def)
  have 3: ( ((SStar SSkip) $\cdot$ (-X))  $\subseteq$  ((-X)  $\cup$  ((SStar SSkip) $\cdot$ (X  $\cap$  (SSkip $\cdot$ (-X))))) )
    using SStar51 SStar50 by blast
  from 1 2 3 show ?thesis by auto
qed

```

```

lemma SChopstarEqv:
  (SStar X) = SEmpty  $\cup$  (X  $\cap$  SMore) $\cdot$ (SStar X)
using SStar52 SStar53 by blast

```

16.5 Extra Laws

16.5.1 Boolean Laws

```

lemma B02:
  assumes -Y  $\subseteq$  -X
  shows (X:: 'a intervals)  $\subseteq$  Y
using assms by auto

```

```

lemma B03:
  ((X:: 'a intervals) = Y)  $\longleftrightarrow$  (-X = -Y)
by auto

```

```

lemma B05:
  assumes (X:: 'a intervals)  $\cup$  Y  $\subseteq$  Z
  shows X  $\subseteq$  Z  $\wedge$  Y  $\subseteq$  Z
using assms by auto

```

```

lemma B06:
  assumes X  $\subseteq$  Z  $\wedge$  Y  $\subseteq$  Z
  shows (X:: 'a intervals)  $\cup$  Y  $\subseteq$  Z
using assms by auto

```

```

lemma B07:
  (X:: 'a intervals)  $\cup$  Y  $\subseteq$  Z  $\longleftrightarrow$ 
  X  $\subseteq$  Z  $\wedge$  Y  $\subseteq$  Z
by auto

```

```

lemma B08:
  assumes (X:: 'a intervals)  $\subseteq$  Y
  shows -X  $\cup$  Y = STrue
using assms
using strue-def by auto

```

```

lemma B10:
  (X:: 'a intervals)  $\subseteq$  Y  $\longleftrightarrow$  -X  $\cup$  Y = STrue

```

using *strue-def* **by** *auto*

lemma *B11*:

assumes $(X:: 'a \text{ intervals}) \subseteq Y$

shows $X \cap -Y = SFalse$

using *assms sfalse-def* **by** *auto*

lemma *B12*:

assumes $X \cap -Y = SFalse$

shows $(X:: 'a \text{ intervals}) \subseteq Y$

using *assms sfalse-def* **by** *auto*

lemma *B13*:

$(X:: 'a \text{ intervals}) \subseteq Y \longleftrightarrow X \cap -Y = SFalse$

using *sfalse-def* **by** *auto*

lemma *B14*:

assumes $(X:: 'a \text{ intervals}) \subseteq Y$

shows $X \cap Y = X$

using *assms* **by** *auto*

lemma *B15*:

assumes $(X:: 'a \text{ intervals}) \subseteq Y \cap Z$

shows $X \subseteq Y \wedge X \subseteq Z$

using *assms* **by** *auto*

lemma *B16*:

assumes $X \subseteq Y \wedge X \subseteq Z$

shows $(X:: 'a \text{ intervals}) \subseteq Y \cap Z$

using *assms* **by** *auto*

lemma *B17*:

$(X:: 'a \text{ intervals}) \subseteq Y \cap Z \longleftrightarrow X \subseteq Y \wedge X \subseteq Z$

by *auto*

lemma *B18*:

assumes $(X:: 'a \text{ intervals}) \subseteq Y \cup Z$

shows $X \cap -Y \subseteq Z$

using *assms* **by** *auto*

lemma *B19*:

assumes $X \cap -Y \subseteq Z$

shows $(X:: 'a \text{ intervals}) \subseteq Y \cup Z$

using *assms* **by** *auto*

lemma *B21*:

$(X:: 'a \text{ intervals}) \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z) \longleftrightarrow$

$X \cup (Y \cap Z) \subseteq (X \cup Y) \wedge X \cup (Y \cap Z) \subseteq (X \cup Z)$

by *auto*

lemma B22:

$$(X:: 'a \text{ intervals}) \cup (Y \cap Z) \subseteq X \cup Y$$

by auto

lemma B23:

$$(X:: 'a \text{ intervals}) \cup (Y \cap Z) \subseteq X \cup Z$$

by auto

lemma B24:

$$\begin{aligned} ((X:: 'a \text{ intervals}) \cup Y) \cap (X \cup Z) &\subseteq X \cup (Y \cap Z) \longleftrightarrow \\ ((X \cup Y) \cap (X \cup Z)) \cap \neg X &\subseteq Y \cap Z \end{aligned}$$

by auto

lemma B25:

$$\begin{aligned} (((X:: 'a \text{ intervals}) \cup Y) \cap (X \cup Z)) \cap \neg X &\subseteq Y \cap Z \longleftrightarrow \\ ((X \cup Y) \cap (X \cup Z)) \cap \neg X &\subseteq Y \wedge \\ ((X \cup Y) \cap (X \cup Z)) \cap \neg X &\subseteq Z \end{aligned}$$

by auto

lemma B26:

$$(((X:: 'a \text{ intervals}) \cup Y) \cap (X \cup Z)) \cap \neg X \subseteq Y$$

by auto

lemma B27:

$$(((X:: 'a \text{ intervals}) \cup Y) \cap (X \cup Z)) \cap \neg X \subseteq Z$$

by auto

lemma B29:

$$(X:: 'a \text{ intervals}) \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

by auto

16.5.2 Chop

lemma CH02:

$$X \cdot Y \cap \neg(X \cdot Z) \subseteq X \cdot (Y \cap \neg Z)$$

by (metis B20 FusionRuleR FusionUnionDistR inf.right-idem inf-le1)

lemma CH03:

$$X \cdot Y \cap \neg(Z \cdot Y) \subseteq (X \cap \neg Z) \cdot Y$$

by (metis B20 FusionRuleL FusionUnionDistL inf.right-idem inf-le1)

lemma CH04:

$$X \cdot Y \cap \neg(X \cdot \neg Z) \subseteq X \cdot (Y \cap Z)$$

using CH02 by fastforce

lemma CH05:

$$X \cdot Y \cap \neg(\neg Z \cdot Y) \subseteq (X \cap Z) \cdot Y$$

using CH03 by fastforce

lemma CH06:

assumes $X \subseteq X1$
 $Y \subseteq Y1$
shows $X \cdot Y \subseteq X1 \cdot Y1$
using *assms*
by (*metis FusionRuleL FusionRuleR order-trans*)

lemma *CH11*:

$((X \cap (SPower SSkip\ n)) \cdot STRue) \cap ((SPower SSkip\ n) \cdot Y) = (X \cap (SPower SSkip\ n)) \cdot Y$
using *PwrFusionInterL[of n X STRue STRue Y]*
by (*simp add: inf-commute strue-def*)

lemma *CH12*:

$(STRue \cdot (X \cap (SPower SSkip\ n))) \cap (Y \cdot (SPower SSkip\ n)) = (Y \cdot (X \cap (SPower SSkip\ n)))$
using *PwrFusionInterR[of STRue n X Y STRue]*
by (*metis STRueTop inf-commute inf-sup-absorb*)

lemma *CH13*:

$(SPower SSkip\ n) \cdot (SPower SSkip\ m) = (SPower SSkip\ (n+m))$
proof
(induct n arbitrary: m)
case 0
then show ?case **by** (*simp add: FusionSEmptyL*)
next
case (*Suc n*)
then show ?case
by (*metis FusionAssoc add-Suc pwr-Suc*)
qed

16.5.3 Next

lemma *N01*:

$(SNext SEmpty) = SSkip$
by (*simp add: FusionSEmptyR snext-def*)

lemma *N02*:

$(SNext SFalse) = SFalse$
by (*simp add: FusionSFalse snext-def*)

lemma *N03*:

$(SNext X) \cdot Y = (SNext (X \cdot Y))$
by (*simp add: snext-def FusionAssoc*)

lemma *N04*:

$(SNext (X \cup Y)) = (SNext X) \cup (SNext Y)$
by (*simp add: FusionUnionDistR snext-def*)

lemma *N05*:

$(SNext (X \cap Y)) = (SNext X) \cap (SNext Y)$
by (*metis CH07 SStar16 inf.orderE snext-def*)

lemma N06:
assumes $X \subseteq Y$
shows $(SNext\ X) \subseteq (SNext\ Y)$
using *assms*
by (*metis FusionUnionDistR Subsumption snext-def*)

lemma N07:
 $(SNext\ ((-X) \cup Y)) = (SNext\ (-X)) \cup (SNext\ Y)$
by (*simp add: N04*)

lemma N08:
 $SMore \subseteq SSkip \cdot STrue$
by (*simp add: smore-def*)
(metis B10 SStarSkip UnfoldL double-complement)

lemma N23:
 $(SWprev\ X) \subseteq (SEmpty \cup (SPrev\ X))$
proof –
have $X \cdot SSkip \cup -\ X \cdot SSkip = SStar\ SSkip \cdot SSkip$
by (*metis (no-types) Compl-empty-eq FusionUnionDistL SStarSkip strue-def sup-compl-top*)
then have $- SWprev\ X \cup (SEmpty \cup SPrev\ X) = STrue$
by (*metis (no-types) SStar19 SStarSkip UnfoldL UnionAssoc double-compl sprev-def sup-commute suprev-def*)
then show *?thesis*
by (*meson B09*)
qed

lemma N24:
 $(SEmpty) \subseteq (SWprev\ X)$
by (*metis B10 B02 FusionRuleL SSkipFusionImpSMore SStar30 SStarSkip UnfoldL compl-bot-eq double-compl smore-def strue-def subset-antisym suprev-def top-greatest*)

lemma N25:
 $(SPrev\ X) \subseteq (SWprev\ X)$
proof –
have $1: ((SPrev\ X) \subseteq (SWprev\ X)) = (((SPrev\ X) \cap (SPrev\ (-X))) \subseteq SFalse)$
by (*simp add: B10 sfalse-def sprev-def suprev-def*)
have $2: ((SPrev\ X) \cap (SPrev\ (-X))) = (X \cap (-X)) \cdot SSkip$
by (*metis CH08 SStar16 inf.orderE sprev-def*)
have $3: (X \cap (-X)) \cdot SSkip = SFalse \cdot SSkip$
by (*simp add: sfalse-def*)
have $4: SFalse \cdot SSkip = SFalse$
by (*simp add: SFalseFusion*)
from $1\ 2\ 3\ 4$ **show** *?thesis* **by** *auto*
qed

lemma N26:
 $(SWprev\ X) = (SEmpty \cup (SPrev\ X))$
using *N23 N24 N25 by blast*

lemma N09:

$SSkip \cup SMore \cdot SSkip \subseteq SMore$

proof –

have 1: $SSkip \subseteq SMore$ **by** (*simp add: smore-def sskip-def*)

have 2: $SMore \cdot SSkip \subseteq SMore$

by (*metis N26 UnionCommute compl-le-swap1 smore-def sup-ge2 swprev-def*)

from 1 2 **show** *?thesis* **by** *auto*

qed

lemma N10:

assumes $SSkip \cup SMore \cdot SSkip \subseteq SMore$

shows $SSkip \cdot (SStar SSkip) \subseteq SMore$

using *assms*

using *SStarInductR N09* **by** *blast*

lemma N11:

$SSkip \cdot STrue \subseteq SMore$

by (*metis SStarSkip N09 N10*)

lemma N12:

$(SNext X) = -(SWnext (-X))$

by (*simp add: snext-def swnext-def*)

lemma N13:

$SMore \cdot STrue = SMore$

by (*metis FusionAssoc N11 N08 SStar48 SStarSkip ST47 UnfoldL subset-antisym sup.right-idem*)

lemma N14:

$STrue \cdot SSkip \subseteq SMore$

by (*metis N11 SStar19 SStarSkip*)

lemma N15:

$SMore \subseteq STrue \cdot SSkip$

by (*metis N08 SStar19 SStarSkip*)

lemma N19:

$(SWnext X) \subseteq (SEmpty \cup (SNext X))$

proof –

have $SSkip \cdot X \cup SSkip \cdot (-X) = SSkip \cdot SStar SSkip$

using *FusionUnionDistR[of SSkip X -X] SStarSkip*

by (*metis STrueTop*)

then have – $SWnext X \cup (SEmpty \cup SNext X) = STrue$

by (*metis SStarSkip UnfoldL UnionCommute double-compl inf-sup-aci(7) snext-def swnext-def*)

then show *?thesis* **by** (*simp add: B09*)

qed

lemma N20:

$(SEmpty) \subseteq (SWnext X)$

proof –

have 1: $((SEmpty) \subseteq (SWnext X)) = (-(SWnext X) \subseteq SMore)$

by (simp add: smore-def)
 have 2: $(\neg(SWnext\ X) \subseteq SMore) = ((SNext\ \neg X) \subseteq SMore)$
 by (simp add: snext-def swnext-def)
 have 3: $(SNext\ \neg X) \subseteq SSkip \cdot STrue$
 by (metis FusionUnionDistR STrueTop snext-def sup.orderI sup.right-idem)
 hence 4: $(SNext\ \neg X) \subseteq SMore$ using SSkipFusionImpSMore by auto
 from 1 2 4 show ?thesis by auto
 qed

lemma N21:
 $(SEmpty \cup (SNext\ X)) \subseteq (SWnext\ X)$
 using N20
 by (metis B06 SNextImpNotNextNot snext-def swnext-def)

lemma N22:
 $(SWnext\ X) = (SEmpty \cup (SNext\ X))$
 using N21 N19 by blast

lemma N16:
 $(SNext\ X) = SMore \cap (SWnext\ X)$
 using N12 N22 smore-def by blast

lemma N17:
 $(SWnext\ (X \cap Y)) = (SWnext\ X) \cap (SWnext\ Y)$
 by (simp add: N05 N22 Un-Int-distrib)

lemma N18:
 $(SWnext\ (X \cup Y)) = (SWnext\ X) \cup (SWnext\ Y)$
 by (simp add: swnext-def)
 (metis (no-types, lifting) CH07 SStar16 compl-inf inf.orderE)

lemma N27:
 $(SNext\ ((\neg X) \cup Y)) \subseteq (\neg(SNext\ X) \cup (SNext\ Y))$
 using N04 SNextImpNotNextNot by blast

lemma N28:
 $(SPrev\ ((\neg X) \cup Y)) \subseteq (\neg(SPrev\ X) \cup (SPrev\ Y))$
 unfolding sprev-def
 proof -
 have $\bigwedge I. (SEmpty) \cup I \cdot SSkip = -(\neg I \cdot SSkip)$
 using N26 sprev-def suprev-def by blast
 then have $(Y \cup \neg X) \cdot SSkip \subseteq Y \cdot SSkip \cup - (X \cdot SSkip)$
 using FusionUnionDistL by blast
 then show $(\neg X \cup Y) \cdot SSkip \subseteq - (X \cdot SSkip) \cup Y \cdot SSkip$
 by (simp add: UnionCommute)
 qed

lemma N29:
 $(SPrev\ X) = -(SWprev\ \neg X)$
 by (simp add: sprev-def suprev-def)

16.5.4 SInit

lemma *ST01*:

$$(X \cap S\text{Empty}) \cdot Y \subseteq Y$$

by (*metis Int-commute FusionUnionDistL FusionSEmptyL le-iff-sup sup-inf-absorb UnionCommute*)

lemma *ST02*:

$$(X \cap S\text{Empty}) \cdot Y \subseteq (X \cap S\text{Empty}) \cdot S\text{True}$$

by (*simp add: FusionRuleR strue-def*)

lemma *ST03*:

$$(X \cap S\text{Empty}) \cdot (X \cap S\text{Empty}) \subseteq (X \cap S\text{Empty})$$

using *ST01* **by** *auto*

lemma *ST04*:

$$(X \cap S\text{Empty}) \subseteq (X \cap S\text{Empty}) \cdot (X \cap S\text{Empty})$$

by (*metis (no-types, lifting) B04 CH10 FusionSEmptyL FusionSEmptyR Int-Un-eq(2) Int-Un-eq(4) inf-commute inf-sup-absorb*)

lemma *ST05*:

$$(X \cap S\text{Empty}) \subseteq -((-X) \cap S\text{Empty})$$

by *blast*

lemma *ST06*:

$$((-X) \cap S\text{Empty}) \subseteq -(X \cap S\text{Empty})$$

by *auto*

lemma *ST07*:

$$(X \cap S\text{Empty}) \cap (Y \cap S\text{Empty}) \subseteq (X \cap S\text{Empty}) \cdot S\text{True}$$

using *ST02 FusionSEmptyR* **by** *blast*

lemma *ST08*:

$$(X \cap S\text{Empty}) \cap (Y \cap S\text{Empty}) \subseteq (S\text{True} \cap S\text{Empty}) \cdot (Y \cap S\text{Empty})$$

by (*metis FusionSEmptyL FusionSEmptyR ST33 inf.cobounded2*)

lemma *ST09*:

$$((X \cap S\text{Empty}) \cdot S\text{True}) \cap (S\text{True} \cap S\text{Empty}) \cdot (Y \cap S\text{Empty}) \subseteq (X \cap S\text{Empty}) \cdot (Y \cap S\text{Empty})$$

by (*metis compl-bot-eq eq-refl FusionAssoc FusionSEmptyR inf commute inf-top.left-neutral CH09 strue-def*)

lemma *ST10*:

$$(X \cap S\text{Empty}) \cdot (Y \cap S\text{Empty}) \subseteq (X \cap S\text{Empty})$$

by (*metis FusionRuleR FusionSEmptyR inf-le2*)

lemma *ST11*:

$$(X \cap S\text{Empty}) \cdot (Y \cap S\text{Empty}) \subseteq (Y \cap S\text{Empty})$$

using *ST01* **by** *blast*

lemma *ST12*:

$$(X \cap S\text{Empty}) \cap (Y \cap S\text{Empty}) = (X \cap S\text{Empty}) \cdot S\text{Empty} \cap (Y \cap S\text{Empty}) \cdot S\text{Empty}$$

by (*simp add: FusionSEmptyR*)

lemma ST14:

$((X \cap Y) \cap SEmpty) \cdot SEmpty = ((X \cap Y) \cap SEmpty)$
by (*simp add: FusionSEmptyR*)

lemma ST16:

$(X \cap SEmpty) \cap (Y \cap SEmpty) \subseteq SEmpty$
by (*simp add: le-infI2*)

lemma ST17:

$(X \cap SEmpty) \cdot (Y \cap SEmpty) \subseteq SEmpty$
using ST10 **by** *auto*

lemma ST18:

$\neg((X \cap SEmpty) \cup (Y \cap SEmpty)) = \neg(X \cap SEmpty) \cap \neg(Y \cap SEmpty)$
by *auto*

lemma ST19:

$(X \cap SEmpty) \cdot (\neg X \cap SEmpty) \subseteq (X \cap SEmpty)$
using ST10 **by** *blast*

lemma ST20:

$(X \cap SEmpty) \cdot (\neg X \cap SEmpty) \subseteq (\neg X \cap SEmpty)$
using ST01 **by** *auto*

lemma ST22:

$((X \cap SEmpty) \cdot SSkip) \cdot (Y \cap SEmpty) \subseteq (X \cap SEmpty) \cdot SSkip$
using *FusionRuleR FusionSEmptyR* **by** *blast*

lemma ST23:

$((X \cap SEmpty) \cdot SSkip) \cdot (Y \cap SEmpty) \subseteq SSkip \cdot (Y \cap SEmpty)$
by (*simp add: ST01 FusionRuleL*)

lemma ST27:

$(SInit X) \cap (Y \cdot Z) \subseteq ((SInit X) \cap Y) \cdot Z$
by (*metis B04 compl-bot-eq FusionAssoc FusionSEmptyL inf-commute inf-top.left-neutral CH09 sinit-def strue-def*)

lemma ST29:

$(SInit X) \cdot Y \subseteq (SInit X)$
using ST02 *FusionAssoc sinit-def* **by** *fastforce*

lemma ST30:

$(SInit X) \cap (SDi Y) = (SDi ((SInit X) \cap Y))$
unfolding *sdi-def sinit-def strue-def*
by (*metis CH09 FusionAssoc FusionSEmptyL compl-bot-eq inf-top.left-neutral*)

lemma ST31:

$(X \cdot (STrue \cap SEmpty)) \cap (STrue \cdot (Y \cap SEmpty)) = X \cdot (Y \cap SEmpty)$

by (*metis Int-commute compl-bot-eq inf-top.right-neutral CH10 strue-def*)

lemma *ST32*:

$$(STrue \cap SEmpty) \cdot SEmpty \cap (SInit X) = (X \cap SEmpty)$$

by (*metis Compl-empty-eq Int-commute CH09 ST14 inf-top.right-neutral sinit-def strue-def*)

lemma *ST34*:

$$((X \cap SEmpty) \cdot Y) = (SInit X) \cap Y$$

by (*metis FusionSEmptyL Int-commute CH09 compl-bot-eq inf-top-right sinit-def strue-def*)

lemma *ST35*:

$$((SInit X) \cap Y) \cdot Z \subseteq (SInit X) \cap (Y \cdot Z)$$

by (*metis B04 ST34 FusionAssoc*)

lemma *ST39*:

$$SEmpty \cap (SInit X) \subseteq (X \cap SEmpty)$$

using *ST32* **by** *blast*

lemma *ST40*:

$$(X \cap SEmpty) \subseteq SEmpty \cap (SInit X)$$

using *ST32* **by** *auto*

lemma *ST41*:

$$SEmpty \cap (SInit X) = (X \cap SEmpty)$$

using *ST40 ST39* **by** *auto*

lemma *ST42*:

$$(X \cap SEmpty) \subseteq ((X \cup Y) \cap SEmpty)$$

by *blast*

lemma *ST43*:

$$(Y \cap SEmpty) \subseteq ((X \cup Y) \cap SEmpty)$$

by *blast*

lemma *ST44*:

$$(X \cap SEmpty) \cap ((-X) \cap SEmpty) = SFalse$$

by (*simp add: sfalse-def*)

lemma *ST45*:

$$((X \cup Y) \cap SEmpty) \subseteq (X \cap SEmpty) \cup (Y \cap SEmpty)$$

by *auto*

lemma *ST46*:

$$(SInit X) \cup (SInit Y) = (SInit (X \cup Y))$$

by (*simp add: Int-Un-distrib2 FusionUnionDistL sinit-def*)

lemma *ST48*:

$$-(STrue \cdot (X \cap SEmpty)) \subseteq STrue \cdot ((-X) \cap SEmpty)$$

by (*metis B09 FusionSEmptyR FusionUnionDistR ST21 double-compl*)

lemma *ST49*:

$STrue.((-X) \cap SEmpty) \subseteq -(STrue.(X \cap SEmpty))$

by (*metis CH10 Compl-disjoint2 FusionSFalse disjoint-eq-subset-Compl inf.idem inf-compl-bot-left2 sfalse-def*)

lemma *ST50*:

$-(STrue.(X \cap SEmpty)) = STrue.((-X) \cap SEmpty)$

using *ST48 ST49 by blast*

16.5.5 SStar

lemma *SStar02*:

assumes $X \subseteq Y$

shows $X.(SStar Y) \cup SEmpty \subseteq (SStar Y)$

using *assms*

by (*metis FusionUnionDistL Int-lower1 SStar15 Un-commute Un-mono UnfoldL inf.orderE sup.orderE sup.orderI*)

lemma *SStar04*:

$(SStar X) \subseteq (SStar X).(SStar X)$

by (*metis Un-absorb UnfoldL UnionAssoc ST47 sup.absorb-iff2*)

lemma *SStar09*:

assumes $(X.(SEmpty \cup (X.(SStar X)))) \cup SEmpty \subseteq (SEmpty \cup (X.(SStar X)))$

shows $(SStar X) \subseteq SEmpty \cup (X.(SStar X))$

using *assms*

by (*simp add: UnfoldL*)

lemma *SStar10*:

$(X.(SEmpty \cup (X.(SStar X)))) \subseteq (SEmpty \cup (X.(SStar X)))$

by (*metis UnfoldL sup-ge2*)

lemma *SStar11*:

$SEmpty \subseteq (SEmpty \cup (X.(SStar X)))$

by *auto*

lemma *SStar13*:

$(SStar SSkip) = STrue$

by (*simp add: SStarSkip*)

lemma *SStar14*:

$(SSometime X) = (SStar SSkip).X$

by (*simp add: SStarSkip ssometime-def*)

lemma *SStar20*:

$(SStar SEmpty) = SEmpty$

by (*metis FusionSEmptyR ST15 ST33*)

lemma *SStar21*:

$(SStar (SEmpty \cap X)) \cdot (SEmpty \cap X) = (SEmpty \cap X)$

by (*metis ST15 FusionSEmptyL inf-commute*)

lemma *SStar24*:

$(SStar SFalse) = SEmpty$

by (*metis SStar20 SStar47 inf-compl-bot sfalse-def smore-def*)

lemma *SStar26*:

$X \subseteq (SStar X)$

by (*metis FusionSEmptyR FusionUnionDistR SStar08 UnCI UnfoldL subsetI subset-iff*)

lemma *SStar27*:

$SEmpty \subseteq (SStar X)$

using *UnfoldL* **by** *blast*

lemma *SStar31*:

assumes $X \cup (X \cdot Y) \cdot (X \cdot (SStar (Y \cdot X))) \subseteq X \cdot (SStar (Y \cdot X))$

shows $(SStar (X \cdot Y)) \cdot X \subseteq X \cdot (SStar (Y \cdot X))$

using *assms SStarInductL* **by** *blast*

lemma *SStar32*:

$X \cup (X \cdot Y) \cdot (X \cdot (SStar (Y \cdot X))) \subseteq X \cdot (SStar (Y \cdot X))$

by (*metis B06 SStar10 SStar11 FusionAssoc FusionRuleR FusionSEmptyR UnfoldL*)

lemma *SStar33*:

$(SStar (X \cdot Y)) \cdot X \subseteq X \cdot (SStar (Y \cdot X))$

using *SStar31 SStar32* **by** *blast*

lemma *SStar37*:

assumes $X \cdot Z \subseteq Z \cdot Y$

shows $(SStar X) \cdot Z \subseteq Z \cdot (SStar Y)$

proof –

have $Z \cdot SStar Y = Z \cdot SEmpty \cup Z \cdot (Y \cdot SStar Y)$

by (*metis FusionUnionDistR UnfoldL*)

then have $Z \cdot SStar Y \cup (Z \cup X \cdot (Z \cdot SStar Y)) = Z \cup Z \cdot Y \cdot SStar Y \cup X \cdot Z \cdot SStar Y$

using *FusionAssoc FusionSEmptyR* **by** *blast*

then have $Z \cdot SStar Y \cup (Z \cup X \cdot (Z \cdot SStar Y)) = Z \cdot SStar Y$

by (*metis (no-types) FusionAssoc FusionSEmptyR FusionUnionDistL FusionUnionDistR UnfoldL UnionAssoc*)

assms sup.absorb-iff1)

then show *?thesis*

by (*meson SStarInductL sup.absorb-iff1*)

qed

lemma *SStar38*:

assumes $Z \cdot X \subseteq Y \cdot Z$

shows $Z \cdot (SStar X) \subseteq (SStar Y) \cdot Z$

using *assms*

proof –
have $f1$: $Z \cup SStar\ Y \cdot Y \cdot Z = SStar\ Y \cdot Z$
by (*metis* (*no-types*) *SStar30 ST47 UnfoldL*)
have $SStar\ Y \cdot Y \cdot Z = SStar\ Y \cdot Z \cdot X \cup SStar\ Y \cdot Y \cdot Z$
by (*metis* *FusionAssoc FusionUnionDistR assms subset-Un-eq*)
then have $Z \cup SStar\ Y \cdot Z \cdot X \subseteq SStar\ Y \cdot Z$
using $f1$ **by** *blast*
then show *?thesis*
by (*simp add: SStarInductR*)
qed

lemma *SStar39*:
 $Y \cdot (SStar\ ((SStar\ X) \cdot Y)) \subseteq (SStar\ (Y \cdot (SStar\ X))) \cdot Y$
by (*simp add: SStar38 FusionAssoc*)

lemma *SStar40*:
 $(SStar\ (Y \cdot (SStar\ X))) \cdot Y \subseteq Y \cdot (SStar\ ((SStar\ X) \cdot Y))$
by (*simp add: SStar33*)

lemma *SStar41*:
 $Y \cdot (SStar\ ((SStar\ X) \cdot Y)) = (SStar\ (Y \cdot (SStar\ X))) \cdot Y$
using *SStar39 SStar40* **by** *blast*

lemma *SStar42*:
 $Z \cdot (SStar\ (Y \cdot Z)) \subseteq (SStar\ (Z \cdot Y)) \cdot Z$
by (*simp add: SStar38 FusionAssoc*)

lemma *SStar43*:
 $(SStar\ (Z \cdot Y)) \cdot Z \subseteq Z \cdot (SStar\ (Y \cdot Z))$
by (*simp add: SStar33*)

lemma *SStar44*:
 $Z \cdot (SStar\ (Y \cdot Z)) = (SStar\ (Z \cdot Y)) \cdot Z$
using *SStar42 SStar43* **by** *blast*

lemma *SStar49*:
 $(SStar\ X) = SEmpty \cup (SStar\ X) \cdot X$
using *SStar30 UnfoldL* **by** *blast*

16.5.6 Box and Diamond

lemma *BD01*:
 $(SSometime\ SEmpty) = STrue$
by (*simp add: ssometime-def FusionSEmptyR*)

lemma *BD02*:
 $X \subseteq (SSometime\ X)$
unfolding *ssometime-def*
by (*metis* *CH01 SStar25 ST47 UnfoldL subset-Un-eq*)

lemma *BD03*:

$$(SNext (SSometime X)) \subseteq (SSometime X)$$

by (*metis FusionUnionDistL N03 SStar16 SStar03 SStar04 SStarSkip snext-def ssometime-def sup.absorb-iff2 sup.orderE*)

lemma *BD04*:

$$(SSometime (SNext X)) \subseteq (SSometime X)$$

by (*metis CH01 FusionAssoc FusionUnionDistL FusionUnionDistR SStar16 SStarSkip snext-def ssometime-def sup.absorb-iff2*)

lemma *BD05*:

$$(SSometime X) \cup (SSometime Y) = (SSometime (X \cup Y))$$

by (*simp add: FusionUnionDistR ssometime-def*)

lemma *BD06*:

$$(SSometime STrue) = STrue$$

by (*simp add: CH01 ssometime-def*)

lemma *BD07*:

$$(SSometime (X \cap Y)) \subseteq (SSometime X) \cap (SSometime Y)$$

by (*simp add: FusionRuleR ssometime-def*)

lemma *BD08*:

$$(SAlways STrue) = STrue$$

by (*simp add: SBoxGen*)

lemma *BD09*:

$$\neg(SAlways X) = (SSometime (\neg X))$$

by (*simp add: salways-def*)

lemma *BD10*:

$$(SAlways X) \subseteq (SSometime X)$$

by (*metis B02 BD02 BD09 subset-trans*)

lemma *BD11*:

$$(SSometime (SSometime X)) = (SSometime X)$$

by (*simp add: CH01 ssometime-def FusionAssoc*)

lemma *BD12*:

$$(SAlways X) \subseteq X$$

by (*simp add: B02 BD02 BD09*)

lemma *BD13*:

$$(SDi STrue) = STrue$$

by (*simp add: CH01 sdi-def*)

lemma *BD14*:

$$(SDi SEmpty) = STrue$$

by (*simp add: sdi-def FusionSEmptyL*)

lemma *BD15*:

$(S\text{Bi } S\text{True}) = S\text{True}$

by (*simp add: SBiGen*)

lemma *BD16*:

$(S\text{Di } (X \cup Y)) = (S\text{Di } X) \cup (S\text{Di } Y)$

by (*simp add: FusionUnionDistL sdi-def*)

lemma *BD17*:

assumes $X \subseteq Y$

shows $(S\text{Di } X) \subseteq (S\text{Di } Y)$

using *assms*

by (*metis FusionUnionDistL Subsumption sdi-def*)

lemma *BD18*:

$(S\text{Di } (S\text{Di } X)) = (S\text{Di } X)$

by (*metis CH01 FusionAssoc sdi-def*)

lemma *BD19*:

$(S\text{Da } S\text{Empty}) = S\text{True}$

by (*simp add: CH01 sda-def FusionSEmptyR*)

lemma *BD20*:

$(S\text{Da } S\text{True}) = S\text{True}$

by (*simp add: CH01 sda-def*)

lemma *BD21*:

$(S\text{Ba } S\text{True}) = S\text{True}$

by (*metis BD15 BD08 BD09 sba-def sbi-def sda-def sdi-def ssometime-def*)

lemma *BD22*:

$(S\text{Da } (X \cup Y)) = (S\text{Da } X) \cup (S\text{Da } Y)$

by (*simp add: FusionUnionDistL FusionUnionDistR sda-def*)

lemma *BD23*:

assumes $X \subseteq Y$

shows $(S\text{Da } X) \subseteq (S\text{Da } Y)$

using *assms*

by (*metis BD22 Subsumption*)

lemma *BD24*:

assumes $X \subseteq Y$

shows $(S\text{Da } (-Y)) \subseteq (S\text{Da } (-X))$

using *assms*

by (*simp add: BD23*)

lemma *BD25*:

$(S\text{Di } X) \subseteq (S\text{Da } X)$

by (*metis BD02 FusionAssoc sda-def sdi-def ssometime-def*)

lemma *BD26*:

$(SSometime\ X) \subseteq (SDa\ X)$

by (*metis BD01 BD02 FusionSEmptyR FusionUnionDistR SStar14 le-iff-sup sda-def*)

lemma *BD27*:

$(SBa\ X) \subseteq (SBi\ X)$

by (*simp add: BD25 sba-def sbi-def*)

lemma *BD28*:

$(SBa\ X) \subseteq (SAlways\ X)$

by (*simp add: B02 BD26 BD09 sba-def*)

lemma *BD29*:

$(SAlways\ X) \cap (SAlways\ Y) = (SAlways\ (X \cap Y))$

by (*metis BD05 BD09 Morgan compl-inf salways-def*)

lemma *BD30*:

$(SAlways\ X) \cup (SAlways\ Y) \subseteq (SAlways\ (X \cup Y))$

using *BD07*

by (*metis B02 BD09 compl-sup*)

lemma *BD31*:

$(SDi\ (X \cap Y)) \subseteq (SDi\ X) \cap (SDi\ Y)$

by (*simp add: BD17*)

lemma *BD32*:

$(SBi\ X) \cup (SBi\ Y) \subseteq (SBi\ (X \cup Y))$

using *BD31*

by (*metis (mono-tags, lifting) B02 compl-sup double-compl sbi-def*)

lemma *BD33*:

$(SDa\ (X \cap Y)) \subseteq (SDa\ X) \cap (SDa\ Y)$

by (*simp add: BD23*)

lemma *BD34*:

$(SBa\ X) \cup (SBa\ Y) \subseteq (SBa\ (X \cup Y))$

using *BD33*

by (*metis (mono-tags, lifting) B02 compl-sup double-compl sba-def*)

lemma *BD35*:

$(SAlways\ SEmpty) = SEmpty$

by (*metis N13 SStar14 SStar30 SStar48 SStarSkip double-complement salways-def smore-def*)

lemma *BD36*:

$(SBi\ SEmpty) = SEmpty$

using *N13 sbi-def sdi-def smore-def* **by** *fastforce*

lemma *BD37*:

$(SBa\ SEmpty) = SEmpty$
by (*metis N13 SStar30 SStar48 double-complement sba-def sda-def smore-def*)

lemma *BD38*:
assumes $X \subseteq Y$
shows $(SAlways\ X) \subseteq (SAlways\ Y)$
using *assms*
by (*simp add: BD29 inf.absorb-iff2*)

lemma *BD39*:
assumes $X \subseteq Y$
shows $(SBi\ X) \subseteq (SBi\ Y)$
using *assms*
by (*simp add: BD17 sbi-def*)

lemma *BD40*:
assumes $X \subseteq Y$
shows $(SBa\ X) \subseteq (SBa\ Y)$
using *assms*
by (*simp add: BD24 sba-def*)

lemma *BD41*:
 $(SBi\ (SBi\ X)) = (SBi\ X)$
by (*simp add: BD18 sbi-def*)

lemma *BD42*:
 $(SAlways\ (SAlways\ X)) = (SAlways\ X)$
by (*simp add: BD11 salways-def*)

lemma *BD43*:
 $(SDa\ (SDa\ X)) = (SDa\ X)$
by (*metis CH01 FusionAssoc sda-def*)

lemma *BD44*:
 $(SBa\ (SBa\ X)) = (SBa\ X)$
by (*simp add: BD43 sba-def*)

lemma *BD47*:
 $(SAlways\ ((-X) \cup Y)) \subseteq ((- (SAlways\ X)) \cup (SAlways\ Y))$
by (*metis B20 BD12 BD29 BD38 BD42 double-compl*)

lemma *BD48*:
 $(SAlways\ X) \subseteq X \cap (SNext\ (SAlways\ X))$
by (*metis B02 B16 BD03 BD09 BD12 N12 salways-def*)

lemma *BD49*:
 $(SBi\ ((-X) \cup Y)) \subseteq ((- (SBi\ X)) \cup (SBi\ Y))$
by (*metis B20 BD45 Un-commute double-complement sbi-def sdi-def*)

lemma *BD50*:

$(SPrev (SDi X)) \subseteq (SDi X)$
by (*metis B04 FusionAssoc FusionUnionDistR N08 SSkipFusionImpSMore SStar19 SStarSkip*
STrueTop sdi-def smore-def sprev-def sup-ge2)

lemma BD51:
 $-(SBi X) = (SDi (-X))$
by (*simp add: sbi-def*)

lemma BD52:
 $X \subseteq (SDi X)$
by (*metis FusionSEmptyR FusionUnionDistR ST33 Subsumption UnionCommute sdi-def sup-inf-absorb*)

lemma BD53:
 $(SBi X) \subseteq X$
by (*simp add: B02 BD51 BD52*)

lemma BD54:
 $(SBi X) \subseteq X \cap (SWprev (SBi X))$
by (*metis B02 B16 BD50 BD51 BD53 N29 sbi-def*)

lemma BD55:
 $(SBi (SMore \cup X)) = (SInit X)$
by (*metis (no-types, lifting) ST38 compl-sup double-complement inf-commute sbi-def sdi-def*
sinit-def smore-def)

lemma BD56:
 $(SAlways (SMore \cup X)) = STrue \cdot (X \cap SEmpty)$
by (*simp add: SStar14 SStarSkip ST50 UnionCommute salways-def smore-def*)

16.6 Time Reversal

16.6.1 Time Reversal Axioms

lemma SRevSEmpty:
 $(SRev SEmpty) = SEmpty$
using *set-eqI*[*of (SRev SEmpty) SEmpty*]
by (*simp add: sempty-elim srev-elim*)

lemma SRevSNot:
 $(SRev (- X)) = (- (SRev X))$
using *set-eqI*[*of (SRev (- X)) (- (SRev X))*]
by (*simp add: srev-elim*)

lemma fuse-irev:
assumes *ilast xs = ifirst ys*
shows $irev (fuse xs ys) = fuse (irev ys) (irev xs)$
using *assms fuse-prefix-suffix*
 $prefix-fuse[*of xs ys*]$
 $suffix-fuse[*of xs ys*]$
by (*metis irev-ilen irev-prefix irev-suffix prefix-ilen-bound suffix-ilen*)

lemma *SRevFusionsem*:

$x \in (SRev (X \cdot Y)) = (x \in ((SRev Y) \cdot (SRev X)))$

by (*auto simp add: srev-elim fusion-iff*)

(*metis fuse-irev ilast-irev irev-irev-ident, metis fuse-irev ifirst-irev ilast-irev*)

lemma *SRevFusion*:

$(SRev (X \cdot Y)) = (SRev Y) \cdot (SRev X)$

using *set-eqI*[*of* ($(SRev (X \cdot Y))$) ($(SRev Y) \cdot (SRev X)$)]

using *SRevFusionsem* **by** *auto*

lemma *SRevUnion*:

$(SRev (X \cup Y)) = (SRev X) \cup (SRev Y)$

using *set-eqI*[*of* ($(SRev (X \cup Y))$) ($(SRev X) \cup (SRev Y)$)]

using *srev-elim* **by** *auto*

lemma *SRevSPower*:

$(SRev (SPower X n)) = (SPower (SRev X) n)$

proof (*induct n*)

case 0

then show ?*case* **by** (*simp add: SRevSEmpty*)

next

case (*Suc n*)

then show ?*case*

proof –

have $SRev X \cap SMore = SRev (X \cap SMore)$

by (*metis (no-types) Morgan SRevSEmpty SRevSNot SRevUnion smore-def*)

then show ?*thesis*

by (*simp add: SRevFusion Suc.hyps spower-commutes*)

qed

qed

lemma *SRevSStar*:

$(SRev (SStar X)) = (SStar (SRev X))$

proof –

have 1: $(SRev (SStar X)) = (SRev (\bigcup n. SPower X n))$ **by** (*simp add: sstar-def*)

have 2: $(SRev (\bigcup n. SPower X n)) = (\bigcup n. SPower (SRev X) n)$

using *set-eqI*[*of* ($(SRev (\bigcup n. SPower X n))$) ($(\bigcup n. SPower (SRev X) n)$)]
SRevSPower[*of* X]

by (*metis srev-elim sstar-def sstar-elim*)

have 3: $(\bigcup n. SPower (SRev X) n) = (SStar (SRev X))$ **by** (*simp add: sstar-def*)

from 1 2 3 **show** ?*thesis* **by** *auto*

qed

lemma *SRevSRev*:

$(SRev (SRev X)) = X$

using *set-eqI*[*of* ($(SRev (SRev X))$) X]

by (*simp add: srev-elim*)

16.6.2 Time Reversal Laws

lemma *TR01*:

$$(SRev\ SMore) = SMore$$

by (*simp add: SRevSEmpty SRevSNot smore-def*)

lemma *TR02*:

$$(SRev\ SSkip) = SSkip$$

by (*metis SRevFusion SRevSEmpty SRevSNot SRevUnion TR01 sskip-def*)

lemma *TR03*:

$$(SRev\ STrue) = STrue$$

by (*metis SRevSStar SStarSkip TR02*)

lemma *TR04*:

$$(SRev\ SFalse) = SFalse$$

by (*metis Compl-eq-Compl-iff SRevSNot TR03 sfalse-def strue-def*)

lemma *TR05*:

$$(SRev\ (SSometime\ X)) = (SDi\ (SRev\ X))$$

by (*simp add: SRevFusion TR03 sdi-def ssometime-def*)

lemma *TR06*:

$$(SRev\ (SAlways\ X)) = (SBi\ (SRev\ X))$$

by (*simp add: SRevSNot TR05 salways-def sbi-def*)

lemma *TR07*:

$$(SRev\ (SDi\ X)) = (SSometime\ (SRev\ X))$$

by (*simp add: SRevFusion TR03 sdi-def ssometime-def*)

lemma *TR08*:

$$(SRev\ (SBi\ X)) = (SAlways\ (SRev\ X))$$

by (*metis SRevSRev TR06*)

lemma *TR09*:

$$(SRev\ (SNext\ X)) = (SPrev\ (SRev\ X))$$

by (*simp add: SRevFusion TR02 snext-def sprev-def*)

lemma *TR10*:

$$(SRev\ (SWnext\ X)) = (SWprev\ (SRev\ X))$$

by (*simp add: SRevFusion SRevSNot TR02 swnext-def swprev-def*)

lemma *TR11*:

$$(SRev\ (SPrev\ X)) = (SNext\ (SRev\ X))$$

by (*simp add: SRevFusion TR02 snext-def sprev-def*)

lemma *TR12*:

$$(SRev\ (SWprev\ X)) = (SWnext\ (SRev\ X))$$

by (*metis SRevSRev TR10*)

lemma *TR13*:

$(SRev (SDa X)) = (SDa (SRev X))$
by (*simp add: SRevFusion TR03 sda-def FusionAssoc*)

lemma TR14:
 $(SRev (SBa X)) = (SBa (SRev X))$
by (*simp add: SRevSNot TR13 sba-def*)

lemma TR15:
 $(SRev (SPower SSkip n)) = (SPower SSkip n)$
by (*simp add: SRevSPower TR02*)

lemma TR16:
assumes $X \subseteq Y$
shows $(SRev X) \subseteq (SRev Y)$
using *assms* **by** (*metis SRevUnion le-iff-sup*)

lemma TR17:
assumes $X = Y$
shows $(SRev X) = (SRev Y)$
using *assms TR16* **by** *auto*

16.7 Link between Set of Intervals and ITL

lemma interval-lan [simp]:
 $\sigma \in (lan f) \longleftrightarrow (\sigma \models f)$
by (*simp add: lan-def*)

lemma valid-lan-equiv :
 $((lan f) = (lan g)) \longleftrightarrow (\vdash f = g)$
using *interval-lan lan-def Valid-def* **by** *fastforce*

lemma valid-lan-imp :
 $((lan f) \subseteq (lan g)) \longleftrightarrow (\vdash f \longrightarrow g)$
using *interval-lan lan-def Valid-def*
by (*simp add: Valid-def lan-def Collect-mono-iff*)

lemma valid-strue :
 $((lan f) = STrue) \longleftrightarrow (\vdash f)$
using *strue-def* **by** *fastforce*

lemma strue-true:
 $\sigma \in STrue \longleftrightarrow (\sigma \models \#True)$
by (*simp add: strue-elim*)

lemma strue-true-1:
 $STrue = (lan (LIFT \#True))$
using *lan-def strue-true* **by** *fastforce*

lemma sfalse-false:
 $\sigma \in SFalse \longleftrightarrow (\sigma \models \#False)$

by (*simp add: sfalse-def*)

lemma *sfalse-false-1*:

$SFalse = (\text{lan } (LIFT(\#False)))$

using *sfalse-false using lan-def by fastforce*

lemma *not-negation*:

$\sigma \in (\neg(\text{lan } f)) \longleftrightarrow (\sigma \models \neg f)$

by *simp*

lemma *not-negation-1*:

$\neg(\text{lan } f) = (\text{lan } (LIFT(\neg f)))$

using *interval-lan lan-def by fastforce*

lemma *inter-and*:

$(\sigma \in ((\text{lan } f) \cap (\text{lan } g))) \longleftrightarrow (\sigma \models f \wedge g)$

by (*simp add: lan-def*)

lemma *inter-and-1*:

$((\text{lan } f) \cap (\text{lan } g)) = (\text{lan } (LIFT(f \wedge g)))$

using *inter-and lan-def by fastforce*

lemma *union-or*:

$(\sigma \in ((\text{lan } f) \cup (\text{lan } g))) \longleftrightarrow (\sigma \models f \vee g)$

by (*simp add: lan-def*)

lemma *union-or-1*:

$((\text{lan } f) \cup (\text{lan } g)) = (\text{lan } (LIFT(f \vee g)))$

using *union-or lan-def by fastforce*

lemma *subset-impl*:

$(\sigma \in (\neg(\text{lan } f) \cup (\text{lan } g))) \longleftrightarrow (\sigma \models f \longrightarrow g)$

by *simp*

lemma *subset-impl-1*:

$(\neg(\text{lan } f) \cup (\text{lan } g)) = (\text{lan } (LIFT(f \longrightarrow g)))$

using *subset-impl lan-def by fastforce*

lemma *fusion-chop*:

$(\sigma \in ((\text{lan } f) \cdot (\text{lan } g))) \longleftrightarrow (\sigma \models f;g)$

by (*metis fusion-iff chop-fuse interval-lan*)

lemma *fusion-chop-1*:

$((\text{lan } f) \cdot (\text{lan } g)) = (\text{lan } (LIFT(f;g)))$

using *fusion-chop lan-def by blast*

lemma *empty-empty*:

$\sigma \in SEmpty \longleftrightarrow (\sigma \models \text{empty})$

by (*simp add: empty-defs empty-elim*)

lemma *empty-empty-1*:

$SEmpty = (\text{lan } (LIFT \text{ empty}))$

using *empty-empty lan-def* **by** *fastforce*

lemma *smore-more*:

$\sigma \in SMore \longleftrightarrow (\sigma \models \text{more})$

by (*simp add: more-defs smore-elim*)

lemma *smore-more-1*:

$SMore = (\text{lan } (LIFT \text{ more}))$

using *smore-more lan-def* **by** *fastforce*

lemma *sskip-skip*:

$\sigma \in SSkip = (\sigma \models \text{skip})$

by (*simp add: skip-defs sskip-elim*)

lemma *sskip-skip-1*:

$SSkip = (\text{lan } (LIFT \text{ skip}))$

using *sskip-skip lan-def* **by** *fastforce*

lemma *snext-next*:

$\sigma \in (SNext (\text{lan } f)) \longleftrightarrow (\sigma \models \circ f)$

by (*metis snext-def fusion-chop next-d-def sskip-skip-1*)

lemma *snext-next-1*:

$(SNext (\text{lan } f)) = (\text{lan } (LIFT(\circ f)))$

using *snext-next lan-def* **by** *fastforce*

lemma *swnext-wnext*:

$\sigma \in (SWnext (\text{lan } f)) \longleftrightarrow (\sigma \models \text{wnext } f)$

by (*simp add: swnext-def fusion-chop-1 next-d-def not-negation-1 sskip-skip-1 wnext-d-def*)

lemma *swnext-wnext-1*:

$(SWnext (\text{lan } f)) = (\text{lan } (LIFT(\text{wnext } f)))$

using *swnext-wnext lan-def* **by** *fastforce*

lemma *sprev-prev*:

$\sigma \in (SPrev (\text{lan } f)) \longleftrightarrow (\sigma \models \text{prev } f)$

by (*metis fusion-chop prev-d-def sprev-def sskip-skip-1*)

lemma *sprev-prev-1*:

$(SPrev (\text{lan } f)) = (\text{lan } (LIFT(\text{prev } f)))$

using *sprev-prev lan-def* **by** *fastforce*

lemma *suprev-wprev*:

$\sigma \in (SWprev (\text{lan } f)) \longleftrightarrow (\sigma \models \text{wprev } f)$

by (*simp add: fusion-chop-1 not-negation-1 prev-d-def sskip-skip-1 suprev-def wprev-d-def*)

lemma *suprev-wprev-1*:

$(SWprev (\text{lan } f)) = (\text{lan } (LIFT(\text{wprev } f)))$

using *suprev-wprev lan-def* **by** *fastforce*

lemma *sinit-init*:

$\sigma \in SInit\ (lan\ f) \longleftrightarrow (\sigma \models init\ f)$

by (*simp add: Int-commute fusion-chop-1 init-d-def inter-and-1 empty-empty-1 sinit-def strue-true-1*)

lemma *sinit-init-1*:

$SInit\ (lan\ f) = (lan\ (LIFT(init\ f)))$

using *sinit-init lan-def* **by** *fastforce*

lemma *and-inter-more*:

$\sigma \in (((lan\ f) \cap SMore)) \longleftrightarrow (\sigma \models (f \wedge more))$

using *smore-more inter-and* **by** *auto*

lemma *and-inter-more-1*:

$\sigma \in (((lan\ f) \cap SMore)) \longleftrightarrow (\sigma \in (lan\ (LIFT(f \wedge more))))$

using *and-inter-more lan-def* **by** (*simp add: smore-more-1*)

lemma *and-inter-more-2*:

$((lan\ f) \cap SMore) = (lan\ (LIFT(f \wedge more)))$

using *and-inter-more-1* **by** *blast*

lemma *and-chop*:

$\sigma \in (((lan\ f) \cap SMore) \cdot (lan\ g)) \longleftrightarrow (\sigma \models (f \wedge more);g)$

by (*metis fusion-chop inter-and-1 smore-more-1*)

lemma *and-chop-1*:

$((lan\ f) \cap SMore) \cdot (lan\ g) = (lan\ (LIFT((f \wedge more);g)))$

using *and-chop lan-def* **by** *blast*

lemma *spower-chop-power*:

$(SPower\ (lan\ f)\ n) = (lan\ (LIFT(power\ (f \wedge more)\ n)))$

proof (*induct n*)

case 0

then show ?case **by** (*simp add: empty-empty-1*)

next

case (*Suc n*)

then show ?case **by** (*metis and-chop-1 pow-Suc pwr-Suc*)

qed

lemma *sstar-spower*:

$\sigma \in SStar\ (lan\ f) \longleftrightarrow (\exists\ n. \sigma \in SPower\ (lan\ f)\ n)$

by (*simp add: sstar-def*)

lemma *sstar-chopstar*:

$\sigma \in (SStar\ (lan\ f)) \longleftrightarrow \sigma \in (lan\ (LIFT(f^*)))$

proof –

have 1: $\sigma \in (SStar\ (lan\ f)) = (\exists\ n. \sigma \in SPower\ (lan\ f)\ n)$

using *sstar-spower* **by** *blast*

have 2: $(\exists\ n. \sigma \in SPower\ (lan\ f)\ n) =$

```

      ( $\exists n. \sigma \in \text{lan} (\text{LIFT}(\text{power} (f \wedge \text{more}) n))$ )
using spower-chop-power by blast
have 3: ( $\exists n. \sigma \in \text{lan} (\text{LIFT}(\text{power} (f \wedge \text{more}) n))$ ) =
      ( $\exists n. (\text{LIFT}(\text{power} (f \wedge \text{more}) n)) \sigma$ )
using interval-lan by simp
have 4: ( $\exists n. (\text{LIFT}(\text{power} (f \wedge \text{more}) n)) \sigma$ ) =
      ( $\sigma \in (\text{lan} (\text{LIFT}(f^*))$ ))
by (simp add: chopstar-d-def powerstar-d-def)
show ?thesis by (simp add: 1 2 4)
qed

```

```

lemma chopstar-sstar-1:
  ( $\text{SStar} (\text{lan } f) = (\text{lan} (\text{LIFT}(f^*))$ ))
using sstar-chopstar lan-def by blast

```

```

lemma chopstar-seqv:
   $\sigma \in (\text{lan} (\text{LIFT}(f^*)) \longleftrightarrow$ 
     $\sigma \in (\text{lan} (\text{LIFT}(\text{empty} \vee (f \wedge \text{more}); f^*))$ ))
using ChopstarEqv by fastforce

```

```

lemma chopstar-seqv-1:
  ( $\text{lan} (\text{LIFT}(f^*)) = (\text{lan} (\text{LIFT}(\text{empty} \vee (f \wedge \text{more}); f^*))$ ))
using chopstar-seqv lan-def by blast

```

```

lemma srev-irev:
   $\sigma \in (\text{SRev} (\text{lan } f) \longleftrightarrow \sigma \in (\text{lan} (\text{LIFT}(f^r))$ ))
by (simp add: reverse-d-def srev-elim)

```

```

lemma srev-irev-1:
  ( $\text{SRev} (\text{lan } f) = (\text{lan} (\text{LIFT}(f^r))$ ))
using srev-irev lan-def by blast

```

end

17 Executability of ITL formulae

```

theory Executability
imports Semantics TimeReversal Fuse
begin

```

In this section we will discuss the notion of executability. It is used to determine whether an ITL formula represents a “programming construct”. We first formalise the notion of forward executability of a formula which corresponds to generating a sequence of states in a particular fashion: we first generate the first state and then generate the next until the final state is generated. This sequence of state constitutes the behaviour of the system described by the formula. We then investigate the reflection of forward executable formula and this requires the introduction of the notion of backward executability. This notion corresponds to generating a sequence of states but now we first generate the final state and then generate the previous state until we generate the first state. This sequence corresponds to the reversed behaviour of the system described by the formula. Forward and backward

executability are related by the reflection operator.

17.1 Forward Executability

17.1.1 Common Prefix Value Trace Definitions

definition $\text{sat-d} :: ('a :: \text{world}) \text{ formula} \Rightarrow \text{bool}$
where $\text{sat-d } F \equiv (\exists s. s \models F)$

The following definition is a constraint on the intervals which satisfy a formula. Only intervals that share a common prefix of the value trace are allowed

definition $\text{commonpfx} :: ('a :: \text{world}) \text{ formula} \Rightarrow ('a, 'b) \text{ stfun} \Rightarrow \text{bool}$
where $\text{commonpfx } F v \equiv$
 $(\forall s1\ s. (s1 \models F) \wedge (s \models F) \wedge \text{ilen } s \leq \text{ilen } s1 \longrightarrow$
 $\text{imap } v\ s = (\text{imap } v (\text{prefix } (\text{ilen } s) s1))$
 $)$

The following definition introduces the notion of forward executability.

definition $\text{fexec} :: ('a :: \text{world}) \text{ formula} \Rightarrow ('a, 'b) \text{ stfun} \Rightarrow \text{bool}$
where $\text{fexec } F v \equiv ((\text{sat-d } F) \wedge (\text{commonpfx } F v))$

syntax

$\text{-sat} \quad \quad \quad :: \text{lift} \Rightarrow \text{bool} \quad \quad \quad ((\text{sat } (-))\ 5)$
 $\text{-commonpfx} \quad :: \text{lift} \Rightarrow \text{lift} \Rightarrow \text{bool} \quad \quad ((\ddagger[-]'\text{-})\ 5)$
 $\text{-fexec} \quad \quad \quad :: \text{lift} \Rightarrow \text{lift} \Rightarrow \text{bool} \quad \quad ((\dagger[-]'\text{-})\ 5)$

translations

$\text{-sat} \quad \quad \quad == \text{CONST sat-d}$
 $\text{-commonpfx} == \text{CONST commonpfx}$
 $\text{-fexec} \quad \quad \quad == \text{CONST fexec}$

17.1.2 Semantic lemmas

lemma imap-eq-inth-ilen :

$(\text{imap } v\ s) = (\text{imap } v\ s1) \longleftrightarrow \text{ilen } s = \text{ilen } s1 \wedge (\forall i. i \leq \text{ilen } s \longrightarrow v (\text{inth } s\ i) = v (\text{inth } s1\ i))$
using $\text{interval-eq-inth-eq[of } (\text{imap } v\ s) (\text{imap } v\ s1)]$
by $(\text{simp add: inth-imap})$

lemma $\text{interval-prefix-suffix-help}$:

$n \leq \text{ilen } s1 \implies (\text{suffix } (\text{min } n (\text{ilen } s1)) (\text{prefix } n\ s1)) = \langle \text{inth } s1\ n \rangle$
unfolding min-def
by $(\text{metis (full-types) ilast-prefix prefix-ilen-good suffix-ilast})$

lemma ExistsilenNhelp : $\exists sa. \text{ilen } sa = \text{Suc } (\text{ilen } s)$
by $(\text{metis Suc-eq-plus1-left ilen-imap ilen.simps}(2))$

lemma ExistsilenN :

$\exists s. \text{ilen } s = n$
proof $(\text{induct } n)$
case 0

then show $?case$ **by** (*meson* *ilen.simps(1)*)
next
case (*Suc n*)
then show $?case$ **using** *ExistsilenNhelp* **by** *blast*
qed

lemma *unithelp1*:

$(v (inth\ s1\ 0) = c \vee v (inth\ s1\ 0) = d) \wedge ilen\ s1 = 0$
 $\implies (imap\ v\ s1) = (INil\ c) \vee (imap\ v\ s1) = (INil\ d)$

by (*simp add: INil-ilen ifirst-imap*)

lemma *unithelp2*:

$v (inth\ (INil\ a)\ 0) = c \vee v (inth\ (INil\ a)\ 0) = d \implies$
 $(imap\ v\ (INil\ a)) = (INil\ c) \vee (imap\ v\ (INil\ a)) = (INil\ d)$

by *auto*

lemma *skiphelp1*:

$v (inth\ s1\ 0) = c \wedge ilen\ s1 = 1 \wedge v (inth\ s1\ (Suc\ 0)) = d \implies (imap\ v\ s1) = (ICons\ c\ (INil\ d))$

by (*metis One-nat-def interval-hd-tail ilen-imap inth-imap suffix-ilast less-numeral-extra(1)*)

lemma *skipnoteqnext*:

$v (inth\ s1\ 0) = c \wedge ilen\ s1 = (Suc\ n) \wedge v (inth\ s1\ (Suc\ 0)) = d \wedge$
 $v (inth\ s\ 0) = c \wedge ilen\ s = (Suc\ n) \wedge v (inth\ s\ (Suc\ 0)) = d1 \wedge d \neq d1 \wedge s \neq s1 \implies$
 $(imap\ v\ s1) \neq (imap\ v\ s)$

by (*metis inth-imap*)

lemma *skipnoteqfirst*:

$v (inth\ s1\ 0) = c \wedge ilen\ s1 = (Suc\ n) \wedge v (inth\ s1\ (Suc\ 0)) = d \wedge$
 $v (inth\ s\ 0) = c1 \wedge ilen\ s = (Suc\ n) \wedge v (inth\ s\ (Suc\ 0)) = d \wedge c \neq c1 \wedge s \neq s1 \implies$
 $(imap\ v\ s1) \neq (imap\ v\ s)$

by (*metis inth-imap*)

lemma *skipnoteqfirstnext*:

$v (inth\ s1\ 0) = c \wedge ilen\ s1 = (Suc\ n) \wedge v (inth\ s1\ (Suc\ 0)) = d \wedge$
 $v (inth\ s\ 0) = c1 \wedge ilen\ s = (Suc\ n) \wedge v (inth\ s\ (Suc\ 0)) = d1 \wedge c \neq c1 \wedge d \neq d1 \wedge s \neq s1 \implies$
 $(imap\ v\ s1) \neq (imap\ v\ s)$

by (*metis inth-imap*)

lemma *skiphelp2*:

$v (inth\ s1\ 0) = c \wedge ilen\ s1 = 1 \implies (imap\ v\ s1) = (ICons\ c\ (INil\ (v (inth\ s1\ 1))))$

by (*metis interval-hd-tail ilen-imap inth-imap suffix-ilast less-numeral-extra(1)*)

lemma *lenkhelp3*:

$surj\ v \implies \exists s. v (inth\ s\ 0) = c \wedge ilen\ s = n$

by (*metis ExistsilenN inth.simps(1) Suc-eq-plus1-left inth-zero ilen.simps(1) ilen.simps(2)*
not0-implies-Suc surjD)

lemma *lenkhelp4*:

$surj\ v \implies 0 < n \implies \exists s. v (inth\ s\ 0) = c \wedge v (inth\ s\ 1) = d \wedge ilen\ s = n$

by (*metis One-nat-def inth-Suc inth-zero ilen.simps(2) lenkhelp3 less-numeral-extra(3)*)

not0-implies-Suc plus-1-eq-Suc)

lemma *lenkhelph1*:

$v \text{ (inth } s1 \ 0) = c \wedge \text{ilen } s1 = n \wedge v \text{ (inth } s1 \text{ (Suc } 0)) = d \wedge n = 1 \implies$
 $(\text{imap } v \ s1) = (\text{ICons } c \ (\text{INil } d))$
by (*metis One-nat-def skiphelp2*)

lemma *len0help*:

assumes *surj v*

shows $(\exists s. v \text{ (inth } s \ 0) = c \wedge \text{ilen } s = 0 \wedge (\forall i. i \leq \text{ilen } s \longrightarrow v \text{ (inth } s \ i) = v \text{ (inth } s \ 0)))$

proof –

have *0*: *surj v* \implies

$(\exists s. v \text{ (inth } s \ 0) = c \wedge \text{ilen } s = 0 \wedge (\forall i. i \leq \text{ilen } s \longrightarrow v \text{ (inth } s \ i) = v \text{ (inth } s \ 0))) =$
 $(\exists s. v \text{ (inth } s \ 0) = c \wedge \text{ilen } s = (\text{Suc } 0) \wedge v \text{ (inth } s \ 0) = v \text{ (inth } s \ 0))$

using *lenkhelph3[of v c]* **by** (*metis inth.simps(1) ilen.simps(1)*)

have *01*: *surj v* $\implies (\exists s. v \text{ (inth } s \ 0) = c \wedge \text{ilen } s = (\text{Suc } 0) \wedge v \text{ (inth } s \ 0) = v \text{ (inth } s \ 0))$

by (*simp add: lenkhelph3*)

show *?thesis* **using** *0 01 assms* **by** *auto*

qed

lemma *getsmorehelp*:

assumes $\forall i < \text{ilen } s. v \text{ (inth } s \text{ (Suc } i)) = \text{Suc } (v \text{ (inth } s \ i))$

$v \text{ (ifirst } s) = 0$

$j \leq \text{ilen } s$

shows $\text{inth } (\text{imap } v \ s) \ j = j$

using *assms*

proof (*induct j arbitrary: s*)

case *0*

then show *?case* **by** (*simp add: ifirst-imap*)

next

case (*Suc j*)

then show *?case* **by** (*simp add: Suc.premis(1) Suc.premis(3) Suc-leD Suc-le-lessD inth-imap*)

qed

lemma *initonlyfinhelp*:

assumes $n \leq \text{ilen } s1$

shows $(\forall na \leq \text{ilen } s1 - n. (na = 0) = w \text{ (suffix } na \text{ (prefix } na \text{ (suffix } n \ s1)))) =$
 $(w \text{ (inth } s1 \ n) \wedge (\forall k. n < k \wedge k \leq \text{ilen } s1 \longrightarrow \neg w \text{ (inth } s1 \ k)))$

proof –

have *1*: $(\forall na \leq \text{ilen } s1 - n. (na = 0) = w \text{ (suffix } na \text{ (prefix } na \text{ (suffix } n \ s1)))) =$

$(\forall na . na + n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \text{ (suffix } na \text{ (prefix } na \text{ (suffix } n \ s1))))$

by (*simp add: Nat.le-diff-conv2 assms*)

have *2*: $(\forall na . na + n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \text{ (suffix } na \text{ (prefix } na \text{ (suffix } n \ s1)))) =$

$(\forall na . na + n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \text{ (suffix } na \text{ (suffix } n \text{ (prefix } (na + n) \ s1))))$

by (*simp add: suffix-prefix-swap*)

have *3*: $(\forall na . na + n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \text{ (suffix } na \text{ (suffix } n \text{ (prefix } (na + n) \ s1)))) =$

$(\forall na . na + n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \text{ (suffix } (na + n) \text{ (prefix } (na + n) \ s1)))$

by *simp*

have *4*: $(\forall na . na + n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \text{ (suffix } (na + n) \text{ (prefix } (na + n) \ s1))) =$

$(\forall na . na+n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \langle \text{inth } s1 (na+n) \rangle)$
by (*metis interval-prefix-suffix-help min.orderE*)
have 5: $(\forall na . na+n \leq \text{ilen } s1 \longrightarrow (na = 0) = w \langle \text{inth } s1 (na+n) \rangle) =$
 $(\forall k . n \leq k \wedge k \leq \text{ilen } s1 \longrightarrow (k-n=0) = w \langle \text{inth } s1 k \rangle)$
by (*metis add-diff-cancel-right' le-add2 le-add-diff-inverse2*)
have 6: $(\forall k . n \leq k \wedge k \leq \text{ilen } s1 \longrightarrow (k-n=0) = w \langle \text{inth } s1 k \rangle) =$
 $(w \langle \text{inth } s1 n \rangle \wedge (\forall k . n < k \wedge k \leq \text{ilen } s1 \longrightarrow \neg w \langle \text{inth } s1 k \rangle))$
using *assms le0 le-eq-less-or-eq less-numeral-extra(3)* **by** *auto*
show *?thesis* **using** 1 2 4 5 6 **by** *auto*
qed

17.1.3 Common prefix value trace theorems

lemma *commonpfx-symmetric*:

$(\forall s1 s . (s1 \models F) \wedge (s \models F) \wedge \text{ilen } s \leq \text{ilen } s1 \longrightarrow$
 $\text{imap } v s = (\text{imap } v (\text{prefix } (\text{ilen } s) s1))$
 $) =$
 $(\forall s1 s . (s1 \models F) \wedge (s \models F) \longrightarrow$
 $(\text{ilen } s \leq \text{ilen } s1 \longrightarrow \text{imap } v s = (\text{imap } v (\text{prefix } (\text{ilen } s) s1))) \wedge$
 $(\text{ilen } s1 \leq \text{ilen } s \longrightarrow \text{imap } v s1 = (\text{imap } v (\text{prefix } (\text{ilen } s1) s)))$
 $)$

by *auto*

lemma *CommonPfxFixedTrace*:

assumes *basevars v*
 $\text{sat } f \wedge \text{len } k$
 $\ddagger[f]\text{-}v$
shows $\text{card } \{ (\text{imap } v s) \mid s . (s \models f) \wedge \text{ilen } s = k \} = 1$
proof –
obtain *s0* **where** 1: $(s0 \models f) \wedge \text{ilen } s0 = k$
by (*metis (mono-tags, lifting) assms(2) len-defs sat-d-def unl-lift2*)
have 3: $\forall s1 . (s1 \models f) \wedge \text{ilen } s1 = k \longrightarrow (\text{imap } v s1) = (\text{imap } v s0)$
by (*metis 1 assms(3) le-cases commonpfx-def*)
have 4: $\{ (\text{imap } v s) \mid s . (s \models f) \wedge \text{ilen } s = k \} = \{ (\text{imap } v s0) \}$
using 3 1 **by** *blast*
show *?thesis* **by** (*simp add: 4*)
qed

lemma *NoTrace*:

assumes *basevars v*
 $\neg(\text{sat } f \wedge \text{len } k)$
shows $\text{card } \{ (\text{imap } v s) \mid s . (s \models f) \wedge \text{ilen } s = k \} = 0$
proof –
have 1: $\forall s . \neg(s \models f) \vee \text{ilen } s \neq k$
using *assms unfolding sat-d-def len-defs* **by** *auto*
have 2: $\{ (\text{imap } v s) \mid s . (s \models f) \wedge \text{ilen } s = k \} = \{ \}$
using 1 **by** *blast*
show *?thesis* **by** (*simp add: 2*)
qed

The following theorem states that the combination of satisfiability with the notion of common prefix value trace can be used to determine whether a formula is executable or not, i.e., satisfiable and deterministic.

lemma *CommonPfxTrace*:

```

assumes basevars v
          sat f
          ‡[f]-v
shows card { (imap v s) | s. (s ⊨ f) ∧ ilen s = k } ≤ 1
proof –
have 1: (sat f) = (sat (∃ k. f ∧ len k))
  by (simp add: len-defs sat-d-def)
have 2:  $\bigwedge k. (sat f \wedge len k) \vee \neg (sat f \wedge len k)$ 
  by simp
have 3:  $\bigwedge k. \neg (sat f \wedge len k) \implies card \{imap v s \mid s. f s \wedge ilen s = k\} \leq 1$ 
  using assms NoTrace[of v f] by auto
have 4:  $\bigwedge k. sat f \wedge len k \implies card \{imap v s \mid s. f s \wedge ilen s = k\} \leq 1$ 
  using assms CommonPfxFixedTrace[of v f] by auto
show ?thesis using 2 3 4 by auto
qed

```

lemma *PairCommonPfx*:

```

assumes basevars (v1, v2)
          ‡[f]-v1
          ‡[f]-v2
shows ‡[f]-(v1,v2)
using assms unfolding commonpfx-def
by (simp add: imap-eq-inth-ilen)

```

lemma *ProjCommonPfxA*:

```

assumes basevars (v1, v2)
          ‡[f]-(v1,v2)
shows ‡[f]-v1
using assms unfolding commonpfx-def
by (simp add: imap-eq-inth-ilen)

```

lemma *ProjCommonPfxB*:

```

assumes basevars (v1, v2)
          ‡[f]-(v1,v2)
shows ‡[f]-v2
using assms unfolding commonpfx-def
by (simp add: imap-eq-inth-ilen)

```

lemma *PairSplitCommonPfx*:

```

assumes basevars (v1, v2)
shows (‡[f]-(v1,v2)) = ((‡[f]-v1) ∧ (‡[f]-v2))
using PairCommonPfx ProjCommonPfxA ProjCommonPfxB assms by blast

```

17.1.4 Common prefix value trace and len and assignment

lemma *SatLenK*:

sat (len k)
by (*simp add: Valid-def itl-defs sat-d-def ExistsilenN*)

lemma *NotCommonPfxLenK:*

assumes *basevars (v :: state \Rightarrow nat)*

shows $\neg(\ddagger[\text{len}(k)]-v)$

proof –

have 1: $\exists s. v (\text{ifirst } s) = 0 \wedge \text{ilen } s = k$

by (*meson assms basevars-def lenkhelph3*)

have 2: $\exists s. v (\text{ifirst } s) = 1 \wedge \text{ilen } s = k$

by (*meson assms basevars-def lenkhelph3*)

have 3: $\neg(\ddagger[\text{len } k]-v)$

using *assms 1 2 unfolding commonpfx-def len-defs*

by (*metis inth-imap prefix-ilen imap-prefix order.order-iff-strict zero-neq-one*)

show *?thesis using 3 by blast*

qed

lemma *NotCommonPfxUnitAssignOrAndLenK:*

assumes *basevars (v :: state \Rightarrow nat)*

shows $\neg(\ddagger[(\$v = \#0 \vee \$v = \#1) \wedge \text{len } k]-v)$

using *assms lenkhelph3[of v 0 k] lenkhelph3[of v 1 k]*

by (*simp add: commonpfx-def itl-defs basevars-def inth-imap*)

(*metis One-nat-def inth-imap prefix-ilen-bound prefix-ilen zero-neq-one*)

lemma *CommonPfxUnitAssignAndNextValAndSkip:*

assumes *basevars (v :: state \Rightarrow nat)*

shows $(\ddagger[\$v = \#c \wedge \text{skip} \wedge v\$ = \#d]-v)$

proof –

have 2: *commonpfx (LIFT($\$v = \#c \wedge \text{skip} \wedge v\$ = \#d$)) v*

using *assms lenkhelph1[of v - c 1 d]*

by (*simp add: commonpfx-def next-val-d-def itl-defs basevars-def*)

show *?thesis using 2 by auto*

qed

lemma *NotCommonPfxUnitAssignAndLenK:*

assumes *basevars (v :: state \Rightarrow nat)*

0 < k

shows $\neg(\ddagger[\$v = \#0 \wedge \text{len } k]-v)$

using *assms using lenkhelph4[of v k 0 0] lenkhelph4[of v k 0 1]*

by (*simp add: commonpfx-def next-val-d-def itl-defs basevars-def*)

(*metis One-nat-def inth-imap prefix-ilen prefix-ilen-bound zero-neq-one*)

lemma *CommonPfxStableAndLenK:*

assumes *basevars v*

shows $(\ddagger[\$v = \#(c::nat) \wedge \text{len } k \wedge \text{stable } v]-v)$

using *assms by (simp add: commonpfx-def itl-defs basevars-def)*

(*metis prefix-ilen imap-eq-inth-ilen*)

17.1.5 Tempura

lemma *HaltAlways*:

$\vdash (\text{halt } f) = \Box(\text{if}_i f \text{ then empty else } (\neg \text{empty}))$

proof –

have 1: $\vdash (\text{empty}=f) = (\text{if}_i f \text{ then empty else } (\neg \text{empty}))$

unfolding *ifthenelse-d-def* **by** *auto*

have 2: $\vdash \Box(\text{empty}=f) = \Box(\text{if}_i f \text{ then empty else } (\neg \text{empty}))$

using 1 *BoxEqvBox*[of *LIFT*(*empty*=*f*) *LIFT*(*if*_{*i*} *f* then empty else (\neg empty))] **by** *auto*

show *?thesis* **unfolding** *halt-d-def* **using** 2 **by** *auto*

qed

lemma *ChopEqvExistAndLenKChop*:

$\vdash f;g = (\exists k. (f \wedge \text{len } k);g)$

by (*simp add: Valid-def itl-defs*)

lemma *ChopEqvExistChopAndLenK*:

$\vdash f;g = (\exists k. f;(g \wedge \text{len } k))$

by (*simp add: Valid-def itl-defs*)

lemma *BiSkipImpEqv* :

$\vdash \text{bi } (\text{skip} \longrightarrow f) = (\text{more} \longrightarrow (f \wedge \text{skip});\# \text{True})$

by (*auto simp add: Valid-def itl-defs itl-def min-def*)

lemma *GetsAlways*:

$\vdash a \text{ gets } b = \Box (\text{bi } (\text{skip} \longrightarrow a\$ = b))$

by (*simp add: Valid-def itl-def*)

lemma *GetsAlwaysBox*:

$\vdash a \text{ gets } b = \Box (\text{more} \longrightarrow (a\$ = b \wedge \text{skip});\# \text{True})$

by (*metis BiSkipImpEqv GetsAlways int-eq*)

lemma $\vdash \text{bm } (a\$ = b) = \Box(\text{more} \longrightarrow a\$ = b)$

by (*simp add: Valid-def itl-defs itl-def*)

lemma $\vdash (a \text{ gets } \$a + \#(1::\text{nat})) = (\Box(\text{more} \longrightarrow a\$ = \$a + \#1))$

by (*auto simp add: Valid-def itl-defs*)

17.1.6 Basic theorems

lemma *CommonPfxEqv*:

assumes *basevars v*

$\vdash f = g$

shows $(\ddagger[f]-v) = (\ddagger[g]-v)$

using *assms inteq-reflection* **by** *blast*

lemma *CommonPfxAnd*:

assumes *basevars v*

$\ddagger[f]-v$

$\ddagger[g]-v$

shows $\ddagger[f \wedge g]-v$

using *assms* **unfolding** *commonpfx-def* **by** *simp*

lemma *CommonPfxEmptyOrMore*:

assumes *basevars v*

shows $(\ddagger[f]-v) = (\ddagger[(f \wedge \text{empty}) \vee (f \wedge \text{more})]-v)$

proof –

have $1: \vdash f = ((f \wedge \text{empty}) \vee (f \wedge \text{more}))$

unfolding *empty-d-def* **by** *fastforce*

show *?thesis* **using** *CommonPfxEqv 1 assms* **by** *blast*

qed

lemma *NotCommonPfxEmpty*:

assumes *basevars (v :: state \Rightarrow nat)*

shows $\neg(\ddagger[\text{empty}]-v)$

proof –

obtain *s0* **where** $2: v (\text{ifirst } s0) = 0 \wedge \text{ilen } s0 = 0$

by (*metis inth.simps(1) assms baseE ilen.simps(1)*)

obtain *s1* **where** $3: v (\text{ifirst } s1) = 1 \wedge \text{ilen } s1 = 0$

by (*metis inth.simps(1) assms baseE ilen.simps(1)*)

show *?thesis* **unfolding** *commonpfx-def empty-defs*

by (*metis 2 3 One-nat-def inth-imap n-not-Suc-n order.order-iff-strict*)

qed

lemma *NotCommonPfxMore*:

assumes *basevars (v :: state \Rightarrow nat)*

shows $\neg(\ddagger[\text{more}]-v)$

proof –

obtain *s0* **where** $2: v (\text{ifirst } s0) = 0 \wedge \text{ilen } s0 = 1$

by (*metis One-nat-def assms baseE inth-zero ilen.simps(1) ilen.simps(2) plus-1-eq-Suc*)

obtain *s1* **where** $3: v (\text{ifirst } s1) = 1 \wedge \text{ilen } s1 = 1$

by (*metis One-nat-def assms baseE inth-zero ilen.simps(1) ilen.simps(2) plus-1-eq-Suc*)

show *?thesis* **unfolding** *commonpfx-def more-defs*

by (*metis 2 3 One-nat-def ifirst-imap lessI n-not-Suc-n order-refl*)

qed

lemma *NotCommonPfxTrue*:

assumes *basevars (v :: state \Rightarrow nat)*

shows $\neg(\ddagger[\# \text{True}]-v)$

using *assms* **by** (*metis NotCommonPfxEmpty commonpfx-def unl-con*)

lemma *CommonPfxChoice*:

assumes *basevars v*

shows $(\ddagger[f]-v) = (\ddagger[(f \wedge g) \vee (f \wedge \neg g)]-v)$

proof –

have $1: \vdash f = ((f \wedge g) \vee (f \wedge \neg g))$

by *fastforce*

show *?thesis* **using** *CommonPfxEqv 1 assms* **by** *blast*

qed

lemma *CommonPfxDetChoice:*

assumes *basevars v*

$\ddagger[(f \wedge g) \vee (f \wedge \neg g)]-v$

shows $(\ddagger[f \wedge g]-v) \wedge (\ddagger[(f \wedge \neg g)]-v)$

using *assms unfolding commonpfx-def* **by** *simp*

lemma *CommonPfxBox:*

assumes *basevars v*

shows $(\ddagger[\Box f]-v) = (\ddagger[f \wedge \text{wnext}(\Box f)]-v)$

proof –

have *1*: $\vdash \Box f = (f \wedge \text{wnext}(\Box f))$

by (*simp add: BoxEqvAndWnextBox*)

show *?thesis* **using** *assms 1 CommonPfxEqv[of v LIFT($\Box f$) LIFT($f \wedge \text{wnext}(\Box f)$)]* **by** *simp*

qed

lemma *CommonPfxBi:*

assumes *basevars v*

$\ddagger[f]-v$

shows $\ddagger[bi\ f]-v$

using *assms* **by** (*simp add: commonpfx-def bi-defs*) *force*

lemma *CommonPfxBox1:*

assumes *basevars v*

$\ddagger[f]-v$

shows $\ddagger[\Box f]-v$

using *assms* **by** (*simp add: commonpfx-def itl-defs*) *force*

lemma *CommonPfxAssign:*

assumes *basevars (v :: state \Rightarrow nat)*

shows $\ddagger[\$v = \#c \wedge v := \#d \wedge \text{skip}]-v$

proof –

have *1*: $\vdash (\$v = \#c \wedge v := \#d \wedge \text{skip}) = (\$v = \#c \wedge \text{skip} \wedge v\$ = \#d)$

unfolding *next-assign-d-def* **by** *fastforce*

have *2*: $\ddagger[\$v = \#c \wedge \text{skip} \wedge v\$ = \#d]-v$

using *CommonPfxUnitAssignAndNextValAndSkip[of v c d] assms* **by** *blast*

show *?thesis* **using** *assms 1 2*

CommonPfxEqv[of v LIFT($\$v = \#c \wedge v := \#d \wedge \text{skip}$) LIFT($\$v = \#c \wedge \text{skip} \wedge v\$ = \#d$)]

by *blast*

qed

lemma *SatValid:*

assumes $\vdash f$

shows *sat f*

by (*simp add: assms intD sat-d-def*)

lemma *SatValidNotA:*

$(\vdash f) = (\neg(\text{sat } \neg f))$

unfolding *sat-d-def Valid-def* **by** *auto*

lemma *SatValidNotB:*

$(sat\ f) = (\neg(\vdash \neg f))$
unfolding *sat-d-def Valid-def* **by** *auto*

lemma *ValidAnd*:
 $(\vdash f \wedge g) = ((\vdash f) \wedge (\vdash g))$
unfolding *Valid-def* **by** *auto*

lemma *ValidOrA*:
assumes $\vdash f$
shows $\vdash f \vee g$
using *assms* **unfolding** *Valid-def* **by** *auto*

lemma *ValidOrB*:
assumes $\vdash g$
shows $\vdash f \vee g$
using *assms* **unfolding** *Valid-def* **by** *auto*

lemma *SatAnd*:
assumes $sat\ f \wedge g$
shows $(sat\ f) \wedge (sat\ g)$
using *assms*
by (*metis sat-d-def unl-lift2*)

lemma *SatOr*:
 $(sat\ (f \vee g)) = ((sat\ f) \vee (sat\ g))$
unfolding *sat-d-def* **by** *auto*

lemma *SatAndLenK*:
assumes $sat\ (f \wedge len\ k)$
shows $sat\ f$
using *assms SatAnd* **by** *blast*

lemma *SatChopHaltPrefix*:
assumes $sat\ (f \wedge halt\ w)$
 $sat\ g$
 $\vdash init(w) \longrightarrow g$
shows $sat\ (f \wedge halt\ w);g$
proof –
have 1: $\exists\ s. (s \models f) \wedge (\forall n \leq ilen\ s. (ilen\ s = n) = w\ (suffix\ n\ s))$
by (*metis (mono-tags, lifting) assms(1) halt-defs sat-d-def unl-lift2*)
obtain s **where** 2: $(s \models f) \wedge (\forall n \leq ilen\ s. (ilen\ s = n) = w\ (suffix\ n\ s))$
using 1 **by** *auto*
have 3: $w \langle\ inth\ s\ (ilen\ s) \rangle$
using 2 **by** *auto*
obtain $s1$ **where** 4: $(s1 \models g) \wedge (inth\ s1\ 0) = inth\ s\ (ilen\ s)$
by (*metis 3 inth.simps(1) Prop10 assms(3) init-defs inteq-reflection prefix-zero-ifirst unl-lift2*)
have 5: $(fuse\ s\ s1) \models (f \wedge halt\ w);g$
by (*metis 2 4 halt-defs chop-fuse unl-lift2*)
show ?thesis **using** 5 *sat-d-def* **by** *auto*

qed

lemma *SatChopHaltInitonly*:

assumes *sat f* \wedge *halt w*

sat g

$\vdash \text{initonly}(w) \longrightarrow g$

shows *sat* (*f* \wedge *halt w*);(*g* \wedge *initonly w*)

proof –

have 1: $\exists s. (s \models f) \wedge (\forall n \leq \text{ilen } s. (\text{ilen } s = n) = w (\text{suffix } n \ s))$

by (*metis* (*mono-tags*, *lifting*) *assms*(1) *halt-defs* *sat-d-def* *unl-lift2*)

obtain *s* **where** 2: $(s \models f) \wedge (\forall n \leq \text{ilen } s. (\text{ilen } s = n) = w (\text{suffix } n \ s))$

using 1 **by** *auto*

have 3: $w \langle \text{inth } s \ (\text{ilen } s) \rangle$

using 2 **by** *auto*

obtain *s1* **where** 5: $(s1 \models g) \wedge (\forall n \leq \text{ilen } s1. (n=0) = w (\text{prefix } n \ s1)) \wedge \text{ifirst } s1 = \text{ilast } s$

using *assms* **by** (*simp* *add*: *itl-defs* *Valid-def* *sat-d-def*)

(*metis* 3 *ifirst-suffix* *ilen.simps*(1) *le-zero-eq* *nat-le-linear* *prefix-ilen* *suffix-ilen-last*)

have 6: $w \langle \text{inth } s1 \ 0 \rangle$

using 5 **by** *auto*

have 7: $(\text{fuse } s \ s1) \models (f \wedge \text{halt } w);(g \wedge \text{initonly } w)$

using 2 5 3 6 **by** (*auto* *simp* *add*: *itl-defs* *chop-fuse* *min-def*)

(*metis* *fuse-ilen-a* *le-add1* *prefix-fuse* *suffix-fuse* *suffix-ilen*)

show ?thesis **using** 7 **unfolding** *sat-d-def* **by** *auto*

qed

lemma *SatChopL*:

assumes *sat f*

sat g

$\vdash f \longrightarrow \text{fin } w$

$\vdash \text{init}(w) \longrightarrow g$

shows *sat f*; *g*

using *assms*

proof –

have 1: $\exists s. (s \models f)$

by (*metis* (*mono-tags*, *lifting*) *assms*(1) *sat-d-def*)

obtain *s* **where** 2: $(s \models f)$

using 1 **by** *auto*

have 3: $w \langle \text{inth } s \ (\text{ilen } s) \rangle$

using 2 *assms* **by** (*auto* *simp* *add*: *Valid-def* *fin-defs* *init-defs*)

obtain *s1* **where** 4: $(s1 \models g) \wedge (\text{inth } s1 \ 0) = \text{inth } s \ (\text{ilen } s)$

by (*metis* 3 *inth.simps*(1) *Prop10* *assms*(4) *init-defs* *inteq-reflection* *prefix-zero-ifirst* *unl-lift2*)

have 5: $(\text{fuse } s \ s1) \models (f);g$

by (*metis* 2 4 *chop-fuse*)

show ?thesis **using** 5 *sat-d-def* **by** *auto*

qed

lemma *SatChopR*:

assumes *sat f*

sat g

```

    ⊢ fin w → f
    ⊢ g → init(w)
shows sat f; g
proof –
  have 1: ∃ s. (s ⊨ g)
    by (metis (mono-tags, lifting) assms(2) sat-d-def )
obtain s where 2: (s ⊨ g)
    using 1 by auto
have 3: w ⟨ inth s 0 ⟩
    using 2 assms by (simp add: Valid-def init-defs)
obtain s1 where 4: (s1 ⊨ f) ∧ (inth s1 (ilen s1)) = (inth s 0)
    using 3 assms(3) by (simp add: Valid-def fin-defs) (metis ilast-irev)
have 5: (fuse s1 s) ⊨ f; g
    by (meson 2 4 chop-fuse)
show ?thesis using 5 sat-d-def by auto
qed

```

lemma *SatChopExistsL*:

```

  (sat f; g) = (sat (∃ k. (f ∧ len k); g))
by (metis ChopEqvExistAndLenKChop inteq-reflection)

```

lemma *SatChopExistsR*:

```

  (sat f; g) = (sat (∃ k. f; (g ∧ len k)))
by (metis ChopEqvExistChopAndLenK inteq-reflection)

```

lemma *SatExistsChopL*:

```

  (sat f; g) = (∃ k. (sat (f ∧ len k); g))
by (metis (mono-tags, lifting) SatChopExistsL sat-d-def unl-Rex )

```

lemma *SatExistsChopR*:

```

  (sat f; g) = (∃ k. (sat f; (g ∧ len k)))
by (metis (mono-tags, lifting) SatChopExistsR sat-d-def unl-Rex )

```

lemma *CommonPfxChopFixed*:

```

assumes basevars v
    ‡[f]-v
    ‡[g]-v
    ⊢ f → fin w
    ⊢ init(w) → g
shows ‡[ (f ∧ len k); g ]-v
proof –
  have 4: ‡[f ∧ fin w]-v
    using assms
    by ( simp add: itl-defs min-def interval-prefix-suffix-help commonpfx-def split: if-split-asm )
have 5: ‡[init w ∧ g]-v
    using assms
    by ( simp add: itl-defs min-def interval-prefix-suffix-help commonpfx-def split: if-split-asm )
have 6: ‡[ (f ∧ len k); g ]-v
    using assms 4 5

```

```

proof (auto simp add: itl-defs min-def Valid-def interval-prefix-suffix-help commonpfx-def)
  fix s1 :: state interval
  fix s :: state interval
  assume a0: basevars v
  assume a1:  $\forall s1\ s. f\ s1 \wedge f\ s \wedge \text{ilen}\ s \leq \text{ilen}\ s1 \longrightarrow \text{imap}\ v\ s = \text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ s)\ s1)$ 
  assume a2:  $\forall s1\ s. g\ s1 \wedge g\ s \wedge \text{ilen}\ s \leq \text{ilen}\ s1 \longrightarrow \text{imap}\ v\ s = \text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ s)\ s1)$ 
  assume a3:  $\forall wa. f\ wa \longrightarrow w\ \langle \text{ilast}\ wa \rangle$ 
  assume a4:  $\forall wa. w\ \langle \text{ifirst}\ wa \rangle \longrightarrow g\ wa$ 
  assume a5:  $\text{ilen}\ s \leq \text{ilen}\ s1$ 
  assume a6:  $k \leq \text{ilen}\ s$ 
  assume a7:  $f\ (\text{prefix}\ k\ s1)$ 
  assume a8:  $f\ (\text{prefix}\ k\ s)$ 
  assume a9:  $g\ (\text{suffix}\ k\ s1)$ 
  assume a10:  $g\ (\text{suffix}\ k\ s)$ 
  show  $\text{imap}\ v\ s = \text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ s)\ s1)$ 
proof –
  have 61:  $\text{ilen}\ (\text{prefix}\ k\ s) \leq \text{ilen}\ (\text{prefix}\ k\ s1)$ 
    using a5 by auto
  have 62:  $\text{imap}\ v\ (\text{prefix}\ k\ s) = \text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ (\text{prefix}\ k\ s))\ (\text{prefix}\ k\ s1))$ 
    using 61 a1 a7 a8 by blast
  have 63:  $\text{ilen}\ (\text{suffix}\ k\ s) \leq \text{ilen}\ (\text{suffix}\ k\ s1)$ 
    by (simp add: a5 diff-le-mono)
  have 64:  $\text{imap}\ v\ (\text{suffix}\ k\ s) = \text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ (\text{suffix}\ k\ s))\ (\text{suffix}\ k\ s1))$ 
    using 63 a10 a2 a9 by blast
  have 65:  $\text{fuse}\ (\text{prefix}\ k\ s)\ (\text{suffix}\ k\ s) = s$ 
    by (simp add: a6 fuse-prefix-suffix)
  have 66:  $\text{fuse}\ (\text{prefix}\ k\ s1)\ (\text{suffix}\ k\ s1) = s1$ 
    using a5 a6 fuse-prefix-suffix order-trans by blast
  have 67:  $\text{imap}\ v\ (\text{fuse}\ (\text{prefix}\ k\ s)\ (\text{suffix}\ k\ s)) =$ 
     $\text{fuse}\ (\text{imap}\ v\ (\text{prefix}\ k\ s))\ (\text{imap}\ v\ (\text{suffix}\ k\ s))$ 
    by (simp add: a6 fuse-prefix-suffix imap-prefix imap-suffix)
  have 68:  $\text{fuse}\ (\text{imap}\ v\ (\text{prefix}\ k\ s))\ (\text{imap}\ v\ (\text{suffix}\ k\ s)) =$ 
     $\text{fuse}\ (\text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ (\text{prefix}\ k\ s))\ (\text{prefix}\ k\ s1)))$ 
     $(\text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ (\text{suffix}\ k\ s))\ (\text{suffix}\ k\ s1))))$ 
    by (simp add: 62 64)
  have 69:  $(\text{fuse}\ (\text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ (\text{prefix}\ k\ s))\ (\text{prefix}\ k\ s1)))$ 
     $(\text{imap}\ v\ (\text{prefix}\ (\text{ilen}\ (\text{suffix}\ k\ s))\ (\text{suffix}\ k\ s1)))) =$ 
     $(\text{fuse}\ (\text{prefix}\ (\text{ilen}\ (\text{prefix}\ k\ s))\ (\text{imap}\ v\ (\text{prefix}\ k\ s1)))$ 
     $(\text{prefix}\ (\text{ilen}\ (\text{suffix}\ k\ s))\ (\text{imap}\ v\ (\text{suffix}\ k\ s1))))$ 
    by (metis 61 63 imap-prefix)
  have 70:  $(\text{fuse}\ (\text{prefix}\ k\ (\text{imap}\ v\ (\text{prefix}\ k\ s1)))$ 
     $(\text{prefix}\ (\text{ilen}\ s - k)\ (\text{imap}\ v\ (\text{suffix}\ k\ s1)))) =$ 
     $(\text{prefix}\ (\text{ilen}\ (\text{fuse}\ (\text{prefix}\ k\ s)\ (\text{suffix}\ k\ s)))$ 
     $(\text{fuse}\ (\text{imap}\ v\ (\text{prefix}\ k\ s1))\ (\text{imap}\ v\ (\text{suffix}\ k\ s1))))$ 
proof –
  have f1:  $\forall n\ na. \neg (n::\text{nat}) \leq na \vee \min\ n\ na = n$ 
    by (meson min-def-raw)
  have f2:  $\forall n\ i. \text{ilen}\ (\text{prefix}\ n\ (i::\text{state}\ \text{interval})) = \min\ n\ (\text{ilen}\ i)$ 
    using prefix-ilen-min by blast
  have f3:  $k + \text{ilen}\ (\text{suffix}\ k\ s) = \text{ilen}\ (\text{fuse}\ (\text{prefix}\ k\ s)\ (\text{suffix}\ k\ s))$ 

```

```

    using f1 by (simp add: a6 fuse-ilen-a)
  have f4:  $k \leq \text{ilen } s1$ 
    by (meson a5 a6 order.trans)
  have  $\forall n \text{ na. } \neg (n::\text{nat}) \leq \text{na} \vee \text{na} - n + n = \text{na}$ 
    using le-add-diff-inverse2 by blast
  then have  $\text{ilen } s - k + k = \text{ilen } s$ 
    by (meson a6)
  then show ?thesis
    using f4 f3 f2 f1
    by (metis a5 a6 fuse-prefix-suffix ilen-imap imap-prefix
      imap-suffix pref-pref-3 prefix-ilen suffix-prefix-swap)
qed
have 71:  $(\text{prefix } (\text{ilen } (\text{fuse } (\text{prefix } k \ s) (\text{suffix } k \ s))))$ 
   $(\text{fuse } (\text{imap } v (\text{prefix } k \ s1)) (\text{imap } v (\text{suffix } k \ s1)))) =$ 
   $(\text{imap } v (\text{prefix } (\text{ilen } s) \ s1))$ 
  by (metis a5 a6 fuse-prefix-suffix ilen-imap imap-prefix imap-suffix le-trans)
show ?thesis using 65 67 68 69 70 71
using a6 prefix-ilen-good by fastforce
qed
qed
show ?thesis by (simp add: 6)
qed

```

lemma *CommonPfxChopFixedExists:*

```

assumes basevars v
   $\ddagger[f]-v$ 
   $\ddagger[g]-v$ 
   $\vdash f \longrightarrow \text{fin } w$ 
   $\vdash \text{init}(w) \longrightarrow g$ 
shows  $(\exists k. (\ddagger[(f \wedge \text{len } k);g]-v))$ 
using CommonPfxChopFixed assms by blast

```

lemma *CommonPfxChopExists:*

```

 $(\ddagger[(\exists k. (f \wedge \text{len } k);g)]-v) \implies (\exists k. (\ddagger[(f \wedge \text{len } k);g]-v))$ 
unfolding commonpfx-def
by auto

```

lemma *CommonPfxExistsChop:*

```

 $(\ddagger[(\exists k. (f \wedge \text{len } k);g)]-v) = (\ddagger[f;g]-v)$ 
by (metis ChopEqvExistAndLenKChop inteq-reflection)

```

17.1.7 Forward Executability Theorems

lemma *FExecStableInitial:*

```

assumes basevars v
shows  $\ddagger[\$v = \#c \wedge \text{stable } v]-v$ 
proof -
  have 1:  $\text{sat } (\$v = \#c \wedge \text{stable } v)$ 
    using assms by (simp add: sat-d-def commonpfx-def itl-defs basevars-def) (meson len0help)
  have 2:  $\ddagger[\$v = \#c \wedge \text{stable } v]-v$ 

```

```

using assms by (simp add: min-def sat-d-def commonpfx-def itl-defs basevars-def imap-eq-inth-ilen)
show ?thesis unfolding fexec-def using 1 2 by auto
qed

```

lemma *FExecAlwaysConstant*:

```

assumes basevars v
shows  $\dagger[\Box(\$v = \#c)]$ -v
proof –
  have 1: sat  $\Box(\$v = \#c)$ 
    using assms by (simp add: sat-d-def commonpfx-def itl-defs basevars-def) (metis len0help)
  have 2:  $\dagger[\Box(\$v = \#c)]$ -v
    using assms by (simp add: min-def sat-d-def commonpfx-def itl-defs basevars-def imap-eq-inth-ilen)
show ?thesis unfolding fexec-def using 1 2 by auto
qed

```

lemma *FExecGetsAndMore*:

```

assumes basevars v
shows  $\dagger[\$v = \#(0::nat) \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)]$ -v
proof –
  have 10: sat  $\$v = \#0 \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)$ 
    using assms by (simp add: sat-d-def itl-defs)
    (metis inth.simps(1) basevars inth-Suc inth-zero ilen.simps(1) ilen.simps(2) neq0-conv
      not-less-eq plus-1-eq-Suc)
  have 20:  $\dagger[\$v = \#0 \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)]$ -v
    using assms
  proof (auto simp add: commonpfx-def itl-defs)
    show  $\bigwedge s1 \ s.$ 
      basevars v  $\implies$ 
      v (ifirst s1) = 0  $\implies$ 
       $\forall i < \text{ilen } s1. \ v \ ( \text{inth } s1 \ (\text{Suc } i)) = \text{Suc } (v \ ( \text{inth } s1 \ i)) \implies$ 
      v (ifirst s) = 0  $\implies$ 
      0 < ilen s  $\implies$ 
       $\forall i < \text{ilen } s. \ v \ ( \text{inth } s \ (\text{Suc } i)) = \text{Suc } (v \ ( \text{inth } s \ i)) \implies$ 
      ilen s  $\leq$  ilen s1  $\implies$  imap v s = (imap v (prefix (ilen s) s1))
  proof –
    fix s1
    fix s
    assume a0: basevars v
    assume a1: v (ifirst s1) = 0
    assume a2:  $\forall i < \text{ilen } s1. \ v \ ( \text{inth } s1 \ (\text{Suc } i)) = \text{Suc } (v \ ( \text{inth } s1 \ i))$ 
    assume a3: v (ifirst s) = 0
    assume a4: 0 < ilen s
    assume a5:  $\forall i < \text{ilen } s. \ v \ ( \text{inth } s \ (\text{Suc } i)) = \text{Suc } (v \ ( \text{inth } s \ i))$ 
    assume a6: ilen s  $\leq$  ilen s1
    show imap v s = (imap v (prefix (ilen s) s1))
    proof –
      have 1: ilen(imap v s) = ilen (imap v ((prefix (ilen s) s1)))
        using a6 by auto
      have 2:  $\bigwedge j. \ j \leq \text{ilen } s \longrightarrow \text{inth } (\text{imap } v \ s) \ j = \text{inth } (\text{imap } v \ ((\text{prefix } (\text{ilen } s) \ s1))) \ j$ 

```

```

    using a1 a2 a3 a5 a6 getsmorehelp le-trans by (metis imap-prefix inth-prefix)
    show ?thesis by (metis 1 2 interval-eq-inth-eq ilen-imap)
  qed
qed
qed
show ?thesis unfolding fexec-def using 10 20 by simp
qed

```

lemma *FExecGetsAndEmpty*:

```

assumes basevars v
shows  $\dagger[\$v = \#(0::nat) \wedge \text{empty} \wedge v \text{ gets } (\$v + \#1)] - v$ 
proof –
  have 10:  $\text{sat } \$v = \#0 \wedge \text{empty} \wedge v \text{ gets } (\$v + \#1)$ 
    using assms by (simp add: sat-d-def itl-defs)
    (metis baseE ilen.simps(1) inth.simps(1) not-less-zero)
  have 20:  $\dagger[\$v = \#0 \wedge \text{empty} \wedge v \text{ gets } (\$v + \#1)] - v$ 
    using assms
    by (auto simp add: commonpfx-def itl-defs ifirst-imap INil-ilen)
show ?thesis unfolding fexec-def using 10 20 by simp
qed

```

lemma *FExecChopFixedExists*:

```

assumes basevars v
   $\dagger[f] - v$ 
   $\dagger[g] - v$ 
   $\vdash f \longrightarrow \text{fin } w$ 
   $\vdash \text{init}(w) \longrightarrow g$ 
shows  $(\exists k. (\dagger[(f \wedge \text{len } k); g] - v))$ 
using assms SatChopL[of f g w]
unfolding fexec-def
using SatExistsChopL CommonPfxChopFixed by blast

```

lemma *FExecNextRule*:

```

assumes basevars v
   $\dagger[g] - v$ 
   $\dagger[\text{init } w \wedge \text{empty}] - v$ 
shows  $\dagger[\text{init } w \wedge \bigcirc g] - v$ 
proof –
  obtain s0 where 1:  $w \langle \text{ifirst } s0 \rangle \wedge \text{ilen } s0 = 0$ 
    using assms unfolding fexec-def sat-d-def
    by (auto simp add: itl-defs)
  obtain s1 where 2:  $g \ s1$ 
    using assms unfolding fexec-def sat-d-def
    by (auto simp add: itl-defs)
  have 3:  $(\langle \text{ifirst } s0 \rangle \ominus s1) \models \text{init } w \wedge \bigcirc g$ 
    using 1 2 by (auto simp add: itl-defs)
  have 4:  $\text{sat } \text{init } w \wedge \bigcirc g$ 
    unfolding sat-d-def using 3 by blast
  have 5: basevars v  $\implies$ 
     $(\text{sat } g) \wedge (\forall s1 \ s. g \ s1 \wedge g \ s \wedge \text{ilen } s \leq \text{ilen } s1 \longrightarrow \text{imap } v \ s = \text{imap } v \ (\text{prefix } (\text{ilen } s) \ s1)) \implies$ 

```

```

(sat init w ∧ empty) ∧
(∀ s1 s.
  w ⟨ifirst s1⟩ ∧ ilen s1 = 0 ∧ w ⟨ifirst s⟩ ∧ ilen s = 0 ∧ ilen s ≤ ilen s1 →
  (∃ y. s = ⟨y⟩ ∧ v y = v (ifirst s1))) ⇒
∀ s1 s.
  w ⟨ifirst s1⟩ ∧
  0 < ilen s1 ∧ g (suffix (Suc 0) s1) ∧ w ⟨ifirst s⟩ ∧ 0 < ilen s ∧ g (suffix (Suc 0) s) ∧ ilen s ≤ ilen
s1 →
  imap v s = imap v (prefix (ilen s) s1)
proof auto
fix s1
fix s
assume a0: basevars v
assume a1: ∀ s1 s. g s1 ∧ g s ∧ ilen s ≤ ilen s1 → imap v s = (imap v (prefix (ilen s) s1))
assume a3: ∀ s1 s. w ⟨ifirst s1⟩ ∧ ilen s1 = 0 ∧ w ⟨ifirst s⟩ ∧ ilen s = 0 ∧ ilen s ≤ ilen s1
  → (∃ y. s = ⟨y⟩ ∧ v y = v (ifirst s1))
assume a4: w ⟨ifirst s1⟩
assume a5: g (suffix (Suc 0) s1)
assume a6: w ⟨ifirst s⟩
assume a7: 0 < ilen s
assume a8: g (suffix (Suc 0) s)
assume a9: ilen s ≤ ilen s1
show imap v s = (imap v (prefix (ilen s) s1))
proof -
  have 51: imap v ⟨ifirst s⟩ = (imap v (prefix (ilen ⟨ifirst s⟩) ⟨ifirst s1⟩))
  by (metis a3 a4 a6 ilast-prefix ilen.simps(1) ilen-gr-zero interval.simps(9) prefix-zero-ifirst)
  have 52: imap v (suffix (Suc 0) s) =
    (imap v (prefix (ilen (suffix (Suc 0) s)) (suffix (Suc 0) s1)))
    by (metis a1 a5 a8 a9 diff-le-mono suffix-ilen)
  have 53: s = ((ifirst s) ⊖ (suffix (Suc 0) s))
    using a7 interval-hd-tail by auto
  have 54: imap v ((ifirst s) ⊖ (suffix (Suc 0) s)) =
    (imap v ⟨ifirst s⟩) ⊖ (imap v (suffix (Suc 0) s))
    by simp
  have 55: (imap v ⟨ifirst s⟩) ⊖ (imap v (suffix (Suc 0) s)) =
    (imap v ((prefix (ilen ⟨ifirst s⟩) ⟨ifirst s1⟩))) ⊖
    (imap v ((prefix (ilen (suffix (Suc 0) s)) (suffix (Suc 0) s1))))
    using 51 52 by auto
  have 56: (imap v ((prefix (ilen ⟨ifirst s⟩) ⟨ifirst s1⟩))) =
    (imap v ((prefix (ilen ⟨ifirst s⟩) (⟨ifirst s1⟩ ⊖ (suffix (Suc 0) s1)))))
    by auto
  have 57: (imap v (prefix (ilen ((ifirst s) ⊖ (suffix (Suc 0) s))))
    ((ifirst s1) ⊖ (suffix (Suc 0) s1))) =
    (imap v ((prefix (ilen ⟨ifirst s⟩) ⟨ifirst s1⟩))) ⊖
    (imap v ((prefix (ilen (suffix (Suc 0) s)) (suffix (Suc 0) s1))))
    by simp
  have 58: imap v ((ifirst s) ⊖ (suffix (Suc 0) s)) =
    (imap v (prefix (ilen ((ifirst s) ⊖ (suffix (Suc 0) s))))
    ((ifirst s1) ⊖ (suffix (Suc 0) s1)))
    using 55 by auto

```



```

have 59:  $s1 = (\langle ifirst\ s1 \rangle \ominus (\text{suffix}\ (Suc\ 0)\ s1))$ 
  using a7 a9 interval-hd-tail by force
show ?thesis using 51 52 53 54 57 59 by metis
qed
qed
have 6:  $\dagger[init\ w \wedge \bigcirc\ g] \cdot v$ 
  using assms 5 unfolding commonpfx-def itl-defs Valid-def fexec-def by force
show ?thesis unfolding fexec-def using 6 4 by auto
qed

```

The rewriting process of Tempura corresponds to the notion of forward executability.

lemma *FExecWNextRule*:

```

assumes basevars v
   $\dagger[g] \cdot v$ 
   $\dagger[init\ w \wedge empty] \cdot v$ 
shows  $\dagger[init\ w \wedge wnext\ g] \cdot v$ 
proof -
have 1:  $\vdash (init\ w \wedge wnext\ g) = ((init\ w \wedge empty) \vee (init\ w \wedge \bigcirc\ g))$ 
  using WnextEqvEmptyOrNext by fastforce
have 2:  $sat\ init\ w \wedge wnext\ g$ 
  using 1 SatOr assms FExecNextRule[of v g w] unfolding fexec-def
  by (metis int-eq)
have 3:  $\dagger[init\ w \wedge \bigcirc\ g] \cdot v$ 
  using assms FExecNextRule[of v g w] unfolding fexec-def
  by blast
have 4:  $\dagger[init\ w \wedge wnext\ g] \cdot v$ 
  using 1 3 assms StateAndEmptyChop[of w LIFT( $\bigcirc\ g$ )] unfolding commonpfx-def fexec-def
proof (auto simp add: itl-defs, metis INil-ilen ilen-gr-zero)
fix s1 :: state interval and s :: state interval
assume a1:  $\forall s1\ s. w\ \langle ifirst\ s1 \rangle \wedge 0 < ilen\ s1 \wedge g\ (\text{suffix}\ (Suc\ 0)\ s1) \wedge w\ \langle ifirst\ s \rangle$ 
   $\wedge 0 < ilen\ s \wedge g\ (\text{suffix}\ (Suc\ 0)\ s) \wedge ilen\ s \leq ilen\ s1 \longrightarrow$ 
   $imap\ v\ s = imap\ v\ (\text{prefix}\ (ilen\ s)\ s1)$ 
assume a2:  $ilen\ s \leq ilen\ s1$ 
assume a3:  $w\ \langle ifirst\ s1 \rangle$ 
assume a4:  $w\ \langle ifirst\ s \rangle$ 
assume a5:  $g\ (\text{suffix}\ (Suc\ 0)\ s1)$ 
assume a6:  $g\ (\text{suffix}\ (Suc\ 0)\ s)$ 
assume  $\forall s1\ s. w\ \langle ifirst\ s1 \rangle \wedge ilen\ s1 = 0 \wedge w\ \langle ifirst\ s \rangle \wedge ilen\ s = 0 \wedge ilen\ s \leq ilen\ s1 \longrightarrow$ 
   $(\exists y. s = \langle y \rangle \wedge v\ y = v\ (\langle ifirst\ s1 \rangle))$ 
then have  $\forall i\ ia. (((imap\ v\ ia = imap\ v\ \langle ifirst\ i \rangle \vee ilen\ i \neq 0) \vee ilen\ ia \neq 0) \vee$ 
   $\neg w\ \langle ifirst\ i \rangle) \vee \neg w\ \langle ifirst\ ia \rangle) \vee \neg ilen\ ia \leq ilen\ i$ 
  by auto
then show  $imap\ v\ s = imap\ v\ (\text{prefix}\ (ilen\ s)\ s1)$ 
  using a6 a5 a4 a3 a2 a1
  by (metis (no-types) ifirst-prefix le0 le-antisym neq0-conv prefix-ilen-good)
qed
show ?thesis unfolding fexec-def using 2 4 by auto
qed

```

The following lemma states the conditions required to strengthen a forward executable specification.

```

lemma FExecStrengthen:
  assumes basevars v
     $\text{sat } f \wedge g$ 
     $\dagger[f]-v$ 
  shows  $\dagger[f \wedge g]-v$ 
using assms unfolding fexec-def commonpfx-def
by simp

```

17.2 Backward Executability

17.2.1 Common Suffix Value Trace Definitions

Reflection relates the notion of prefix intervals with that of suffix intervals. So we need to introduce the “mirror image” of common prefix value traces, i.e. the notion of common suffix value trace.

```

definition commonsfx :: ('a :: world) formula  $\Rightarrow$  ('a,'b) stfun  $\Rightarrow$  bool
where commonsfx F v  $\equiv$ 
  ( $\forall$  s1 s. (s1  $\models F$ )  $\wedge$  (s  $\models F$ )  $\wedge$  ilen s  $\leq$  ilen s1  $\longrightarrow$ 
     $\text{imap } v \text{ } s = (\text{imap } v (\text{suffix } ((\text{ilen } s1) - (\text{ilen } s)) \text{ } s1))$ 
  )

```

The notion of common suffix value trace corresponds to notion of generating a satisfying interval for a formula but it “limits” how this is achieved, i.e., one proceeds in a backward manner. The following definition introduces the notion of backward executability.

```

definition bexec :: ('a :: world) formula  $\Rightarrow$  ('a,'b) stfun  $\Rightarrow$  bool
where bexec F v  $\equiv$  ( (sat-d F)  $\wedge$  (commonsfx F v) )

```

syntax

```

-commonsfx :: lift  $\Rightarrow$  lift  $\Rightarrow$  bool          (( $\dagger[-]'-(-)$ ) 5)
-bexec      :: lift  $\Rightarrow$  lift  $\Rightarrow$  bool          (( $\dagger[-]'-(-)$ ) 5)

```

translations

```

-commonsfx == CONST commonsfx
-bexec      == CONST bexec

```

17.2.2 Common suffix value trace theorems

lemma *CommonSfxFixedTrace:*

```

assumes basevars v
   $\text{sat } f \wedge \text{len } k$ 
   $\dagger[f]-v$ 
shows  $\text{card } \{ (\text{imap } v \text{ } s) \mid s. (s \models f) \wedge \text{ilen } s = k \} = 1$ 
proof –
  obtain s0 where 1: (s0  $\models f$ )  $\wedge$  ilen s0 = k
  by (metis (mono-tags, lifting) assms(2) len-defs sat-d-def unl-lift2)
  have 3:  $\forall s1. (s1 \models f) \wedge \text{ilen } s1 = k \longrightarrow (\text{imap } v \text{ } s1) = (\text{imap } v \text{ } s0)$ 
  using 1 assms unfolding commonsfx-def by (metis le-less)
  have 4:  $\{ (\text{imap } v \text{ } s) \mid s. (s \models f) \wedge \text{ilen } s = k \} = \{ (\text{imap } v \text{ } s0) \}$ 
  using 3 1 by blast
show ?thesis by (simp add: 4)
qed

```

The following theorem states that the combination of satisfiability with the notion of common suffix value trace can be used to determine whether a formula is deterministic or not, i.e., is backward executable or not.

lemma *CommonSfxTrace*:

```

assumes basevars  $v$ 
           $\text{sat } f$ 
           $\mathfrak{h}[f]\text{-}v$ 
shows  $\text{card } \{ ( \text{imap } v \ s \mid s. (s \models f) \wedge \text{ilen } s = k ) \} \leq 1$ 
proof –
have 1:  $(\text{sat } f) = (\text{sat } (\exists k. f \wedge \text{len } k))$ 
  by (simp add: len-defs sat-d-def)
have 2:  $\bigwedge k. (\text{sat } f \wedge \text{len } k) \vee \neg (\text{sat } f \wedge \text{len } k)$ 
  by simp
have 3:  $\bigwedge k. \neg (\text{sat } f \wedge \text{len } k) \implies \text{card } \{ \text{imap } v \ s \mid s. f \ s \wedge \text{ilen } s = k \} \leq 1$ 
  using assms NoTrace[of v f] by auto
have 4:  $\bigwedge k. \text{sat } f \wedge \text{len } k \implies \text{card } \{ \text{imap } v \ s \mid s. f \ s \wedge \text{ilen } s = k \} \leq 1$ 
  using assms CommonSfxFixedTrace[of v f] by auto
show ?thesis using 2 3 4 by auto
qed

```

lemma *PairCommonSfx*:

```

assumes basevars  $(v1, v2)$ 
           $\mathfrak{h}[f]\text{-}v1$ 
           $\mathfrak{h}[f]\text{-}v2$ 
shows  $\mathfrak{h}[f]\text{-(}v1, v2\text{)}$ 
using assms unfolding commonsfx-def
by (simp add: imap-eq-inth-ilen)

```

lemma *ProjCommonSfxA*:

```

assumes basevars  $(v1, v2)$ 
           $\mathfrak{h}[f]\text{-(}v1, v2\text{)}$ 
shows  $\mathfrak{h}[f]\text{-}v1$ 
using assms unfolding commonsfx-def
by (simp add: imap-eq-inth-ilen)

```

lemma *ProjCommonSfxB*:

```

assumes basevars  $(v1, v2)$ 
           $\mathfrak{h}[f]\text{-(}v1, v2\text{)}$ 
shows  $\mathfrak{h}[f]\text{-}v2$ 
using assms unfolding commonsfx-def
by (simp add: imap-eq-inth-ilen)

```

lemma *PairSplitCommonSfx*:

```

assumes basevars  $(v1, v2)$ 
shows  $(\mathfrak{h}[f]\text{-(}v1, v2\text{)}) = ((\mathfrak{h}[f]\text{-}v1) \wedge (\mathfrak{h}[f]\text{-}v2))$ 
using PairCommonSfx ProjCommonSfxA ProjCommonSfxB assms by blast

```

17.2.3 Time reversal theorems

lemma *SatRev*:

assumes *basevars v*
shows $(\text{sat } f) = (\text{sat } (f^r))$
using *assms unfolding sat-d-def*
by (*metis irev-irev-ident reverse-d-def*)

lemma *RevCommonPfxSfx*:
assumes *basevars v*
shows $(\mathfrak{h}[f]-v) = (\mathfrak{h}[f^r]-v)$
using *assms unfolding commonpfx-def commonsfx-def reverse-d-def*
by (*metis (no-types, lifting) irev-ilen irev-prefix irev-irev-ident irev-imap*)

lemma *RevCommonSfxPfx*:
assumes *basevars v*
shows $(\mathfrak{h}[f]-v) = (\mathfrak{h}[f^r]-v)$
using *assms*
by (*simp add: RevCommonPfxSfx reverse-d-def*)

17.2.4 Common suffix value trace and len and assignment

lemma *CommonSfxUnitAssignAndNextValAndSkip*:
assumes *basevars (v :: state \Rightarrow nat)*
shows $(\mathfrak{h}[\$v = \#c \wedge \text{skip} \wedge v\$ = \#d]-v)$
proof –
have $\exists: \text{commonsfx } (\text{LIFT}(\$v = \#c \wedge \text{skip} \wedge v\$ = \#d)) \ v$
using *assms using lenkhelp1[of v - c 1 d]*
by (*simp add: commonsfx-def next-val-d-def itl-defs basevars-def*)
show *?thesis using \exists by auto*
qed

lemma *NotCommonSfxUnitAssignAndLenK*:
assumes *basevars (v :: state \Rightarrow nat)*
 $0 < k$
shows $\neg (\mathfrak{h}[\$v = \#0 \wedge \text{len } k]-v)$
using *assms using lenkhelp4[of v k 0 0] lenkhelp4[of v k 0 1]*
by (*simp add: commonsfx-def next-val-d-def itl-defs basevars-def*)
(metis One-nat-def inth-imap order-refl zero-neq-one)

lemma *CommonSfxStableAndLenK*:
assumes *basevars v*
shows $(\mathfrak{h}[\$v = \#(c::\text{nat}) \wedge \text{len } k \wedge \text{stable } v]-v)$
proof –
have $\exists: \text{commonsfx } (\text{LIFT}(\$v = \#(c::\text{nat}) \wedge \text{len } k \wedge \text{stable } v)) \ v$
using *assms by (simp add: commonsfx-def itl-defs basevars-def imap-eq-inth-ilen)*
show *?thesis using \exists by simp*
qed

lemma *NotCommonSfxGetsAndMore*:

```

assumes basevars v
shows  $\neg(\text{h}[\$v = \#(0::\text{nat}) \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)] - v)$ 
proof –
  have 10:  $\bigwedge k. \exists s. v \text{ (ifirst } s) = 0 \wedge \text{ilen } s = k \wedge$ 
     $(\forall i < \text{ilen } s. v \text{ ( inth } s \text{ (Suc } i)) = \text{Suc } (v \text{ ( inth } s \text{ } i)))$ 
  using assms
  proof –
    fix k
    show basevars v  $\implies$ 
       $\exists s. v \text{ (ifirst } s) = 0 \wedge \text{ilen } s = k \wedge$ 
       $(\forall i < \text{ilen } s. v \text{ ( inth } s \text{ (Suc } i)) = \text{Suc } (v \text{ ( inth } s \text{ } i)))$ 
    proof (induct k)
    case 0
    then show ?case
    by (metis inth.simps(1) basevars ilen.simps(1) less-nat-zero-code)
    next
    case (Suc k)
    then show ?case
    proof –
      have 11:  $\exists s. v \text{ (ifirst } s) = 0 \wedge \text{ilen } s = k \wedge$ 
         $(\forall i < \text{ilen } s. v \text{ ( inth } s \text{ (Suc } i)) = \text{Suc } (v \text{ ( inth } s \text{ } i)))$ 
      using Suc.hyps Suc.prem by blast
      obtain ss where 12:  $v \text{ (ifirst } ss) = 0 \wedge \text{ilen } ss = k \wedge$ 
         $(\forall i < \text{ilen } ss. v \text{ ( inth } ss \text{ (Suc } i)) = \text{Suc } (v \text{ ( inth } ss \text{ } i)))$ 
      using 11 by auto
      have 121:  $v \text{ (inth } ss \text{ } k) = k$ 
      by (metis 12 getsmorehelp inth-imap order-refl)
      have 13:  $\exists s. v \text{ (} s) = (\text{Suc } k)$ 
      by (simp add: assms basevars)
      obtain ssl where 141:  $v \text{ ssl} = (\text{Suc } k)$ 
      using 13 by auto
      have 14:  $v \text{ (ifirst (iapp ss (INil ssl)))} = 0 \wedge \text{ilen (iapp ss (INil ssl))} = \text{Suc } k$ 
      by (simp add: 12)
      have 15:  $(\forall i < \text{ilen (iapp ss (INil ssl))}. v \text{ (inth (iapp ss (INil ssl)) (Suc } i)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } i))) \wedge$ 
         $(\forall i < \text{ilen } ss. v \text{ (inth (iapp ss (INil ssl)) (Suc } i)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } i))) \wedge$ 
         $v \text{ (inth (iapp ss (INil ssl)) (Suc } k)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } k)))$ 
      using 14 less-Suc-eq by auto
      have 16:  $v \text{ (inth (iapp ss (INil ssl)) (Suc } k)) = v \text{ ssl}$ 
      by (metis 14 ilast-iapp)
      have 17:  $v \text{ (inth (iapp ss (INil ssl)) } k) = v \text{ (inth } ss \text{ } k)$ 
      by (simp add: 12 iapp-inth)
      have 18:  $v \text{ (inth (iapp ss (INil ssl)) (Suc } k)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } k))$ 
      by (simp add: 121 141 16 17)
      have 19:  $(\forall i < \text{ilen } ss. v \text{ (inth (iapp ss (INil ssl)) (Suc } i)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } i)))$ 
      by (simp add: 12 iapp-inth)
    show ?thesis
    using 14 15 18 19 by blast

```

```

    qed
  qed
  qed
have 20:  $\neg (\text{h}[\$v = \#0 \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)] - v)$ 
using assms
proof (simp add: commonsfx-def itl-defs)
  obtain s0 where 1:  $v (\text{ifirst } s0) = 0 \wedge 1 = \text{ilen } s0 \wedge$ 
     $(\forall i < \text{ilen } s0. v (\text{inth } s0 (\text{Suc } i)) = \text{Suc } (v (\text{inth } s0 i)))$ 
    using 10[of 1]
    by fastforce
  obtain s1 where 2:  $v (\text{ifirst } s1) = 0 \wedge 2 = \text{ilen } s1 \wedge$ 
     $(\forall i < \text{ilen } s1. v (\text{inth } s1 (\text{Suc } i)) = \text{Suc } (v (\text{inth } s1 i)))$ 
    using 10[of 2] by fastforce
  have 3:  $\text{ilen } s0 \leq \text{ilen } s1$ 
    using 1 2 by auto
  have 4:  $\text{imap } v s0 \neq (\text{imap } v (\text{suffix } (\text{ilen } s1 - \text{ilen } s0) s1))$ 
    by (metis 1 2 3 One-nat-def Suc-1 ilast-suffix ilast-imap lessI less-le-trans n-not-Suc-n)
  have 5:  $0 < \text{ilen } s0$ 
    using 1 by linarith
  have 6:  $0 < \text{ilen } s1$ 
    using 3 5 order.strict-trans2 by blast
  show basevars  $v \implies$ 
 $\exists s1. v (\text{ifirst } s1) = 0 \wedge$ 
 $0 < \text{ilen } s1 \wedge$ 
 $(\forall i < \text{ilen } s1. v (\text{inth } s1 (\text{Suc } i)) = \text{Suc } (v (\text{inth } s1 i))) \wedge$ 
 $(\exists s. v (\text{ifirst } s) = 0 \wedge$ 
 $0 < \text{ilen } s \wedge$ 
 $(\forall i < \text{ilen } s. v (\text{inth } s (\text{Suc } i)) = \text{Suc } (v (\text{inth } s i))) \wedge$ 
 $\text{ilen } s \leq \text{ilen } s1 \wedge \text{imap } v s \neq (\text{imap } v (\text{suffix } (\text{ilen } s1 - \text{ilen } s) s1)))$ 
    using 1 2 3 4 5 6 by blast
  qed
show ?thesis using 20 by auto
qed

```

lemma *CommonSfxGetsAndLenK*:

assumes *basevars v*

shows $(\text{h}[\$v = \#(0::\text{nat}) \wedge \text{len } k \wedge v \text{ gets } (\$v + \#1)] - v)$

proof –

have 10: $\bigwedge k. \exists s. v (\text{ifirst } s) = 0 \wedge \text{ilen } s = k \wedge$
 $(\forall i < \text{ilen } s. v (\text{inth } s (\text{Suc } i)) = \text{Suc } (v (\text{inth } s i)))$

using *assms*

proof –

fix k

show *basevars v* \implies

$\exists s. v (\text{ifirst } s) = 0 \wedge \text{ilen } s = k \wedge$
 $(\forall i < \text{ilen } s. v (\text{inth } s (\text{Suc } i)) = \text{Suc } (v (\text{inth } s i)))$

proof (*induct k*)

case 0

then show ?case

by (*metis inth.simps(1) basevars ilen.simps(1) less-nat-zero-code*)

```

next
case (Suc k)
then show ?case
proof -
  have 11:  $\exists s. v \text{ (ifirst } s) = 0 \wedge \text{ilen } s = k \wedge$ 
     $(\forall i < \text{ilen } s. v \text{ (inth } s \text{ (Suc } i)) = \text{Suc } (v \text{ (inth } s \text{ } i)))$ 
    using Suc.hyps Suc.premis by blast
  obtain ss where 12:  $v \text{ (ifirst } ss) = 0 \wedge \text{ilen } ss = k \wedge$ 
     $(\forall i < \text{ilen } ss. v \text{ (inth } ss \text{ (Suc } i)) = \text{Suc } (v \text{ (inth } ss \text{ } i)))$ 
    using 11 by auto
  have 121:  $v \text{ (inth } ss \text{ } k) = k$ 
    by (metis 12 getsmorehelp inth-imap order-refl)
  have 13:  $\exists s. v \text{ (s)} = (\text{Suc } k)$ 
    by (simp add: assms basevars)
  obtain ssl where 141:  $v \text{ (s)} = (\text{Suc } k)$ 
    using 13 by auto
  have 14:  $v \text{ (ifirst (iapp ss (INil ssl)))} = 0 \wedge \text{ilen (iapp ss (INil ssl))} = \text{Suc } k$ 
    by (simp add: 12)
  have 15:  $(\forall i < \text{ilen (iapp ss (INil ssl))}. v \text{ (inth (iapp ss (INil ssl)) (Suc } i)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } i))) =$ 
     $((\forall i < \text{ilen } ss. v \text{ (inth (iapp ss (INil ssl)) (Suc } i)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } i))) \wedge$ 
     $v \text{ (inth (iapp ss (INil ssl)) (Suc } k)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } k)))$ 
    using 14 less-Suc-eq by auto
  have 16:  $v \text{ (inth (iapp ss (INil ssl)) (Suc } k)) = v \text{ (s)}$ 
    by (metis 14 ilast-iapp)
  have 17:  $v \text{ (inth (iapp ss (INil ssl)) } k) = v \text{ (inth } ss \text{ } k)$ 
    by (simp add: 12 iapp-inth)
  have 18:  $v \text{ (inth (iapp ss (INil ssl)) (Suc } k)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } k))$ 
    by (simp add: 121 141 16 17)
  have 19:  $(\forall i < \text{ilen } ss. v \text{ (inth (iapp ss (INil ssl)) (Suc } i)) = \text{Suc } (v \text{ (inth (iapp ss (INil ssl)) } i)))$ 
    by (simp add: 12 iapp-inth)
  show ?thesis
    using 14 15 18 19 by blast
qed
qed
qed
have 20 :  $(\nexists [ \$v = \#0 \wedge \text{len } k \wedge v \text{ gets } (\$v + \#1) ] -v)$ 
  using assms
  by (simp add: commonsfx-def itl-defs imap-eq-inth-ilen inth-imap)
    (metis getsmorehelp inth-imap)
show ?thesis using 20 by auto
qed

```

```

lemma RevCommonPfxStableAndLenK:
assumes basevars v
shows  $(\nexists [ (\$v = \#(c::nat) \wedge \text{len } k \wedge \text{stable } v)^r ] -v)$ 
using assms
using RevCommonPfxSfx CommonSfxStableAndLenK by blast

```

lemma *RevCommonSfxStableAndLenK*:
assumes *basevars v*
shows $\mathbb{H}[(\$v = \#(c::nat) \wedge len\ k \wedge stable\ v)^r]-v)$
using *assms*
using *RevCommonSfxPfx CommonPfxStableAndLenK* **by** *blast*

lemma *StableAndLenKRev*:
 $\vdash (\$v = \#(c::nat) \wedge len\ k \wedge stable\ v) = (\$v = \#(c::nat) \wedge len\ k \wedge stable\ v)^r$
by (*simp add: Valid-def itl-defs current-val-d-def reverse-d-def*)
(metis diff-le-self ifirst-irev irev-ilen irev-inth
prefix-ilen-bound prefix-ilen irev-irev-ident)

17.2.5 Basic theorems

lemma *RevAndWeakNext*:
 $\vdash (init\ w \wedge wnext\ f \wedge fin\ w1)^r = (fin\ w \wedge wprev\ (f^r) \wedge init\ w1)$
by (*metis (no-types, lifting) RInitEqvFin RRand RRWPrevEqvWNext all-rev-eq(3) inteq-reflection*)

lemma *CommonSfxChopFixed*:
assumes *basevars v*
 $\mathbb{H}[f]-v$
 $\mathbb{H}[g]-v$
 $\vdash fin\ w \longrightarrow f$
 $\vdash g \longrightarrow init\ w$
shows $\mathbb{H}[f;(g \wedge len\ k)]-v$
proof –
have 1: $\mathbb{H}[g^r]-v$
using *assms RevCommonPfxSfx* **by** *blast*
have 2: $\mathbb{H}[f^r]-v$
using *assms RevCommonPfxSfx* **by** *blast*
have 3: $\vdash g^r \longrightarrow fin\ w$
by (*metis RImpRule RInitEqvFin assms(5) int-eq*)
have 4: $\vdash init\ w \longrightarrow f^r$
by (*metis RFinEqvInit RImpRule assms(4) inteq-reflection*)
have 5: $\mathbb{H}[(g^r \wedge len\ k);f^r]-v$
using *assms using CommonPfxChopFixed[of v LIFT(g^r) LIFT(f^r) w k]*
using 1 2 3 4 **by** *blast*
have 6: $\vdash (g^r \wedge len\ k)^r = (g \wedge len\ k)$
by (*simp add: intI len-defs reverse-d-def*)
have 7: $\vdash ((g^r \wedge len\ k);f^r)^r = f;(g \wedge len\ k)$
by (*metis 6 EqvReverseReverse all-rev-eq(13) int-eq*)
have 8: $(\mathbb{H}[(g^r \wedge len\ k);f^r]-v) = (\mathbb{H}[f;(g \wedge len\ k)]-v)$
using 7 *RevCommonSfxPfx assms(1) int-eq* **by** *force*
show *?thesis* **using** 5 8 **by** *auto*
qed

lemma *CommonSfxEqv*:
assumes *basevars v*
 $\vdash f = g$

shows $(\mathbb{h}[f]-v) = (\mathbb{h}[g]-v)$
using *assms inteq-reflection* **by** *blast*

lemma *CommonSfxEmptyOrMore:*

assumes *basevars v*
shows $(\mathbb{h}[f]-v) = (\mathbb{h}[(f \wedge \text{empty}) \vee (f \wedge \text{more})]-v)$
proof –
have $1: \vdash f = ((f \wedge \text{empty}) \vee (f \wedge \text{more}))$
unfolding *empty-d-def* **by** *fastforce*
show *?thesis* **using** *CommonSfxEqv 1 assms* **by** *blast*
qed

lemma *CommonSfxChoice:*

assumes *basevars v*
shows $(\mathbb{h}[f]-v) = (\mathbb{h}[(f \wedge g) \vee (f \wedge \neg g)]-v)$
proof –
have $1: \vdash f = ((f \wedge g) \vee (f \wedge \neg g))$
by *fastforce*
show *?thesis* **using** *CommonSfxEqv 1 assms* **by** *blast*
qed

lemma *CommonSfxDetChoice:*

assumes *basevars v*
 $\mathbb{h}[(f \wedge g) \vee (f \wedge \neg g)]-v$
shows $(\mathbb{h}[f \wedge g]-v) \wedge (\mathbb{h}[f \wedge \neg g]-v)$
using *assms unfolding commonsfx-def* **by** *simp*

lemma *CommonSfxBox:*

assumes *basevars v*
shows $(\mathbb{h}[\Box f]-v) = (\mathbb{h}[f \wedge \text{wnext } (\Box f)]-v)$
proof –
have $1: \vdash \Box f = (f \wedge \text{wnext } (\Box f))$
by (*simp add: BoxEqvAndWnextBox*)
show *?thesis* **using** *assms 1 CommonSfxEqv[of v LIFT($\Box f$) LIFT($f \wedge \text{wnext } (\Box f)$)]* **by** *simp*
qed

lemma *CommonSfxBi:*

assumes *basevars v*
 $\mathbb{h}[f]-v$
shows $\mathbb{h}[bi\ f]-v$
using *assms* **by** (*simp add: commonsfx-def itl-defs*) *force*

lemma *CommonSfxBox1:*

assumes *basevars v*
 $\mathbb{h}[f]-v$
shows $\mathbb{h}[\Box f]-v$
using *assms* **by** (*simp add: commonsfx-def itl-defs*) *force*

lemma *CommonSfxAssign*:
assumes *basevars* ($v :: \text{state} \Rightarrow \text{nat}$)
shows $\models [\$v = \#c \wedge v := \#d \wedge \text{skip}] - v$
proof –
have 1: $\vdash (\$v = \#c \wedge v := \#d \wedge \text{skip}) = (\$v = \#c \wedge \text{skip} \wedge v\$ = \#d)$
unfolding *next-assign-d-def* **by** *fastforce*
have 2: $\models [\$v = \#c \wedge \text{skip} \wedge v\$ = \#d] - v$
using *CommonSfxUnitAssignAndNextValAndSkip*[of $v \ c \ d$] *assms* **by** *blast*
show *?thesis* **using** *assms* 1 2
 CommonSfxEqv [of $v \ \text{LIFT}(\$v = \#c \wedge v := \#d \wedge \text{skip}) \ \text{LIFT}(\$v = \#c \wedge \text{skip} \wedge v\$ = \#d)$]
by *blast*
qed

17.2.6 Backward Executability theorems

lemma *RStableEqvStable*:
 $\vdash (\text{stable } v)^r = (\text{stable } v)$
proof –
have 1: $\vdash (v \text{ gets } \$v)^r = \text{ba } (\text{skip} \longrightarrow (\$v = (\$v)^r))$
using *RGetsEqvBaSkipImp* **by** *blast*
have 2: $\vdash (\text{skip} \longrightarrow (\$v = (\$v)^r)) =$
 $(\text{skip} \longrightarrow (v\$ = \$v))$
by (*auto simp add: Valid-def all-rev-eq itl-defs*)
have 3: $\vdash \text{ba } (\text{skip} \longrightarrow (\$v = (\$v)^r)) =$
 $\text{ba } (\text{skip} \longrightarrow (v\$ = \$v))$
using 2 **by** (*simp add: BaEqvBa*)
have 4: $\vdash \text{ba } (\text{skip} \longrightarrow (v\$ = \$v)) = (v \text{ gets } \$v)$
using *BaEqvBtBi GetsAlways* **by** *fastforce*
show *?thesis* **unfolding** *stable-d-def* **using** 1 3 4 **by** *fastforce*
qed

lemma *RStableInitialEqvStableFin*:
 $\vdash (\$v = \#c \wedge \text{stable } v)^r = (!v = \#c \wedge \text{stable } v)$
proof –
have *f1*: $\vdash (!v = \#c) = (\$v = \#c)^r$
by (*simp add: all-rev-eq(1) all-rev-eq(3) all-rev-eq(8)*)
have $\vdash (\text{stable } v)^r = (\text{stable } v)$
by (*simp add: RStableEqvStable int-eq*)
then show *?thesis*
using *f1 RAnd* **by** *fastforce*
qed

lemma *StableInitialEqvStableFin*:
 $\vdash (\$v = \#c \wedge \text{stable } v) = (!v = \#c \wedge \text{stable } v)$
by (*simp add: Valid-def itl-defs*)
 $(\text{metis order-refl})$

lemma *BExecStableInitial*:
assumes *basevars v*
shows $\vdash [\$v = \#c \wedge \text{stable } v] \text{-} v$
proof –
have 1: *sat* ($\$v = \#c \wedge \text{stable } v$)
using *assms* **by** (*simp add: sat-d-def commonpfx-def itl-defs basevars-def*)
(*meson len0help*)
have 2: $\ddagger [\$v = \#c \wedge \text{stable } v] \text{-} v$
by (*meson FExecStableInitial assms fexec-def*)
have 3: $\natural [\$v = \#c \wedge \text{stable } v] \text{-} v$
using 2
by (*metis RStableInitialEqvStableFin RevCommonSfxPfx StableInitialEqvStableFin assms int-eq*)
show ?thesis **unfolding** *bexec-def* **using** 1 3 **by** *auto*
qed

lemma *RAlwaysConstantEqvAlwaysConstant*:
 $\vdash (\Box(\$v = \#c))^r = \Box(\$v = \#c)$
by (*auto simp add: Valid-def itl-defs reverse-d-def irev-inth*)
(*metis diff-diff-cancel diff-le-self*)

lemma *BExecAlwaysConstant*:
assumes *basevars v*
shows $\vdash [\Box(\$v = \#c)] \text{-} v$
proof –
have 1: *sat* $\Box(\$v = \#c)$
using *assms* **by** (*simp add: sat-d-def commonpfx-def itl-defs basevars-def*)
(*metis len0help*)
have 2: $\ddagger [\Box(\$v = \#c)] \text{-} v$
by (*meson FExecAlwaysConstant assms fexec-def*)
have 3: $\natural [\Box(\$v = \#c)] \text{-} v$
by (*meson 2 RAlwaysConstantEqvAlwaysConstant RevCommonSfxPfx CommonSfxEqv assms*)
show ?thesis **unfolding** *bexec-def* **using** 1 3 **by** *auto*
qed

lemma *NotBExecGetsAndMore*:
assumes *basevars v*
shows $\neg (\vdash [\$v = \#(0::\text{nat}) \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)] \text{-} v)$
proof –
have 10: *sat* $\$v = \#0 \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)$
using *assms* **by** (*simp add: sat-d-def itl-defs*)
(*metis inth.simps(1) basevars inth-Suc inth-zero ilen.simps(1) ilen.simps(2) neq0-conv*
not-less-eq plus-1-eq-Suc)
have 20: $\neg (\natural [\$v = \#0 \wedge \text{more} \wedge v \text{ gets } (\$v + \#1)] \text{-} v)$
using *assms*
using *NotCommonSfxGetsAndMore* **by** *auto*
show ?thesis **unfolding** *bexec-def* **using** 10 20 **by** *simp*
qed

```

lemma BExecGetsAndEmpty:
assumes basevars v
shows  $\mathsf{b}[\$v = \#(0::\mathsf{nat}) \wedge \mathsf{empty} \wedge v \text{ gets } (\$v + \#1)] - v$ 
proof –
  have 10:  $\mathsf{sat} \$v = \#0 \wedge \mathsf{empty} \wedge v \text{ gets } (\$v + \#1)$ 
    using assms by (simp add: sat-d-def itl-defs)
    (metis baseE ilen.simps(1) inth.simps(1) not-less-zero)
  have 20:  $\mathsf{b}[\$v = \#0 \wedge \mathsf{empty} \wedge v \text{ gets } (\$v + \#1)] - v$ 
    using assms
    by (auto simp add: commonsfx-def itl-defs imap-eq-inth-ilen)
show ?thesis unfolding bexec-def using 10 20 by simp
qed

```

```

lemma BExecChopFixedExists:
assumes basevars v
   $\mathsf{b}[f] - v$ 
   $\mathsf{b}[g] - v$ 
   $\vdash \mathsf{fin} w \longrightarrow f$ 
   $\vdash g \longrightarrow \mathsf{init}(w)$ 
shows  $(\exists k. (\mathsf{b}[f; (g \wedge \mathsf{len} k)] - v))$ 
using assms SatChopR[of f g w]
unfolding bexec-def
using SatExistsChopR CommonSfxChopFixed
by metis

```

```

lemma BExecPrevRule:
assumes basevars v
   $\mathsf{b}[g] - v$ 
   $\mathsf{b}[\mathsf{fin} w \wedge \mathsf{empty}] - v$ 
shows  $\mathsf{b}[\mathsf{fin} w \wedge \mathsf{prev} g] - v$ 
proof –
  have 1:  $\dagger[g^r] - v$ 
    using RevCommonPfxSfx SatRev assms fexec-def bexec-def by blast
  have 2:  $\dagger[(\mathsf{init} w \wedge \mathsf{empty})^r] - v$ 
    by (metis RAndEmptyEqvAndEmpty REmptyEqvEmpty RInitEqvFin RevCommonPfxSfx all-rev-eq(3)
      assms(1) assms(3) fexec-def bexec-def inteq-reflection)
  have 3:  $\dagger[(\mathsf{init} w) \wedge \mathsf{empty}] - v$ 
    using 2 RAndEmptyEqvAndEmpty SatRev CommonPfxEqv assms(1) fexec-def by blast
  have 4:  $\dagger[(\mathsf{init} w) \wedge (\bigcirc (g^r))] - v$ 
    by (simp add: 1 3 FExecNextRule assms(1))
  have 5:  $\dagger[(\mathsf{fin} w \wedge \mathsf{prev} g)^r] - v$ 
    by (metis 4 RFinEqvInit RPrevEqvNext all-rev-eq(3) inteq-reflection)
  show ?thesis
    using 5 RevCommonPfxSfx SatRev assms(1) fexec-def bexec-def by blast
qed

```

In Tempura we have unfortunately no rules for backward execution. But we can define a mirror image

of lemma `FExecWNextRule`. So we first generate the last state of the interval and then proceed to determine the previous states if there are any.

lemma *BExecWPrevRule*:

```

assumes basevars v
          b[g]-v
          b[fin w  $\wedge$  empty]-v
shows    b[fin w  $\wedge$  wprev g]-v
proof -
have 1:  $\vdash$  (fin w  $\wedge$  wprev g) = ( (fin w  $\wedge$  empty)  $\vee$  (fin w  $\wedge$  prev g))
using WprevEqvEmptyOrPrev by fastforce
have 2: sat fin w  $\wedge$  wprev g
      using 1 SatOr assms BExecPrevRule[of v g w] unfolding bexec-def
      by (metis int-eq)
have 3:  $\mathbb{b}$ [fin w  $\wedge$  prev g]-v
      using assms BExecPrevRule[of v g w] unfolding bexec-def
      by blast
have 5:  $\mathbb{b}$ [fin w  $\wedge$  wprev g]-v
      using assms 1 3 AndFinEqvChopAndEmpty[of LIFT(prev g) w] unfolding bexec-def commonsfx-def
      by (simp add: itl-defs)
      (metis diff-self-eq-0 gr-implies-not0 le-less neq0-conv suffix-ilen suffix-ilen-last suffix-zero)
show ?thesis unfolding bexec-def using 2 5 by auto
qed

```

lemma *BExecStrengthen*:

```

assumes basevars v
          sat f  $\wedge$  g
           $\mathbb{b}$ [f]-v
shows    b[f  $\wedge$  g]-v
using assms unfolding bexec-def commonsfx-def
by simp

```

lemma *RevBExecFExec*:

```

assumes basevars v
           $\vdash f = f^r$ 
shows    (b[f]-v) = ( $\dagger$ [f]-v)
proof -
have 1: (sat f) = (sat fr)
      using SatRev assms(1) by blast
have 2: ( $\mathbb{b}$ [f]-v) = ( $\dagger$ [f]-v)
      using RevCommonPfxSfx assms(1) assms(2) inteq-reflection by force
show ?thesis unfolding bexec-def fexec-def using 1 2 by auto
qed

```

17.3 Reversing bad computations

The following lemmas gives the conditions necessary to “undo” a bad computation.

lemma *FExecChopLenKRev*:

```

assumes basevars v
          sat f  $\wedge$  len k

```

```

 $\dagger[f]-v$ 
 $\mathbb{h}[f]-v$ 
shows  $\dagger[(f \wedge \text{len } k); (f^r \wedge \text{len } k)]-v$ 
proof –
have 1:  $\vdash (f \wedge \text{len } k)^r = (f^r \wedge \text{len } k)$ 
  by (simp add: intI len-defs reverse-d-def)
have 2:  $\text{sat } (f^r \wedge \text{len } k)$ 
  by (metis 1 SatRev assms(1) assms(2) int-eq)
obtain s0 where 3:  $s0 \models (f \wedge \text{len } k)$ 
  using assms(2) sat-d-def by blast
obtain s1 where 4:  $(s1 \models (f^r \wedge \text{len } k)) \wedge \text{ilast } s0 = \text{ifirst } s1$ 
  using 2
  by (metis 1 3 RREnd all-rev-eq(3) ifirst-irev int-eq reverse-d-def)
have 5:  $\text{fuse } s0 \ s1 \models (f \wedge \text{len } k); (f^r \wedge \text{len } k)$ 
  by (meson 3 4 chop-fuse)
have 6:  $\text{sat } (f \wedge \text{len } k); (f^r \wedge \text{len } k)$ 
  using 5 sat-d-def by auto
have 7:  $\dagger[f \wedge \text{len } k]-v$ 
  using FExecStrengthen assms fexec-def by blast
have 8:  $\dagger[f^r \wedge \text{len } k]-v$ 
  using 2 FExecStrengthen RevCommonPfxSfx assms fexec-def by blast
have 9:  $\dagger[(f \wedge \text{len } k); (f^r \wedge \text{len } k)]-v$ 
  using 7 8
proof (auto simp add: itl-defs min-def Valid-def reverse-d-def interval-prefix-suffix-help
  commonpfx-def)
fix s1 :: state interval
fix s :: state interval
assume a0:  $\forall s1 \ s. f \ s1 \wedge \text{ilen } s1 = k \wedge f \ s \wedge \text{ilen } s = k \wedge \text{ilen } s \leq \text{ilen } s1$ 
   $\longrightarrow \text{imap } v \ s = \text{imap } v \ (\text{prefix } k \ s1)$ 
assume a1:  $\forall s1 \ s. f \ (\text{irev } s1) \wedge \text{ilen } s1 = k \wedge f \ (\text{irev } s) \wedge \text{ilen } s = k \wedge \text{ilen } s \leq \text{ilen } s1$ 
   $\longrightarrow \text{imap } v \ s = \text{imap } v \ (\text{prefix } k \ s1)$ 
assume a2:  $\text{ilen } s \leq \text{ilen } s1$ 
assume a3:  $k \leq \text{ilen } s$ 
assume a4:  $f \ (\text{prefix } k \ s1)$ 
assume a5:  $f \ (\text{prefix } k \ s)$ 
assume a6:  $f \ (\text{irev } (\text{suffix } k \ s1))$ 
assume a7:  $\text{ilen } s1 - k = k$ 
assume a8:  $f \ (\text{irev } (\text{suffix } k \ s))$ 
assume a9:  $\text{ilen } s - k = k$ 
show  $\text{imap } v \ s = \text{imap } v \ (\text{prefix } (\text{ilen } s) \ s1)$ 
proof –
have 91:  $\text{imap } v \ (\text{prefix } k \ s) = \text{imap } v \ (\text{prefix } k \ s1)$ 
  using a0 a3 a4 a5 a7 by force
have 92:  $\text{imap } v \ (\text{suffix } k \ s) = \text{imap } v \ (\text{prefix } k \ (\text{suffix } k \ s1))$ 
  by (simp add: a1 a6 a7 a8 a9)
have 93:  $\text{fuse } (\text{prefix } k \ s) \ (\text{suffix } k \ s) = s$ 
  by (simp add: a3 fuse-prefix-suffix)
have 94:  $\text{fuse } (\text{prefix } k \ s1) \ (\text{suffix } k \ s1) = s1$ 
  using a2 a3 fuse-prefix-suffix le-trans by blast
have 95:  $\text{imap } v \ (\text{fuse } (\text{prefix } k \ s) \ (\text{suffix } k \ s)) =$ 

```

```

      fuse (imap v (prefix k s)) (imap v (suffix k s))
    by (simp add: a3 fuse-prefix-suffix imap-prefix imap-suffix)
  have 96: fuse (imap v (prefix k s)) (imap v (suffix k s)) =
    fuse (imap v (prefix (ilen (prefix k s)) (prefix k s1)))
      (imap v (prefix (ilen (suffix k s)) (suffix k s1)))
    using 91 92 a3 a9 prefix-ilen-good by fastforce
  have 97: (fuse (imap v (prefix (ilen (prefix k s)) (prefix k s1)))
    (imap v (prefix (ilen (suffix k s)) (suffix k s1)))) =
    (fuse (prefix (ilen (prefix k s)) (imap v (prefix k s1)))
      (prefix (ilen (suffix k s)) (imap v (suffix k s1))))
    by (metis 91 a3 a7 a9 imap-prefix order-refl pref-pref prefix-ilen-good suffix-ilen)
  have 98: (fuse (prefix (ilen (prefix k s)) (imap v (prefix k s1)))
    (prefix (ilen (suffix k s)) (imap v (suffix k s1)))) =
    (prefix (ilen (fuse (prefix k s) (suffix k s)))
      (fuse (imap v (prefix k s1)) (imap v (suffix k s1))))
    by (metis 91 92 95 a7 ilen-imap prefix-ilen suffix-ilen)
  have 99: (prefix (ilen (fuse (prefix k s) (suffix k s)))
    (fuse (imap v (prefix k s1)) (imap v (suffix k s1)))) =
    (imap v (prefix (ilen s) s1))
    by (metis a2 a3 fuse-prefix-suffix ilen-imap imap-prefix imap-suffix le-trans)
  show ?thesis using 93 95 96 97 98 99 by auto
qed
qed
show ?thesis by (simp add: 6 9 fexec-def)
qed

```

lemma *FExecRevChopLenKRev*:

```

assumes basevars v
  sat f ∧ len k
  ‡[f]-v
  ‡[f]-v
shows ‡[ (fr ∧ len k); (f ∧ len k) ]-v
proof –
  have 1: ⊢ (f ∧ len k)r = (fr ∧ len k)
    by (simp add: intI len-defs reverse-d-def)
  have 2: sat fr ∧ len k
    by (metis 1 SatRev assms(1) assms(2) int-eq)
  have 3: ‡[fr]-v
    using RevCommonPfxSfx assms by blast
  have 4: ‡[fr]-v
    using RevCommonSfxPfx assms by auto
  show ?thesis using assms 1 2 3 4
    by (metis EqvReverseReverse FExecChopLenKRev int-eq)
qed

```

lemma *BExecChopLenKRev*:

```

assumes basevars v
  sat f ∧ len k
  ‡[f]-v
  ‡[f]-v

```

```

shows  $\vdash [(f \wedge \text{len } k); (f^r \wedge \text{len } k)]\text{-}v$ 
using assms
proof –
  have 1:  $\vdash (f \wedge \text{len } k)^r = (f^r \wedge \text{len } k)$ 
    by (simp add: intI len-defs reverse-d-def)
  have 2:  $\text{sat } (f^r \wedge \text{len } k)$ 
    by (metis 1 SatRev assms(1) assms(2) int-eq)
  obtain s0 where 3:  $s0 \models (f \wedge \text{len } k)$ 
    using assms(2) sat-d-def by blast
  obtain s1 where 4:  $(s1 \models (f^r \wedge \text{len } k)) \wedge \text{ilast } s0 = \text{ifirst } s1$ 
    using 2
  by (metis 1 3 RREnd all-rev-eq(3) ifirst-irev int-eq reverse-d-def)
  have 5:  $\text{fuse } s0 \text{ } s1 \models (f \wedge \text{len } k); (f^r \wedge \text{len } k)$ 
    by (meson 3 4 chop-fuse)
  have 6:  $\text{sat } (f \wedge \text{len } k); (f^r \wedge \text{len } k)$ 
    using 5 sat-d-def by auto
  have 7:  $\ddagger[(f \wedge \text{len } k); (f^r \wedge \text{len } k)]\text{-}v$ 
    using assms FExecChopLenKRev[of v f k] unfolding fexec-def by blast
  have 8:  $\vdash ((f \wedge \text{len } k); (f^r \wedge \text{len } k))^r = (f \wedge \text{len } k); (f^r \wedge \text{len } k)$ 
    by (metis 1 LeftChopEqvChop RREnd all-rev-eq(13) all-rev-eq(3) inteq-reflection)
  have 9:  $\natural[(f \wedge \text{len } k); (f^r \wedge \text{len } k)]\text{-}v$ 
    using 7 8 RevCommonPfxSfx assms(1) int-eq by force
  show ?thesis by (simp add: 6 9 bexec-def)
qed

```

lemma *BExecRevChopLenKRev*:

```

assumes basevars v
   $\text{sat } f \wedge \text{len } k$ 
   $\ddagger[f]\text{-}v$ 
   $\natural[f]\text{-}v$ 
shows  $\vdash [(f^r \wedge \text{len } k); (f \wedge \text{len } k)]\text{-}v$ 
proof –
  have 1:  $\vdash (f \wedge \text{len } k)^r = (f^r \wedge \text{len } k)$ 
    by (simp add: intI len-defs reverse-d-def)
  have 2:  $\text{sat } f^r \wedge \text{len } k$ 
    by (metis 1 SatRev assms(1) assms(2) int-eq)
  have 3:  $\ddagger[f^r]\text{-}v$ 
    using RevCommonPfxSfx assms by blast
  have 4:  $\natural[f^r]\text{-}v$ 
    using RevCommonSfxPfx assms by auto
  show ?thesis using assms 1 2 3 4
  by (metis BExecChopLenKRev EquReverseReverse inteq-reflection)
qed

```

end

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